This paper deals with the 2-dimensional harmonic oscillator. It is based on:

http://quantum.phys.unm.edu/521-07/2DharmonicOscillator.pdf

http://ocw.nctu.edu.tw/upload/classbfs120904402944051.pdf

Hope I can help you with learning quantum mechanics.

Inhalt

Introduction

Classic

We use a particle moving in the xy-plane. Its potential energy V depends on x and y:

$$V(x,y) = \frac{m}{2}\omega^2(x^2 + y^2)$$

This is a radial potential: $x^2 + y^2 = r^2$

The classical Hamiltonian:

$$H = H_{xy} + H_z$$

$$H_{xy} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

$$H_z = \frac{1}{2m}p_z^2$$

Note: the oscillation takes place in the xy-plane. In the z-direction the particle is moving with unchanged speed and no potential whatsoever.

Note: H_{xy} is a two-dimensional harmonic oscillator Hamiltonian.

The equation of motion.

The z-direction:

$$p_z(t) = p_z$$

$$z(t) = \frac{p_z}{m} \cdot t + z_0$$

The x-direction:

(1) $$p_x(t) = -m\omega x_0 sin(\omega t - \varphi_x)$$

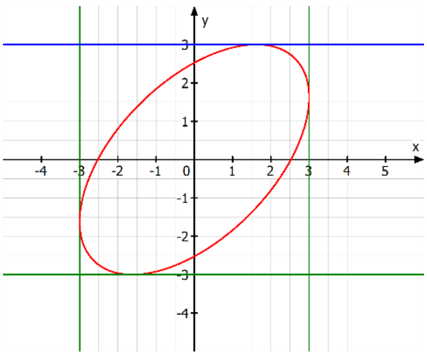
(2) $$x(t) = x_0 cos(\omega t - \varphi_x)$$

The y-direction:

(3) $$p_y(t) = -m\omega y_0 sin\left(\omega t - \varphi_y\right)$$

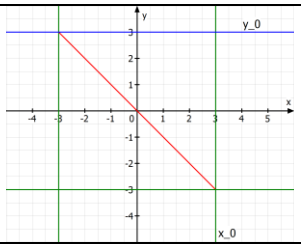
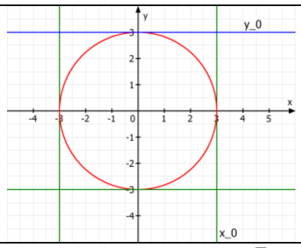
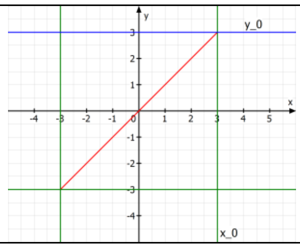
(4) $$y(t) = y_0 cos\left(\omega t - \varphi_y\right)$$

Note: $p_z, z_0, \varphi_x, \varphi_{y,} x_0, y_0$ are constants which depend on the initial conditions. x_0, y_0 are assumed to be positive.



The classical trajectory of a particle in a two-dimensional harmonic potential ($x - y$-plane) shows an ellipse. The x-coordinate is in the range from $-x_0$ to x_0, the y-coordinate from $-y_0$ to y_0.

The shape of the ellipse depends on the phase difference $\varphi_y - \varphi_x$.

		
For $\varphi_x = 0, \varphi_y = \pi$	For $\varphi_x = 0, \varphi_y = \frac{\pi}{2}$	For $\varphi_x = 0, \varphi_y = 0$

The total energy of the motion in the x, y-plane:

$$H_{xy} = \frac{1}{2}m\omega^2(x_0^2 + y_0^2)$$

The sum of the energies in x-direction and y-direction:

$$H_{xy} = H_x + H_y = \frac{1}{2}m\omega^2 x_0^2 + \frac{1}{2}m\omega^2 y_0^2$$

The orbital angular momentum L_z along the z-axis:

$$L_z = x(t)p_y - y(t)p_x$$

We use (1),(2),(3) and (4):

$$L_z = -xm\omega y_0 sin(\omega t - \varphi_y) + ym\omega x_0 sin(\omega t - \varphi_x) =$$

$$m\omega \left(y(t)x_0 sin(\omega t - \varphi_x) - x(t)y_0 sin(\omega t - \varphi_y)\right) =$$

$$m\omega y_0 x_0 \left(cos(\omega t - \varphi_y) sin(\omega t - \varphi_x) - cos(\omega t - \varphi_x) sin(\omega t - \varphi_y)\right) =$$

$$m\omega y_0 x_0 sin(\omega t - \varphi_x - \omega t + \varphi_y) =$$

$$m\omega y_0 x_0 sin(\varphi_y - \varphi_x) \coloneqq m\omega y_0 x_0 sin(\Delta\varphi)$$

According to this result we state:

- L_z is positive: $0 < \Delta\varphi < \pi$, rotation counterclockwise
- L_z is negative: $-\pi < \Delta\varphi < 0$, rotation clockwise
- L_z is zero: $\Delta\varphi = 0 \; or \mp \pi$
- L_z is maximal: $\Delta\varphi = \mp\frac{\pi}{2}$ and $y_0 = x_0$ (circle)

Quantum mechanics

Note: all operators $\widehat{H}_{xy}$ etc. written without hat: H_{xy}.

The quantum Hamiltonian:

$$H_{xy} = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(X^2 + Y^2)$$

$$H_z = \frac{P_z^2}{2m}$$

The orbital angular momentum L_z along the z-axis:

$$L_z = X(t)P_y - Y(t)P_x$$

The stationary states $|\varphi\rangle$ of the particle:

$$H|\varphi\rangle = \left(H_{xy} + H_z\right)|\varphi\rangle = E|\varphi\rangle$$

We choose a composed basis of eigenstates of H:

$$|\varphi\rangle = |\varphi_{xy}\rangle \otimes |\varphi_z\rangle$$

$|\varphi_{xy}\rangle$ is an eigenvector of H_{xy} in the state space S_{xy}:

$$H_{xy}|\varphi_{xy}\rangle = E_{xy}|\varphi_{xy}\rangle$$

$|\varphi_z\rangle$ is an eigenvector of H_z in the state space S_z:

$$H_z|\varphi_z\rangle = E_z|\varphi_z\rangle$$

We get the total energy:

$$E = E_{xy} + E_z$$

Note: the degeneracy in $S \coloneqq S_{xy} \otimes S_z$ is not due solely to the degeneracy of E_{xy} in S_{xy} and of E_z in S_z. Two eigenvectors of H of the form $|\varphi\rangle = |\varphi_{xy}\rangle \otimes |\varphi_z\rangle$ can have the same total energy E without their corresponding values of E_{xy} and of E_z being equal.

Note: E_z describes the stationary states of a free particle in a one-dimensional problem:

$$\langle z|\varphi_z\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip_z z}{\hbar}}$$

Note: p_z is a real constant.

The energy in z-direction:

$$E_z = \frac{{p_z}^2}{2m}$$

Note: The eigenvalues E_z of H_z are two-fold degenerate in the space S_z.

Obviously, the problem is the motion in the plane:

$$H_{xy}|\varphi_{xy}\rangle = E_{xy}|\varphi_{xy}\rangle$$

L_z commutes with neither H_x nor H_y.

H commutes with L_z, but not with L_x and L_y.

A complete set of commuting operators (CSCO) can be built of (H_{xy}, H_x), (H_{xy}, H_y) or (H_{xy}, L_z):

$$\{H_{xy}, H_x\}$$

$$\{H_{xy}, H_y\}$$

$$\{H_{xy}, L_z\}$$

H commutes with L_z but neither with L_x nor L_y because the potential energy

$$V(x, y) = \frac{m}{2}\omega^2(x^2 + y^2)$$

is rotation-invariant about the z-axis only.

We are working with L_z because it acts in S_{xy} only.

Stationary states

Energies and stationary states

We will classify the stationary states by quantum numbers n_x and n_y.

We search the solutions of the eigenvalue equation:

$$H_{xy}|\varphi_{xy}\rangle = E_{xy}|\varphi_{xy}\rangle$$

H_{xy} can be written as a sum of two independent directions x and y:

$$H_{xy} = H_x + H_y$$

H_x and H_y are both Hamiltonians of a one-dimensional harmonic oscillator:

$$H_x = \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2(X^2)$$

$$H_y = \frac{P_y^2}{2m} + \frac{1}{2}m\omega^2(Y^2)$$

We know the eigenstates $|\varphi_{n_x}\rangle$ of H_x in S_x:

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\varphi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}}\, e^{-\frac{m\omega x^2}{2\hbar}}$$

…

The energies:

$$E_x = \left(n_x + \frac{1}{2}\right)\hbar\omega$$

$$E_y = \left(n_y + \frac{1}{2}\right)\hbar\omega$$

With this we get the eigenstates of H_{xy} by use of the tensor product:

$$|\varphi_{n_x,n_y}\rangle = |\varphi_{n_x}\rangle \otimes |\varphi_{n_y}\rangle$$

The energy:

$$E_{xy} = E_x + E_y =$$

$$\left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega =$$

$$(n_x + n_y + 1)\hbar\omega$$

E_x is non-degenerate in S_x, E_y is non-degenerate in S_y.

$\left|\varphi_{n_x n_y}\right\rangle$ corresponds to a pair $\{n_x, n_y\}$.

$\{H_x, H_y\}$ form a complete set of commutating operators (CSCO) in S_{xy}.

We use the lowering operators:

$$a_x = \frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right)$$

$$a_y = \frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right)$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}}$$

The raising operators:

$${a_x}^\dagger = \frac{1}{\sqrt{2}}\left(\beta X - i\frac{P_x}{\beta\hbar}\right)$$

$${a_y}^\dagger = \frac{1}{\sqrt{2}}\left(\beta Y - i\frac{P_y}{\beta\hbar}\right)$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}}$$

$a_x, {a_x}^\dagger$ act in S_x, $a_y, {a_y}^\dagger$ act in S_y.

The only non-zero commutators between the four operators $a_x, a_y, {a_x}^\dagger$ and ${a_y}^\dagger$ are the commutators between lowering and raising operator:

$$\left[a_x, {a_x}^\dagger\right] = 1 = \left[a_y, {a_y}^\dagger\right]$$

Note: depending on the model (wave functions or matrices) "1" can be the identity matrix.

The numbering operators of the one-dimensional harmonic oscillators:

$$N_x = {a_x}^\dagger a_x$$

$$N_y = {a_y}^\dagger a_y$$

We rewrite the Hamiltonian:

$$H_{xy} = H_x + H_y = (N_x + N_y + 1)\omega\hbar$$

We apply the Hamiltonian H_{xy} to the wave function $\left|\varphi_{n_x, n_y}\right\rangle$:

$$H_{xy}\left|\varphi_{n_x,n_y}\right\rangle = \left(N_x + N_y + 1\right)\omega\hbar\left|\varphi_{n_x,n_y}\right\rangle =$$

$$\omega\hbar N_x\left|\varphi_{n_x,n_y}\right\rangle + \omega\hbar N_y\left|\varphi_{n_x,n_y}\right\rangle + \omega\hbar\left|\varphi_{n_x,n_y}\right\rangle =$$

$$\omega\hbar n_x\left|\varphi_{n_x,n_y}\right\rangle + \omega\hbar n_y\left|\varphi_{n_x,n_y}\right\rangle + \omega\hbar\left|\varphi_{n_x,n_y}\right\rangle$$

The ground state:

$$|\varphi_{0,0}\rangle = |\varphi_{n_x=0}\rangle \otimes \left|\varphi_{n_y=0}\right\rangle$$

We apply the operators ${a_x}^\dagger$ and ${a_y}^\dagger$ and get:

$$\left|\varphi_{n_x,n_y}\right\rangle = \frac{1}{\sqrt{n_x!\,n_y!}}\left({a_x}^\dagger\right)^{n_x}\left({a_y}^\dagger\right)^{n_y}|\varphi_{0,0}\rangle$$

The corresponding wave function is the product of $\varphi_{n_x}(x)$ and $\varphi_{n_y}(y)$:

$$\varphi_{n_x,n_y}(x,y) = \frac{\beta}{\sqrt{2^{(n_x+n_y)}\cdot\pi\cdot n_x!\cdot n_y!}}e^{-\frac{1}{2}\beta^2(x^2+y^2)}H_{n_x}(\beta x)H_{n_y}(\beta y) \qquad (5)$$

Note:

$$\beta = \sqrt{\frac{m\omega}{\hbar}}$$

Note: the eigenvalues of the 2D harmonic oscillator are the sum of the two 1D harmonic oscillator eigenvalues, the eigenfunctions are the product of two 1D eigenfunctions.

H_{xy} does not constitute a CSCO in S_{xy}

$$E_{xy} = \left(n_x + n_y + 1\right)\hbar\omega$$

To each value $n = n_x + n_y$ of energy correspond different orthogonal eigenvectors:

$$\left|\varphi_{n_x=n,n_y=0}\right\rangle, \left|\varphi_{n_x=n-1,n_y=1}\right\rangle, \left|\varphi_{n_x=n-2,n_y=2}\right\rangle, \dots, \left|\varphi_{n_x=0,n_y=n}\right\rangle$$

As there are $(n+1)$ of these vectors, the eigenvalue E_n is $(n+1)$-fold degenerate in S_{xy}.

H_{xy} alone does not constitute a CSCO.

$\{H_x, H_y\}$ constitutes a CSCO as well as $\{H_{xy}, H_x\}$ and $\{H_{xy}, H_y\}$.

Stationary states and angular momenta

Significance and properties of the operator L_z

We identified the stationary states by the quantum numbers n_x and n_y. These numbers depend on the axis chosen but are not unique. The energy is invariant under rotation about the z-axis, so we can choose another system of orthogonal axes in the $x-y$ plane. We will obtain different stationary states.

To get rid of this ambiguity, we orient on the angular momentum L_z:

$$L_z = XP_y - YP_x$$

We can express the position operator X and the momentum operator P by the lowering operator and raising operator ${a_x}^\dagger, a_x$ resp. ${a_y}^\dagger, a_y$:

${a_x}^\dagger = \frac{1}{\sqrt{2}}\left(\beta X - i\frac{P_x}{\beta\hbar}\right)$	${a_y}^\dagger = \frac{1}{\sqrt{2}}\left(\beta Y - i\frac{P_y}{\beta\hbar}\right)$	$a_x = \frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right)$	$a_y = \frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right)$

Note: $\beta = \sqrt{\frac{m\omega}{\hbar}}$

The inverse relations:

$X = \sqrt{\frac{\hbar}{2m\omega}}(a_x + {a_x}^\dagger)$	$Y = \sqrt{\frac{\hbar}{2m\omega}}(a_y + {a_y}^\dagger)$	$P_x = i\sqrt{\frac{m\omega\hbar}{2}}({a_x}^\dagger - a_x)$	$P_y = i\sqrt{\frac{m\omega\hbar}{2}}({a_y}^\dagger - a_y)$

We express L_z in terms of operators:

$$L_z = XP_y - YP_x =$$

$$\left(\sqrt{\frac{\hbar}{2m\omega}}(a_x + {a_x}^\dagger)\right)\left(i\sqrt{\frac{m\omega\hbar}{2}}({a_y}^\dagger - a_y)\right) - \left(\sqrt{\frac{\hbar}{2m\omega}}(a_y + {a_y}^\dagger)\right)\left(i\sqrt{\frac{m\omega\hbar}{2}}({a_x}^\dagger - a_x)\right) =$$

$$\frac{i\hbar}{2}\left((a_x + {a_x}^\dagger)({a_y}^\dagger - a_y) - (a_y + {a_y}^\dagger)({a_x}^\dagger - a_x)\right) =;$$

For easier calculation we work with:

$$a \coloneqq a_x, b \coloneqq {a_x}^\dagger, c \coloneqq a_y, d \coloneqq {a_y}^\dagger$$

We get:

$$\frac{i\hbar}{2}\left((a+b)(d-c) - (c+d)(b-a)\right) =$$

$$\frac{i\hbar}{2}(ad - ac + bd - bc - cb + ca - db + da) =$$

$$\frac{i\hbar}{2}(ad + da + ca - ac + bd - db - bc - cb) =;$$

Note: operators to different axes commute (they are independent).

$$\frac{i\hbar}{2}(2ad - 2bc) = i\hbar(ad - bc)$$

We get:

$$L_z = i\hbar\left(a_x {a_y}^\dagger - {a_x}^\dagger a_y\right)$$

We compare with the expression for H_{xy}:

$$H_{xy} = H_x + H_y = \left(N_x + N_y + 1\right)\omega\hbar$$

We use:

$$N_x = {a_x}^\dagger a_x$$

$$N_y = {a_y}^\dagger a_y$$

We get:

$$H_{xy} = \left(a_x{}^{\dagger}a_x + a_y{}^{\dagger}a_y + 1\right)\omega\hbar$$

We build the commutator of L_z and H_{xy}, omitting the factor $i\omega\hbar^2$:

$$\left[H_{xy}, L_z\right] =$$

$$H_{xy}L_z - L_zH_{xy} =;$$

We substitute:

$$a \coloneqq a_x{}^{\dagger}a_x; b \coloneqq a_y{}^{\dagger}a_y; c \coloneqq a_xa_y{}^{\dagger}; d \coloneqq a_x{}^{\dagger}a_y$$

We get:

$$(a + b + 1)(c - d) - (c - d)(a + b + 1) =$$

$$[a + b, c - d] =$$

$$[a + b, c] - [a + b, d]$$

We restitute:

$$\left[H_{xy}, L_z\right] = \left[a_x{}^{\dagger}a_x + a_y{}^{\dagger}a_y, a_xa_y{}^{\dagger}\right] - \left[a_x{}^{\dagger}a_x + a_y{}^{\dagger}a_y, a_x{}^{\dagger}a_y\right]$$

We examine at the first commutator:

$$\left[a_x{}^{\dagger}a_x + a_y{}^{\dagger}a_y, a_xa_y{}^{\dagger}\right] =$$

$$\left[a_x{}^{\dagger}a_x, a_xa_y{}^{\dagger}\right] + \left[a_y{}^{\dagger}a_y, a_xa_y{}^{\dagger}\right] =;$$

The first part:

$$\left[a_x{}^{\dagger}a_x, a_xa_y{}^{\dagger}\right] = a_x{}^{\dagger}a_xa_xa_y{}^{\dagger} - a_xa_y{}^{\dagger}a_x{}^{\dagger}a_x = -a_xa_y{}^{\dagger} + a_xa_y{}^{\dagger} = 0$$

Note: $a_x{}^{\dagger}a_x = -id$

The second part:

$$\left[a_y{}^{\dagger}a_y, a_xa_y{}^{\dagger}\right] = a_y{}^{\dagger}a_y, a_xa_y{}^{\dagger} - a_xa_y{}^{\dagger}a_y{}^{\dagger}a_y = -a_xa_y{}^{\dagger} + a_xa_y{}^{\dagger} = 0$$

We examine the second commutator:

$$\left[a_x{}^{\dagger}a_x + a_y{}^{\dagger}a_y, a_x{}^{\dagger}a_y\right] = \left[a_x{}^{\dagger}a_x, a_x{}^{\dagger}a_y\right] + \left[a_x{}^{\dagger}a_y, a_x{}^{\dagger}a_x\right] =$$

$$a_x{}^{\dagger}a_xa_x{}^{\dagger}a_y - a_x{}^{\dagger}a_ya_x{}^{\dagger}a_x + a_x{}^{\dagger}a_ya_x{}^{\dagger}a_x -, a_x{}^{\dagger}a_xa_x{}^{\dagger}a_y =$$

$$a_x{}^{\dagger}a_y - a_x{}^{\dagger}a_y + a_x{}^{\dagger}a_y - a_x{}^{\dagger}a_y = 0$$

Result:

$$\left[H_{xy}, L_z\right] = 0$$

The component L_z of the angular momentum and the Hamiltonian $H_{xy} = H_x + H_y$ commute.

We have a basis of eigenvectors common to H_{xy} and L_z.

Right and left circular quantum numbers

We are dealing with the two-dimensional harmonic oscillator.

We need lowering and raising operators that are acting in it, raising and lowering the total energy by $\frac{1}{2}\hbar + \frac{1}{2}\hbar = \hbar$.

We define operators a_d and a_g:

$a_d \coloneqq \frac{1}{\sqrt{2}}(a_x - ia_y)$	$a_g \coloneqq \frac{1}{\sqrt{2}}(a_x + ia_y)$	$a_d{}^\dagger \coloneqq \frac{1}{\sqrt{2}}(a_x + ia_y)$	$a_g{}^\dagger \coloneqq \frac{1}{\sqrt{2}}(a_x - ia_y)$

The action of a_d (or a_g) on $|\varphi_{n_x,n_y}\rangle$ yields a state that is a linear combination of $|\varphi_{n_x-1,n_y}\rangle$ and $|\varphi_{n_x,n_y-1}\rangle$, a stationary state that has one less energy quantum $\hbar\omega$.

Analog, the action of $a_d{}^\dagger$ (or $a_g{}^\dagger$) on $|\varphi_{n_x,n_y}\rangle$ yields a state that is a linear combination of $|\varphi_{n_x-1,n_y}\rangle$ and $|\varphi_{n_x,n_y-1}\rangle$, a stationary state that has one more energy quantum $\hbar\omega$.

We remember that the only nonzero commutators between $a_x, a_x{}^\dagger, a_y, a_y{}^\dagger$ were:

$$[a_x, a_x{}^\dagger] = 1 = [a_y, a_y{}^\dagger]$$

Similar we have that the only nonzero commutators between $a_d, a_d{}^\dagger, a_g, a_g{}^\dagger$ were:

$$[a_d, a_d{}^\dagger] = 1 = [a_g, a_g{}^\dagger]$$

We rewrite the Hamiltonian:

$$H_{xy} = (a_x{}^\dagger a_x + a_y{}^\dagger a_y + 1)\omega\hbar \to (a_d{}^\dagger a_d + a_g{}^\dagger a_g + 1)\omega\hbar$$

We rewrite the momentum:

$$L_z = \hbar(a_x a_y{}^\dagger - a_x{}^\dagger a_y) \to i\hbar(a_d a_d{}^\dagger - a_g{}^\dagger a_g)$$

The number operators for left and right circular quantum numbers becomes:

$$N_d = a_d{}^\dagger a_d$$

$$N_g = a_g{}^\dagger a_g$$

We rewrite the Hamiltonian and the momentum:

$$H_{xy} = (N_d + N_g + 1)\omega\hbar$$

$$L_z = \hbar(N_d - N_g)$$

Stationary states of well-defined angular momentum

Using the operators a_d and a_g we could check their behavior - it is analog to a_x and a_y (not done here).

From these results it follows that the spectra of N_d and N_g are composed of all positive integers and zero. The states $|\chi_{n_d,n_g}\rangle$ form an orthonormal basis for an infinite dimensional complex vector space.

In addition, specifying a pair $\{n_d, n_g\}$ of such integers (eigenvalues) determines uniquely (up to a constant factor) the eigenvector common to N_d and N_g:

$$|\chi_{n_d,n_g}\rangle = \frac{1}{\sqrt{n_d!\,n_g!}}\left(a_d{}^\dagger\right)^{n_d}\left(a_g{}^\dagger\right)^{n_g}|\varphi_{0,0}\rangle$$

Note: we will need these eigenvectors when calculating wave functions associated with eigenstates common to H_{xy} and L_z

N_d and N_g form a CSCO in S_{xy}.

$|\chi_{n_d,n_g}\rangle$ is eigenvector of H_{xy} and L_z with eigenvalues:

$$(n_d + n_g + 1)\omega\hbar \text{ for } H_{xy}$$

$$(n_d - n_g)\hbar \text{ for } L_z$$

We set:

$$n_d + n_g := \nu$$

$$n_d - n_g := \mu$$

We rewrite the eigenvalues:

$$(\nu + 1)\omega\hbar \text{ for } H_{xy}$$

$$\mu\hbar \text{ for } L_z$$

The action of the operator $a_d{}^\dagger$: $a_d{}^\dagger\,|\chi_{n_d,n_g}\rangle$ produces a state with one more quantum number, to which, since $n_d + n_g$ grew by one, an additional angular momentum $+\hbar$ must be added. This corresponds to a counterclockwise rotation.

Similar, $a_g{}^\dagger$ produces a state with one more quantum number with an angular momentum $-\hbar$. This corresponds to a clockwise rotation.

n_d and n_g are positive integers (or zero).

The eigenvalues of H_{xy} are of the form $(\nu + 1)\omega\hbar$ with ν being a positive integer or zero. Their degree of degeneracy is $(\nu + 1)$ since, for any ν we have $(\nu + 1)$ possibilities of combination as the left part of the table shows:

$n_d = \nu, n_g = 0$	$\mu = \nu$
$n_d = \nu - 1, n_g = 1$	$\mu = \nu - 2$
$n_d = \nu - 2, n_g = 2$	$\mu = \nu - 4$
$n_d = \nu - 3, n_g = 3$	$\mu = \nu - 6$
…	…
$n_d = 3, n_g = \nu - 3$	$\mu = 6 - \nu$
$n_d = 2, n_g = \nu - 2$	$\mu = 4 - \nu$
$n_d = 1, n_g = \nu - 1$	$\mu = 2 - \nu$
$n_d = 0, n_g = \nu$	$\mu = -\nu$

To any pair μ, ν corresponds a single eigenvector $|\chi_{n_d,n_g}\rangle$ with $n_d = \frac{\nu+\mu}{2}$, $n_g = \frac{\nu-\mu}{2}$.

H_{xy} and L_z form a CSCO in S_{xy}.

For a given value of energy E_ν, the states $|\chi_{\nu,0}\rangle$ and $|\chi_{0,\nu}\rangle$ corresponds to the maximal $(\nu\hbar)$ and minimal $(-\nu\hbar)$ values of L_z.

Common eigenstates of H_{xy} and L_z

We switch to polar coordinates with $r \geq 0$ and $0 \leq \varphi \leq 2\pi$:

$$x := r \cdot cos(\varphi)$$

$$y := r \cdot sin(\varphi)$$

We examine the action of the operators a_d and a_g on a function of r and φ, $f(r, \varphi)$.

We remember:

$a_d := \frac{1}{\sqrt{2}}(a_x - ia_y)$	$a_x = \frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right)$
${a_d}^\dagger := \frac{1}{\sqrt{2}}(a_x + ia_y)$	${a_x}^\dagger = \frac{1}{\sqrt{2}}\left(\beta X - i\frac{P_x}{\beta\hbar}\right)$
$a_g := \frac{1}{\sqrt{2}}(a_x + ia_y)$	$a_y = \frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right)$
${a_g}^\dagger := \frac{1}{\sqrt{2}}(a_x - ia_y)$	${a_y}^\dagger = \frac{1}{\sqrt{2}}\left(\beta Y - i\frac{P_y}{\beta\hbar}\right)$

Note: $\beta = \sqrt{\frac{m\omega}{\hbar}}$

$$a_d := \frac{1}{\sqrt{2}}(a_x - ia_y) =$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\beta X + i\frac{P_x}{\beta\hbar}\right) - i\frac{1}{\sqrt{2}}\left(\beta Y + i\frac{P_y}{\beta\hbar}\right)\right) =$$

$$\frac{1}{2}\left(\beta X + i\frac{P_x}{\beta\hbar} - i\beta Y + \frac{P_y}{\beta\hbar}\right) =$$

$$\frac{1}{2}\left(\beta(X - iY) + \frac{1}{\beta}\left(\frac{P_y}{\hbar} + i\frac{P_x}{\hbar}\right)\right) =$$

$$\frac{1}{2}\left(\beta(x - iy) + \frac{1}{\beta}\left(-i\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\right)$$

First part:

$$\beta(x - iy) = \beta\big(r \cdot cos(\varphi) - r \cdot i \cdot sin(\varphi)\big) =$$

$$\beta r\big(cos(\varphi) - i \cdot sin(\varphi)\big) =$$

$$\beta r\big(cos(-\varphi) + i \cdot sin(-\varphi)\big) =$$

$$\beta r e^{-i\varphi}$$

Note: cos is symmetric, sin is antisymmetric.

For the second part we need some information of how to transform coordinates and partial derivations. You find these in the paper on my website https://www.quantum-abc.de/part_der_coo_transf.pdf

The following holds:

$$\frac{\partial}{\partial x} = cos\varphi \cdot \frac{\partial}{\partial r} - \frac{1}{r} sin\varphi \cdot \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = sin\varphi \cdot \frac{\partial}{\partial r} + \frac{1}{r} cos\varphi \cdot \frac{\partial}{\partial \varphi}$$

We replace:

$$\left(-i\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) \rightarrow$$

$$-i\left(sin\varphi \cdot \frac{\partial}{\partial r} + \frac{1}{r} cos\varphi \cdot \frac{\partial}{\partial \varphi}\right) + \left(cos\varphi \cdot \frac{\partial}{\partial r} - \frac{1}{r} sin\varphi \cdot \frac{\partial}{\partial \varphi}\right) =$$

$$-isin\varphi \cdot \frac{\partial}{\partial r} - \frac{i}{r} cos\varphi \cdot \frac{\partial}{\partial \varphi} + cos\varphi \cdot \frac{\partial}{\partial r} - \frac{1}{r} sin\varphi \cdot \frac{\partial}{\partial \varphi}$$

$$\left(-isin\varphi \cdot \frac{\partial}{\partial r} + cos\varphi \cdot \frac{\partial}{\partial r}\right) - \left(\frac{1}{r} sin\varphi \cdot \frac{\partial}{\partial \varphi} - \frac{i}{r} cos\varphi \cdot \frac{\partial}{\partial \varphi}\right) =$$

$$(cos\varphi - isin\varphi) \cdot \frac{\partial}{\partial r} - \frac{i}{r}(-isin\varphi + cos\varphi)\frac{\partial}{\partial \varphi} =$$

$$(cos\varphi - isin\varphi) \cdot \frac{\partial}{\partial r} - \frac{i}{r}(cos\varphi - isin\varphi)\frac{\partial}{\partial \varphi} =$$

$$e^{-i\varphi} \cdot \frac{\partial}{\partial r} - \frac{i}{r} e^{-i\varphi} \frac{\partial}{\partial \varphi} =$$

$$e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)$$

We assemble the complete result. If we change to polar coordinates, we have:

$$a_d = \frac{1}{2}\left(\beta(x - iy) + \frac{1}{\beta}\left(-i\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\right) \rightarrow \frac{1}{2}\left(\beta r e^{-i\varphi} + \frac{1}{\beta} e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)$$

$$a_d = \frac{1}{2} e^{-i\varphi}\left(\beta r + \frac{1}{\beta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)$$

In the same way we get:

$$a_d{}^{\dagger} = \frac{1}{2} e^{i\varphi}\left(\beta r - \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)$$

$$a_g = \frac{1}{2} e^{i\varphi}\left(\beta r + \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)$$

$$a_g{}^{\dagger} = \frac{1}{2} e^{-i\varphi}\left(\beta r - \frac{1}{\beta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)$$

We need the wave function according to that of the x, y-plane ((5), page 8):

$$\varphi_{n_x,n_y}(x,y) = \frac{\beta}{\sqrt{2^{(n_x+n_y)} \cdot \pi \cdot n_x! \cdot n_y!}} e^{-\frac{1}{2}\beta^2(x^2+y^2)} H_{n_x}(\beta x) H_{n_y}(\beta y)$$

This gives the ground state wave function for polar coordinates:

$$\chi_{0,0}(r,\varphi) = e^{m\cdot i\varphi} \frac{\beta}{\sqrt{\pi}} e^{-\frac{1}{2}\beta^2 r^2} \coloneqq e^{m\cdot i\varphi} F(r)$$

Note: $m = n_d - n_g$. For the ground state we have $n_d = n_g \rightarrow m = 0$.

We examine the action of the operators ${a_d}^\dagger$ and ${a_g}^\dagger$ on a function of type $e^{m\cdot i\varphi} F(r)$:

$${a_d}^\dagger \left(e^{m\cdot i\varphi} F(r)\right) = \frac{1}{2} e^{i\varphi} \left(\beta r - \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)\left(e^{m\cdot i\varphi} F(r)\right) =$$

$$\frac{1}{2} e^{i\varphi} \beta r \left(e^{m\cdot i\varphi} F(r)\right) - \frac{1}{2} e^{i\varphi} \frac{1}{\beta}\frac{\partial}{\partial r}\left(e^{m\cdot i\varphi} F(r)\right) - \frac{1}{2} e^{i\varphi} \frac{1}{\beta}\frac{i}{r}\frac{\partial}{\partial \varphi}\left(e^{m\cdot i\varphi} F(r)\right) =$$

$$\frac{1}{2} e^{i(m+1)\varphi} \beta r F(r) - \frac{1}{2} e^{i(m+1)\varphi} \frac{1}{\beta}\frac{\partial}{\partial r} F(r) - \frac{1}{2} e^{i(m+1)\varphi} \frac{1}{\beta}\frac{i}{r} i m F(r) =$$

$$\frac{1}{2} e^{i(m+1)\varphi} \left(\beta r F(r) - \frac{1}{\beta}\frac{\partial}{\partial r} F(r) + \frac{m}{\beta r} F(r)\right) =$$

$$\frac{1}{2} e^{i(m+1)\varphi} \left(\left(\beta r + \frac{m}{\beta r}\right) F(r) - \frac{1}{\beta}\frac{\partial}{\partial r} F(r)\right)$$

Analog we get:

$${a_g}^\dagger \left(e^{m\cdot i\varphi} F(r)\right) = \frac{1}{2} e^{-i\varphi} \left(\beta r - \frac{1}{\beta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)\left(e^{m\cdot i\varphi} F(r)\right) =$$

$$\frac{1}{2} e^{i(m-1)\varphi} \left(\left(\beta r - \frac{m}{\beta r}\right) F(r) - \frac{1}{\beta}\frac{\partial}{\partial r} F(r)\right)$$

The φ-dependency of $\chi_{n_d,n_g}(r,\varphi)$ is given by $e^{i(n_d-n_g)\varphi}$. This is a general result: the φ -dependency of an eigenfunction of L_z of eigenvalue $m\hbar$ is $e^{m\cdot i\varphi}$.

We choose:

$$F(r) = r^m e^{-\frac{1}{2}\beta^2 r^2}$$

We apply the operator ${a_d}^\dagger$:

$${a_d}^\dagger \left(e^{im\varphi} \cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right) =$$

$$\frac{1}{2} e^{i\varphi} \left(\beta r - \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)\left(e^{im\varphi} \cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right) =$$

$$\frac{1}{2} e^{i\varphi} \beta r e^{im\varphi} \cdot r^m e^{-\frac{1}{2}\beta^2 r^2} - \frac{1}{2} e^{i\varphi} \frac{1}{\beta}\frac{\partial}{\partial r}\left(e^{im\varphi} \cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right) - \frac{1}{2} e^{i\varphi} \frac{1}{\beta}\frac{i}{r}\frac{\partial}{\partial \varphi}\left(e^{im\varphi} \cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right) =;$$

We do this in parts.

Part one:

$$\frac{1}{2}e^{i\varphi}\beta r e^{im\varphi}\cdot r^m e^{-\frac{1}{2}\beta^2 r^2}=\frac{1}{2}e^{i(m+1)\varphi}\cdot r^{m+1}\cdot\beta\cdot e^{-\frac{1}{2}\beta^2 r^2}$$

Part two:

$$\frac{1}{2}e^{i\varphi}\frac{1}{\beta}\frac{\partial}{\partial r}\left(e^{im\varphi}\cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right)=$$

$$\frac{1}{2}e^{i(m+1)\varphi}\frac{1}{\beta}\left(mr^{m-1}\cdot e^{-\frac{1}{2}\beta^2 r^2}-r^m\cdot\beta^2 r\cdot e^{-\frac{1}{2}\beta^2 r^2}\right)=$$

$$\frac{1}{2}e^{i(m+1)\varphi}\cdot e^{-\frac{1}{2}\beta^2 r^2}\frac{1}{\beta}\left(mr^{m-1}-r^{m+1}\cdot\beta^2\right)=$$

Part three:

$$\frac{1}{2}e^{i\varphi}\frac{1}{\beta}\frac{i}{r}\frac{\partial}{\partial\varphi}\left(e^{im\varphi}\cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right)=$$

$$\frac{1}{2}e^{i\varphi}\frac{1}{\beta}\frac{i}{r}\cdot r^m\cdot e^{-\frac{1}{2}\beta^2 r^2}\frac{\partial}{\partial\varphi}\left(e^{im\varphi}\right)=$$

$$\frac{1}{2}e^{i\varphi}\frac{1}{\beta}\frac{i}{r}\cdot r^m\cdot e^{-\frac{1}{2}\beta^2 r^2}\cdot im\cdot e^{im\varphi}=$$

$$-\frac{1}{2}e^{i(m+1)\varphi}\frac{1}{\beta}\cdot r^{m-1}\cdot e^{-\frac{1}{2}\beta^2 r^2}\cdot m$$

Result:

$$\frac{1}{2}e^{i(m+1)\varphi}\cdot r^{m+1}\cdot\beta\cdot e^{-\frac{1}{2}\beta^2 r^2}-\frac{1}{2}e^{i(m+1)\varphi}\cdot e^{-\frac{1}{2}\beta^2 r^2}\frac{1}{\beta}\left(mr^{m-1}-r^{m+1}\cdot\beta^2\right)+\frac{1}{2}e^{i(m+1)\varphi}\frac{1}{\beta}\cdot r^{m-1}\cdot e^{-\frac{1}{2}\beta^2 r^2}\cdot m=$$

$$\frac{1}{2}e^{i(m+1)\varphi}e^{-\frac{1}{2}\beta^2 r^2}\left(r^{m+1}\cdot\beta-\frac{1}{\beta}\left(mr^{m-1}-r^{m+1}\cdot\beta^2-r^{m-1}\cdot m\right)\right)=$$

$$\frac{1}{2}e^{i(m+1)\varphi}e^{-\frac{1}{2}\beta^2 r^2}\left(r^{m+1}\cdot\beta+\beta\cdot r^{m+1}\right)=$$

$$\beta\cdot e^{i(m+1)\varphi}\cdot r^{m+1}\cdot e^{-\frac{1}{2}\beta^2 r^2}$$

We got:

$$a_d{}^{\dagger}\left(e^{im\varphi}\cdot r^m e^{-\frac{1}{2}\beta^2 r^2}\right)=\beta\cdot e^{i(m+1)\varphi}\cdot r^{m+1}\cdot e^{-\frac{1}{2}\beta^2 r^2}$$

If we apply the operator $a_d{}^{\dagger}$ to $\chi_{0,0}(r,\varphi)$ n_d-times, we get (the normalized function):

$$\chi_{n_d,0}(r,\varphi)=\frac{\beta}{\sqrt{\pi\cdot n_d!}}\cdot e^{i(n_d)\varphi}\cdot(\beta r)^{n_d}\cdot e^{-\frac{1}{2}\beta^2 r^2}$$

Analog if we apply the operator ${a_g}^\dagger$ to $\chi_{0,0}(r,\varphi)$ n_g-times, we get (the normalized function):

$$\chi_{0,n_g}(r,\varphi) = \frac{\beta}{\sqrt{\pi \cdot n_g!}} \cdot e^{i(n_g)\varphi} \cdot (\beta r)^{n_g} \cdot e^{-\frac{1}{2}\beta^2 r^2}$$

For a given energy level $(n+1)\hbar\omega$, the wave functions $\chi_{n_d,0}(r,\varphi)$ and $\chi_{0,n_g}(r,\varphi)$ correspond to the limiting values $+n$ and $-n$ of the quantum number m. Their r-dependence is simple, their modulus reaches a maximum for $r = \sqrt{\frac{n}{\beta}}$.

As in the one-dimensional harmonic oscillator, the spatial spread of these wave functions increases with the energy $(n+1)\hbar\omega$.

To construct a wave function of type $\chi_{n_d,n_g}(r,\varphi)$, we apply the operators ${a_d}^\dagger$ resp. ${a_g}^\dagger$ to $\chi_{0,0}(r,\varphi)$.

The result for the first levels:

$n=0$	$m=0$	$\chi_{0,0}(r,\varphi) = e^{0\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 0!}}\cdot(\beta r)^0 e^{-\frac{1}{2}\beta^2 r^2}$
$n=1$	$\begin{cases} m=1 \\ m=-1 \end{cases}$	$\begin{cases} \chi_{1,0}(r,\varphi) = e^{1\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 1!}}\cdot(\beta r)^1 e^{-\frac{1}{2}\beta^2 r^2} \\ \chi_{0,1}(r,\varphi) = e^{-1\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 1!}}\cdot(\beta r)^1 e^{-\frac{1}{2}\beta^2 r^2} \end{cases}$
$n=2$	$\begin{cases} m=2 \\ m=0 \\ m=-2 \end{cases}$	$\begin{cases} \chi_{2,0}(r,\varphi) = e^{2\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 2!}}\cdot(\beta r)^2 e^{-\frac{1}{2}\beta^2 r^2} \\ \chi_{1,1}(r,\varphi) = e^{0\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 0!}}\cdot((\beta r)^2-1) e^{-\frac{1}{2}\beta^2 r^2} \\ \chi_{0,2}(r,\varphi) = e^{-2\cdot i\varphi}\frac{\beta}{\sqrt{\pi\cdot 2!}}\cdot(\beta r)^2 e^{-\frac{1}{2}\beta^2 r^2} \end{cases}$

The functions $\chi_{n_d,0}(r,\varphi)$:

$$\chi_{n_d,0}(r,\varphi) = \frac{\beta}{\sqrt{\pi \cdot n_d!}} \cdot e^{i(n_d)\varphi} \cdot (\beta r)^{n_d} \cdot e^{-\frac{1}{2}\beta^2 r^2}$$

They are proportional to:

$$e^{-\frac{1}{2}\beta^2 r^2}\left(\beta r e^{i\varphi}\right)^{n_d}$$

All their linear combinations are of the form:

$$F(r,\varphi) = e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)$$

Note: f is an arbitrary function of one variable.

The subspace of eigenfunctions of N_g of eigenvalue zero is composed of functions $F(r,\varphi)$.

We remember:

$$a_g = \frac{1}{2}e^{i\varphi}\left(\beta r + \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi}\right)\right)$$

We try:

$$a_g F(r,\varphi) =$$

$$\frac{1}{2}e^{i\varphi}\left(\beta r + \frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) + \frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) + \frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}\left(\frac{\partial}{\partial r}\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) + \frac{1}{2}e^{i\varphi}$$
$$\cdot\frac{1}{\beta}\left(\frac{i}{r}\frac{\partial}{\partial \varphi}\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) =;$$

We do this in parts.

Part one:

$$\frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)$$

Part two:

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}\left(\frac{\partial}{\partial r}\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}\left(-\beta^2 r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) + e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial}{\partial r} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$-\frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) + \frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial}{\partial r} f\left(\beta r e^{i\varphi}\right)$$

Part three:

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}\left(\frac{i}{r}\frac{\partial}{\partial \varphi}\right)\left(e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{i}{\beta r}\left(e^{-\frac{1}{2}\beta^2 r^2}\frac{\partial}{\partial \varphi} f\left(\beta r e^{i\varphi}\right)\right)$$

We combine:

$$\frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) - \frac{1}{2}e^{i\varphi}\cdot\beta r\cdot e^{-\frac{1}{2}\beta^2 r^2} f\left(\beta r e^{i\varphi}\right) + \frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial}{\partial r} f\left(\beta r e^{i\varphi}\right)$$
$$+\frac{1}{2}e^{i\varphi}\cdot\frac{i}{\beta r}\left(e^{-\frac{1}{2}\beta^2 r^2}\frac{\partial}{\partial \varphi} f\left(\beta r e^{i\varphi}\right)\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial}{\partial r} f\left(\beta r e^{i\varphi}\right) + \frac{1}{2}e^{i\varphi}\cdot\frac{i}{\beta r}e^{-\frac{1}{2}\beta^2 r^2}\frac{\partial}{\partial \varphi} f\left(\beta r e^{i\varphi}\right) =$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial f}{\partial\left(\beta r e^{i\varphi}\right)}\cdot\frac{\partial\left(\beta r e^{i\varphi}\right)}{\partial r} + \frac{1}{2}e^{i\varphi}\cdot\frac{i}{\beta r}e^{-\frac{1}{2}\beta^2 r^2}\cdot\frac{\partial f}{\partial\left(\beta r e^{i\varphi}\right)}\cdot\frac{\partial\left(\beta r e^{i\varphi}\right)}{\partial \varphi} =$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot\beta e^{i\varphi}+\frac{1}{2}e^{i\varphi}\cdot\frac{i}{\beta r}e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot i\beta re^{i\varphi}=$$

$$\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta}e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot\beta e^{i\varphi}-\frac{1}{2}e^{i\varphi}\cdot\frac{1}{\beta r}e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot\beta re^{i\varphi}=$$

$$\frac{1}{2}e^{i\varphi}\cdot e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot e^{i\varphi}-\frac{1}{2}e^{i\varphi}\cdot e^{-\frac{1}{2}\beta^2r^2}\cdot\frac{\partial f}{\partial(\beta re^{i\varphi})}\cdot e^{i\varphi}=0$$

Result:

$$a_gF(r,\varphi)=0$$

Similarly, the subspace of eigenfunctions of N_d of eigenvalue zero is composed of functions $G(r,\varphi)$:

$$G(r,\varphi)=e^{-\frac{1}{2}\beta^2r^2}g\left(\beta re^{-i\varphi}\right)$$

Note: g is an arbitrary function of one variable.