## Preface

This file contains the exercises of „Quantum Mechanics, The Theoretical Minimum" and is specific in this respect. On the other hand, the topics generally deal with quantum mechanics and maybe are helpful for reasons of training too.

I tried to make the exercises explicit so most of them can be tackled without specific knowledge of the book itself. I also tried to write a kind of "deep dive" solutions that give you more information than just the correct result.

Hope I can help you with learning quantum mechanics.

I would like to thank Sunjiv Varsani, Dhruv Patel, Kenneth Verbist and Dr. Wolfgang Lindner that helped for minimizing the number of errors in this paper …

Dieter Kriesell

Exercise 1.1

a) Using the axioms for inner products, prove $\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle$.
b) Prove that $\langle A|A\rangle$ is a real number.

Exercise 1.2

Show that the inner product satisfies linearity and interchange.
Inner product:

$$\langle A|B\rangle = \left(a_1^* a_2^* \dots a_{n_1}^*\right)\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$$

Linearity: $\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle$
Interchange: $\langle B|A\rangle = \langle A|B\rangle^*$

Exercise 2.1

Prove that the vector $|r\rangle$ is orthogonal to vector $|l\rangle$:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$
$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

Exercise 2.2

Prove that $|i\rangle$ and $|o\rangle$ satisfy the conditions in Eqs. 2.7, 2.8, and 2.9.
Are they unique in that respect?
Eq. 2.7:

$$\langle i|o\rangle = 0$$

Eqs. 2.8:

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{2} \qquad \langle o|d\rangle\langle d|o\rangle = \frac{1}{2}$$
$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{2} \qquad \langle i|d\rangle\langle d|i\rangle = \frac{1}{2}$$

Eqs. 2.9

$$\langle o|r\rangle\langle r|o\rangle = \frac{1}{2} \qquad \langle o|l\rangle\langle l|o\rangle = \frac{1}{2}$$
$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2} \qquad \langle i|l\rangle\langle l|i\rangle = \frac{1}{2}$$

$$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \text{ and } \langle i| = \langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}$$
$$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle \text{ and } \langle o| = \langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}$$

Exercise 2.3

Assume that $|i\rangle$ and $|o\rangle$ are given as:

$$|i\rangle = \alpha|u\rangle + \beta|d\rangle \text{ and } \langle i| = \langle u|\alpha^* + \langle d|\beta^*$$
$$|o\rangle = \gamma|u\rangle - \delta|d\rangle \text{ and } \langle o| = \langle u|\gamma^* - \langle d|\delta^*$$

a) Use Eqs. 2.8 to show that

$$\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$$

b) Use the above result and Eqs. 2.9 to show that:

$$\alpha^*\beta + \alpha\beta^* = 0$$

c) Show that $\alpha^*\beta$ must be pure imaginary.

Eqs. 2.8:

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{2} \qquad \langle o|d\rangle\langle d|o\rangle = \frac{1}{2}$$
$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{2} \qquad \langle i|d\rangle\langle d|i\rangle = \frac{1}{2}$$

Eqs. 2.9

$$\langle o|r\rangle\langle r|o\rangle = \frac{1}{2} \qquad \langle o|l\rangle\langle l|o\rangle = \frac{1}{2}$$
$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2} \qquad \langle i|l\rangle\langle l|i\rangle = \frac{1}{2}$$

Exercise 3.1

Prove: If a vector space is N-dimensional, an orthonormal basis of N vectors can be constructed from the eigenvectors of a Hermitian operator.

Exercise 3.2

Prove that $\sigma_z$ of Eq. 3.16 is the unique solution to Eqs. 3.14 und 3.15.

Eq. 3.16

$$\sigma_z = \begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eq. 3.14

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eq. 3.15

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Exercise 3.3

Calculate the eigenvectors and eigenvalues of $\sigma_n$.

Assume the eigenvector $\lambda_1$ has the form:

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$\alpha$ is an unknown parameter. Plug this vector into the eigenvalue equation and solve for $\alpha$ in terms of $\theta$.

Why did we use a single parameter $\alpha$?

Notice that our suggested column vector must have unit length.

$$\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

To show:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Exercise 3.4
Let $n_z = cos\theta$, $n_x = sin\theta cos\phi$, and $n_y = sin\theta sin\phi$.
Angles $\theta$ and $\phi$ are defined according to the usual conventions for spherical coordinates.
Compute the eigenvalues and eigenvectors for the matrix $\sigma_n$:

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

Exercise 3.5
Suppose a spin is prepared so that $\sigma_m = +1$. The apparatus is then rotated to the $\hat{n}$ direction and $\sigma_n$ is measured. What is the probability that the result is +1?
Note that $\sigma_m = \sigma \cdot \hat{m}$, using the same convention we used for $\sigma_n$.

Exercise 4.1
Prove that if (the time operator) U is unitary, and if $|A\rangle$ and $|B\rangle$ are any two state-vectors, then the inner product of $U|A\rangle$ and $U|B\rangle$ is the same as the inner product of $|A\rangle$ and $|B\rangle$. One could call this the conservation of overlaps. It expresses the fact that the logical relation between states is preserved with time.
U is unitary:

$$U^\dagger U = I$$
$$U|A\rangle = \langle A|U^\dagger$$

Exercise 4.2
Prove that if $M$ and $L$ are both Hermitian, the (extended) commutator $i[M, L]$ is also Hermitian. Note that the $i$ is important. The commutator is, by itself, not Hermitian.

Hermitian: the diagonal is pure real and: $M = M^\dagger$

Exercise 4.3
With the definition of Poisson brackets check that the identification in Eq. 4.21 is dimensionally consistent. Show that without the factor $\hbar$, it would not be.
Eq. 4.21

$$[F, G] \leftrightarrow i\hbar\{F, G\}$$
$$[\hbar] = J \cdot s = \frac{kg \cdot m^2}{s^2} \cdot s$$

Exercise 4.4
Verify the commutation relations:

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$
$$[\sigma_y, \sigma_z] = 2i\sigma_x$$
$$[\sigma_z, \sigma_x] = 2i\sigma_y$$
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise 4.5

Take any unit 3-vector $\vec{n}$ and form the operator

$$H = \frac{\hbar\omega}{2}\sigma \cdot \vec{n}$$

Find the energy eigenvalues and eigenvectors by solving the time-independent Schrödinger equation. Recall $\sigma \cdot \vec{n}$ in component form:

$$\sigma_n = \sigma \cdot \vec{n} = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

Exercise 4.6

Carry out the Schrödinger Ket recipe for a single spin.

The Hamiltonian is $H = \frac{\hbar\omega}{2}\sigma_z$ and the final observable is $\sigma_x$.

The initial state is given as $|u\rangle$ (the state in which $\sigma_z = \pm 1$).

After time t, an experiment is done to measure $\sigma_y$.

What are the possible outcomes and what are the probabilities for those outcomes?

Exercise 5.1

Verify that any $2 \times 2$ Hermitian matrix L can be written as a sum of four terms,

$$L = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

where a, b, c and d are real numbers.

The four Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{I}= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A general Hermitian matrix $(r, r)$ are real numbers:

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix}$$

Exercise 5.2

1) Show that $(\triangle A)^2 = \langle \bar{A}^2 \rangle$ and $(\triangle B)^2 = \langle \bar{B}^2 \rangle$
2) Show that $[\bar{A}, \bar{B}] = [A, B]$
3) Using these relations, show that $\triangle A \triangle B \geq \frac{1}{2}|\langle \Psi|[A, B]|\Psi\rangle|$

The square of uncertainty (or standard deviation) of A, $(\triangle A)^2$:

$$(\triangle A)^2 = \sum_a \bar{a}^2 P(a) = \sum_a (a - \langle A \rangle)^2 P(A)$$

Exercise 6.1

Prove that if $P(a, b)$ factorizes:

$$P(a, b) = P_A(a)P_b(b)$$

then the correlation between a and b is zero:

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = 0$$

Average:

$$\langle \sigma_A \rangle = \sum_n a_n P(a_n)$$

$$\langle \sigma_B \rangle = \sum_n b_n P(b_n)$$

$$\langle \sigma_A \sigma_B \rangle = \sum_n \sum_n a_n b_n P(a_n b_n)$$

Exercise 6.2

Show that if the two normalization conditions are satisfied:

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$$
$$\beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

then the state-vector is automatically normalized as well:

$$\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

In other words, show that for this product state, normalizing the overall state-vector does not put any additional constraints on the $\alpha's$ and $\beta's$.

Exercise 6.3

Prove that the state $|sing\rangle$ cannot be written as a product state.

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

The shape of a product state:

$$\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

Exercise 6.4

Use the matrix forms of $\sigma_x, \sigma_y, \sigma_z$ and the column vectors for $|u\rangle$ and $|d\rangle$ to verify:

$$\sigma_z|u\rangle = |u\rangle \quad \text{and} \quad \sigma_z|d\rangle = -|d\rangle$$
$$\sigma_x|u\rangle = |d\rangle \quad \text{and} \quad \sigma_x|d\rangle = |u\rangle$$
$$\sigma_y|u\rangle = i|d\rangle \quad \text{and} \quad \sigma_y|d\rangle = -i|u\rangle$$

Then, use:

$$\tau_z|u\rangle = |u\rangle \quad \text{and} \quad \tau_z|d\rangle = -|d\rangle$$
$$\tau_x|u\rangle = |d\rangle \quad \text{and} \quad \tau_x|d\rangle = |u\rangle$$
$$\tau_y|u\rangle = i|d\rangle \quad \text{and} \quad \tau_y|d\rangle = -i|u\rangle$$

to write the equations for all possible combinations of the tensor product states $\sigma_z|uu\rangle = |uu\rangle$ etc.

Exercise 6.5

Prove the following theorem:

When any one of Alice's and Bob's spin operators acts on a product state, the result is still a product state.

Show that in a product state, the expectation value of any component of $\bar{\sigma}$ or $\bar{\tau}$ is the same as it would be in the individual single-spin states.

Exercise 6.6

Assume Charlie has prepared the two spins in the singlet state. This time, Bob measures $\tau_y$ and Alice measures $\sigma_x$. What is the expectation value of $\sigma_x\tau_y$?

What does this say about the correlation between the two measurements?

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$
$$\langle sing| = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)$$

Exercise 6.7

Next *(after Charlie has had prepared the two spins in the singlet state*), Charlie prepares the spins in a different state, called $|T_1\rangle$, where

$$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$$

T stands for triplet. The triplet states are completely different from the states in the coin and die examples.

What are the expectation values of the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$?

Notice what a difference a sign can make.

Exercise 6.8

Calculate the expectation values of the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, *and* $\sigma_y\tau_y$ for the other two entangled triplet states:

$$|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$$

$$|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$$

*(Addendum: at the end of this exercise we check the expectation values for the singlet state too.)*

Exercise 6.9

Prove that the four vectors $|sing\rangle, |T_1\rangle, |T_2\rangle, |T_3\rangle$ are eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$.

What are their eigenvalues?

$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$

$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$ $\quad|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$ $\quad|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$

Exercise 6.10

A system of two spins has the Hamiltonian:

$$\mathcal{H} = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}$$

Question 1)

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian?

Question 2)

Suppose the system starts in the state $|uu\rangle$.

What is the state at any later time?

Answer the same question for initial states of $|ud\rangle$, $|du\rangle$, $|dd$.

Exercise 7.1

Write the tensor product $I \otimes \tau_x$ as a matrix, and apply that matrix to each of the $|uu\rangle$, $|ud\rangle$, $|du\rangle$ and $|dd\rangle$ column vectors. Show that Alice's half of the state-vector is unchanged in each case.

Recall that $I$ is the $2 \times 2$ unit matrix.

Exercise 7.2

Calculate the matrix elements of $\sigma_z \otimes \tau_x$ by forming the inner product analog to:

$$\sigma_z \otimes I = \begin{pmatrix} \langle uu|\sigma_z I|uu\rangle & \langle uu|\sigma_z I|ud\rangle & \langle uu|\sigma_z I|du\rangle & \langle uu|\sigma_z I|dd\rangle \\ \langle ud|\sigma_z I|uu\rangle & \langle ud|\sigma_z I|ud\rangle & \langle ud|\sigma_z I|du\rangle & \langle ud|\sigma_z I|dd\rangle \\ \langle du|\sigma_z I|uu\rangle & \langle du|\sigma_z I|ud\rangle & \langle du|\sigma_z I|du\rangle & \langle du|\sigma_z I|dd\rangle \\ \langle dd|\sigma_z I|uu\rangle & \langle dd|\sigma_z I|ud\rangle & \langle dd|\sigma_z I|du\rangle & \langle dd|\sigma_z I|dd\rangle \end{pmatrix}$$

$\sigma_z$ operates to the left, $I$ to the right.

Exercise 7.3

Consider the equation:

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

$A$ and $B$ represent $2 \times 2$ matrices (or operators), and $A$ and $b$ represent $2 \times 1$ column vectors.

a) Rewrite the equation:

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

in component form, replacing the symbols $A, B, a, b$ with the following matrices and column vectors:

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}$$

b) Perform the matrix multiplication $Aa$ and $Bb$ on the right-hand side. Verify that each result is a $4 \times 1$ matrix.

c) Expand all three Kronecker products.

d) Verify the row and column sizes of each Kronecker product:

- $A \otimes B : 4 \times 4$
- $a \otimes b : 4 \times 1$
- $Aa \otimes Bb : 4 \times 1$

e) Perform the matrix multiplication on the left-hand side, resulting in a $4 \times 1$ column vector. Each row should be the sum of four separate terms

f) Finally, verify that the resulting column vectors on the left and right sides are identical.

Exercise 7.4

Calculate the density matrix for:

$$|\Psi\rangle = \alpha|uu\rangle + \beta|uu\rangle$$

Answer:

$$\psi(u) = \alpha;\ \psi^*(u) = \alpha^*$$
$$\psi(d) = \beta;\ \psi^*(d) = \beta^*$$

$$\rho_{a\prime a} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{pmatrix}$$

Now try plugging some numbers for $\alpha$ and $\beta$. Make sure they are normalized to 1. For example, $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$

Exercise 7.5

a) Show:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

b) Now, suppose:

$$\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Calculate:

$$\rho^2, \qquad Tr(\rho), \qquad Tr(\rho^2)$$

c) If $\rho$ is a density matrix, does it represent a pure state or a mixed state?

Exercise 7.6

By the standard rules of probability, the probability for a:

$$P(a) = \sum_b \psi^*(a,b)\psi(a,b)$$

This is just a diagonal entry in the density matrix:

$$P(a) = \rho_{aa}$$

Use $P(a) = \rho_{aa}$ to show that if $\rho$ is a density matrix, then:

$$Tr(\rho) = 1$$

Exercise 7.7

We have the density matrix:

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Calculate $\rho^2$. How does this result confirm that $\rho$ represents an entangled state?

Exercise 7.8

Consider the following states:

$$|\psi_1\rangle = \frac{1}{2}(|uu\rangle + |ud\rangle + |du\rangle + |dd\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$$
$$|\psi_3\rangle = \frac{1}{5}(3|uu\rangle + 4|dd\rangle)$$

For each one, calculate Alice's density matrix and Bob's density matrix. Check their properties.

Exercise 7.9

Given any Alice observable $A$ and Bo observable $B$, show that for a product state, the correlation $C(A,B)$ is zero.

Exercise 7.10

Given a measuring apparatus with the states $|b\rangle$ for initial blank state, $|+1\rangle$ for "result of spin measurement is up" and $|-1\rangle$ for "result of spin measurement is down".
Verify that the state-vector

$$\alpha_u|u,b\rangle + \alpha_d|d,b\rangle$$

represents a completely unentangled state.

Exercise 7.11

Calculate Alice's density matrix for $\sigma_z$ for the "near-singlet" state.

Exercise 7.12

Verify the numerical values in each rap sheet.
This is a very specific exercise that refers to so called "rap sheets". The solution here is for reasons of completeness only. It is a superset of the solution to exercise 7.11.

Exercise 8.1

Prove that the position operator $X$ and the momentum operator $D$ are linear operators.

$$\boldsymbol{X}: \boldsymbol{X}\varphi(x) = x\varphi(x)$$
$$\boldsymbol{D}: \boldsymbol{D}\varphi(x) = \frac{d\varphi(x)}{dx}$$

Exercise 9.1

Applicate the Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$$

to the wave function:

$$\psi(x) = e^{\frac{ip}{\hbar}x}$$

Show that this wave function is a solution, if we set:

$$E = \frac{p^2}{2m}$$

Exercise 9.2

Prove the following equation by expanding each side and comparing the results:

$$[P^2, X] = P[P, X] + [P, X]P$$

$P$ is the momentum operator, $X$ is the position operator – both are matrices.
$[P, X]$ is the commutator relation: $[P, X] = PX - XP$

Exercise 9.3

Show that the right-hand side of

$$[V(x), P]\psi(x) = V(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) - \left(-i\hbar\frac{d}{dx}\right)V(x)\psi(x)$$

simplifies to the right-hand side of:

$$[V(x), P]\psi(x) = i\hbar\frac{dV(x)}{dx}$$

Hint: First expand the second term by taking the derivative of the product.
Then look for cancellations.

Exercise 10.1

Find the second derivative of $x$:

$$x = A \cdot cos(\omega t) + B \cdot sin(\omega t)$$

Show thereby that it solves:

$$-\omega^2 x = \ddot{x}$$

Exercise 1.1

a) Using the axioms for inner products, prove $\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle$.
b) Prove that $\langle A|A\rangle$ is a real number.

* * * * * * * * * *

a)

$$\{\langle A| + \langle B|\}|C\rangle =$$

$$[\langle C|\{|A\rangle + |B\rangle\}]^* =$$

$$\langle C|A\rangle^* + \langle C|B\rangle^* =$$

$$\langle A|C\rangle^{**} + \langle B|C\rangle^{**} =$$

$$\langle A|C\rangle + \langle B|C\rangle$$

b)

$$\langle A|A\rangle = \left(a_1^* a_2^* \dots a_{n_1}^*\right)\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} =$$

$$a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n =$$

$$|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 \in \mathbb{R}$$

Exercise 1.2

Show that the inner product defined by Eq. 1.2 satisfies all the axioms of inner products.

Inner product:

$$\langle A|B\rangle = \left(a_1^* a_2^* \dots a_{n1}^*\right)\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$$

Linearity: $\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle$

Interchange: $\langle B|A\rangle = \langle A|B\rangle^*$

* * * * * * * * * *

Linearity:

$$\langle C|\{|A\rangle + |B\rangle\} =$$

$$\left(c_1^* c_2^* \dots c_{n1}^*\right)\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_n + b_n \end{pmatrix} =$$

$$c_1^*(a_1 + b_1) + c_2^*(a_2 + b_2) + \dots + c_n^*(a_n + b_n) =$$

$$c_1^* a_1 + c_1^* b_1 + c_2^* a_2 + c_2^* b_2 + \dots + c_n^* a_n + c_n^* b_n =$$

$$c_1^* a_1 + c_2^* a_2 + \dots + c_n^* a_n + c_1^* b_1 + c_2^* b_2 + \dots + c_n^* b_n =$$

$$\left(c_1^* c_2^* \dots c_{n1}^*\right)\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} + \left(c_1^* c_2^* \dots c_{n1}^*\right)\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} =$$

$$\langle C|A\rangle + \langle C|B\rangle$$

Interchange:

$$\langle A|B\rangle^* = \left[\left(a_1^* a_2^* \dots a_{n1}^*\right)\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}\right] =^*$$

$$(a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n)^* =$$

$$a_1^{**} b_1^* + a_2^{**} b_2^* + \dots + a_n^{**} b_n^* =$$

$$a_1 b_1^* + a_2 b_2^* + \dots + a_n b_n^* =$$

$$b_1^* a_1 + b_2^* a_2 + \dots + b_n^* a_n =$$

$$(b_1^* b_2^* \dots b_n^*)\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} =$$

$$\langle B|A\rangle$$

Exercise 2.1

Prove that the vector $|r\rangle$ is orthogonal to vector $|l\rangle$:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$$

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$$

* * * * * * * * * *

If $|r\rangle$ is orthogonal to $|l\rangle$, then the scalar product $\langle r|l\rangle$ must be 0.

$$\langle r|l\rangle =$$

$$\left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|d\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\langle u|u\rangle - \frac{1}{2}\langle u|d\rangle + \frac{1}{2}\langle d|u\rangle - \frac{1}{2}\langle d|d\rangle =$$

$$\frac{1}{2} - 0 + 0 - \frac{1}{2} = 0$$

Exercise 2.2

Prove that $|i\rangle$ and $|o\rangle$ satisfy all the conditions in Eqs. 2.7, 2.8, and 2.9.
Are they unique in that respect?

Eq. 2.7:

$$\langle i|o\rangle = 0$$

Eqs. 2.8:

$$\langle o|u\rangle\langle u|o\rangle = \tfrac{1}{2} \qquad \langle o|d\rangle\langle d|o\rangle = \tfrac{1}{2}$$

$$\langle i|u\rangle\langle u|i\rangle = \tfrac{1}{2} \qquad \langle i|d\rangle\langle d|i\rangle = \tfrac{1}{2}$$

Eqs. 2.9

$$\langle o|r\rangle\langle r|o\rangle = \tfrac{1}{2} \qquad \langle o|l\rangle\langle l|o\rangle = \tfrac{1}{2}$$

$$\langle i|r\rangle\langle r|i\rangle = \tfrac{1}{2} \qquad \langle i|l\rangle\langle l|i\rangle = \tfrac{1}{2}$$

$$|i\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{i}{\sqrt{2}}|d\rangle \text{ and } \langle i| = \langle u|\tfrac{1}{\sqrt{2}} - \langle d|\tfrac{i}{\sqrt{2}}$$

$$|o\rangle = \tfrac{1}{\sqrt{2}}|u\rangle - \tfrac{i}{\sqrt{2}}|d\rangle \text{ and } \langle o| = \langle u|\tfrac{1}{\sqrt{2}} + \langle d|\tfrac{i}{\sqrt{2}}$$

* * * * * * * * * *

Eq. 2.7: $\langle i|o\rangle = 0$

$$\langle i|o\rangle =$$

$$\left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle u|d\rangle - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle + \frac{i}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 - \frac{i}{2}\cdot 0 - \frac{i}{2}\cdot 0 - \frac{1}{2}\cdot 1 = 0$$

Eqs. 2.8 $\langle o|u\rangle\langle u|o\rangle$:

$$\langle o|u\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}\right)(|u\rangle) =$$

$$\left(\frac{1}{\sqrt{2}}\langle u|u\rangle + \frac{i}{\sqrt{2}}\langle d|u\rangle\right) =$$

$$\left(\frac{1}{\sqrt{2}} + 0\right) = \frac{1}{\sqrt{2}}$$

$$\langle u|o\rangle = (\langle u|)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{i}{\sqrt{2}}\langle u|d\rangle =$$

$$\frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

Eqs. 2.8 $\langle o|d\rangle\langle d|o\rangle$:

$$\langle o|d\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}\right)(|d\rangle) =$$

$$\left(\frac{1}{\sqrt{2}}\langle u|d\rangle + \frac{i}{\sqrt{2}}\langle d|d\rangle\right) =$$

$$\left(0 + \frac{i}{\sqrt{2}}\right) = \frac{i}{\sqrt{2}}$$

$$\langle d|o\rangle = (\langle d|)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$0 - \frac{i}{\sqrt{2}} = -\frac{i}{\sqrt{2}}$$

$$\langle o|d\rangle\langle d|o\rangle = \frac{i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2}$$

Eqs. 2.8 $\langle i|u\rangle\langle u|i\rangle$:

$$\langle i|u\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)(|u\rangle) =$$

$$\left(\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{i}{\sqrt{2}}\langle d|u\rangle\right) =$$

$$\left(\frac{1}{\sqrt{2}} - 0\right) = \frac{1}{\sqrt{2}}$$

$$\langle u|i\rangle = (\langle u|)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{i}{\sqrt{2}}\langle u|d\rangle =$$

$$\frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

Eqs. 2.8 $\langle i|d\rangle\langle d|i\rangle$:

$$\langle i|d\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)(|d\rangle) =$$

$$\left(\frac{1}{\sqrt{2}}\langle u|d\rangle - \frac{i}{\sqrt{2}}\langle d|d\rangle\right) =$$

$$\left(0 - \frac{i}{\sqrt{2}}\right) = \frac{-i}{\sqrt{2}}$$

$$\langle d|i\rangle = (\langle d|)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$0 - \frac{i}{\sqrt{2}} = \frac{-i}{\sqrt{2}}$$

$$\langle i|d\rangle\langle d|i\rangle = \frac{-i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = \frac{1}{2}$$

Eqs. 2.9 $\langle o|r\rangle\langle r|o\rangle$:

$|i\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle$ and $\langle i| = \langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}$

$|o\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle$ and $\langle o| = \langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}$

$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$ and $\langle r| = \langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}}$

$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$ and $\langle l| = \langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{1}{\sqrt{2}}$

Eqs. 2.9 $\langle o|r\rangle\langle r|o\rangle$:

$$\langle o|r\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|d\rangle + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 + \frac{i}{2}\cdot 0 + \frac{i}{2}\cdot 1 =$$

$$\frac{1+i}{2}$$

$$\langle r|o\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle u|d\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 - \frac{i}{2}\cdot 0 + \frac{1}{2}\cdot 0 - \frac{i}{2}\cdot 1 =$$

$$\frac{1-i}{2}$$

$$\langle o|r\rangle\langle r|o\rangle = \frac{1+i}{2}\cdot\frac{1-i}{2} = \frac{1-(i\cdot i)^2}{4} = \frac{2}{4} = \frac{1}{2}$$

Eqs. 2.9 $\langle o|l\rangle\langle l|o\rangle$:

$$\langle o|l\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|d\rangle + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0 + \frac{i}{2}\cdot 0 - \frac{i}{2}\cdot 1 =$$

$$\frac{1-i}{2}$$

$$\langle l|o\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle u|d\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 - \frac{i}{2}\cdot 0 + \frac{1}{2}\cdot 0 + \frac{i}{2}\cdot 1 =$$

$$\frac{1+i}{2}$$

$$\langle o|l\rangle\langle l|o\rangle = \frac{1-i}{2}\cdot\frac{1+i}{2} = \frac{1-(i\cdot i)^2}{4} = \frac{2}{4} = \frac{1}{2}$$

Eqs. 2.9 $\langle i|r\rangle\langle r|i\rangle$:

$$\langle i|r\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|d\rangle - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 - \frac{i}{2}\cdot 0 - \frac{i}{2}\cdot 1 =$$

$$\frac{1-i}{2}$$

$$\langle r|i\rangle = \left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle u|d\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 + \frac{i}{2}\cdot 0 + \frac{1}{2}\cdot 0 + \frac{i}{2}\cdot 1 =$$

$$\frac{1+i}{2}$$

$$\langle i|r\rangle\langle r|i\rangle = \frac{1-i}{2}\cdot\frac{1+i}{2} = \frac{1-(i\cdot i)^2}{4} = \frac{2}{4} = \frac{1}{2}$$

Eqs. 2.9 $\langle i|l\rangle\langle l|i\rangle$:

$$\langle i|l\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|d\rangle - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0 - \frac{i}{2}\cdot 0 + \frac{i}{2}\cdot 1 =$$

$$\frac{1+i}{2}$$

$$\langle l|i\rangle = \left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle u|u\rangle + \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle u|d\rangle - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle d|u\rangle - \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{1}{2}\cdot 1 + \frac{i}{2}\cdot 0 - \frac{1}{2}\cdot 0 - \frac{i}{2}\cdot 1 =$$

$$\frac{1-i}{2}$$

$$\langle i|l\rangle\langle l|i\rangle = \frac{1+i}{2}\cdot\frac{1-i}{2} = \frac{1-(i\cdot i)^2}{4} = \frac{2}{4} = \frac{1}{2}$$

Are $|i\rangle$ and $|o\rangle$ unique in that respect?

I think that this question goes to the phase ambiguity. $|i\rangle$ and $|o\rangle$ can be multiplied by any z of the form $z = e^{i\theta}$ without disturbing the relationships 2.7, 2.8 and 2.9.

Exercise 2.3

Assume that $|i\rangle$ and $|o\rangle$ are given as:

$$|i\rangle = \alpha|u\rangle + \beta|d\rangle \text{ and } \langle i| = \langle u|\alpha^* + \langle d|\beta^*$$

$$|o\rangle = \gamma|u\rangle - \delta|d\rangle \text{ and } \langle o| = \langle u|\gamma^* - \langle d|\delta^*$$

a) Use Eqs. 2.8 to show that

$$\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$$

b) Use the above result and Eqs. 2.9 to show that:

$$\alpha^*\beta + \alpha\beta^* = 0$$

c) Show that $\alpha^*\beta$ must be pure imaginary.

Eqs. 2.8:

$$\langle o|u\rangle\langle u|o\rangle = \tfrac{1}{2} \qquad \langle o|d\rangle\langle d|o\rangle = \tfrac{1}{2}$$

$$\langle i|u\rangle\langle u|i\rangle = \tfrac{1}{2} \qquad \langle i|d\rangle\langle d|i\rangle = \tfrac{1}{2}$$

Eqs. 2.9

$$\langle o|r\rangle\langle r|o\rangle = \tfrac{1}{2} \qquad \langle o|l\rangle\langle l|o\rangle = \tfrac{1}{2}$$

$$\langle i|r\rangle\langle r|i\rangle = \tfrac{1}{2} \qquad \langle i|l\rangle\langle l|i\rangle = \tfrac{1}{2}$$

* * * * * * * * * *

a) Use Eqs. 2.8 to show that $\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$

$$\langle o|u\rangle\langle u|o\rangle = \frac{1}{2}$$

$$\langle o|u\rangle = (\langle u|\gamma^* - \langle d|\delta^*)|u\rangle =$$

$$\gamma^*\langle u|u\rangle - \delta^*\langle d|u\rangle =$$

$$\gamma^* \cdot 1 - \delta^* \cdot 0 = \gamma^*$$

$$\langle u|o\rangle = \langle u|(\gamma|u\rangle - \delta|d\rangle) =$$

$$\gamma\langle u|u\rangle - \delta\langle u|d\rangle =$$

$$\gamma \cdot 1 - \delta \cdot 0 = \gamma$$

$$\langle o|u\rangle\langle u|o\rangle = \gamma^*\gamma$$

According to equation 2.8 this gives $\gamma^*\gamma = \frac{1}{2}$.

$$\langle o|d\rangle\langle d|o\rangle = \frac{1}{2}$$

$$\langle o|d\rangle = (\langle u|\gamma^* - \langle d|\delta^*)|d\rangle =$$

$$\gamma^*\langle u|d\rangle - \delta^*\langle d|d\rangle =$$

$$\gamma^* \cdot 0 - \delta^* \cdot 1 = -\delta^*$$

$$\langle d|o\rangle = \langle d|(\gamma|u\rangle - \delta|d\rangle) =$$

$$\gamma\langle d|u\rangle - \delta\langle d|d\rangle =$$

$$\gamma \cdot 0 - \delta \cdot 1 = -\delta$$

$$\langle o|d\rangle\langle d|o\rangle = \delta^*\delta$$

According to equation 2.8 this gives $\delta^*\delta = \frac{1}{2}$.

$$\langle i|u\rangle\langle u|i\rangle = \frac{1}{2}$$

$$\langle i|u\rangle = (\langle u|\alpha^* + \langle d|\beta^*)|u\rangle =$$

$$\alpha^*\langle u|u\rangle + \beta^*\langle d|u\rangle =$$

$$\alpha^* \cdot 1 + \beta^* \cdot 0 = \alpha^*$$

$$\langle u|i\rangle = \langle u|(\alpha|u\rangle + \beta|d\rangle) =$$

$$\alpha\langle u|u\rangle + \beta\langle u|d\rangle =$$

$$\alpha \cdot 1 + \beta \cdot 0 = \alpha$$

$$\langle o|u\rangle\langle u|o\rangle = \alpha^*\alpha$$

According to equation 2.8 this gives $\alpha^*\alpha = \frac{1}{2}$.

$$\langle i|d\rangle\langle d|i\rangle = \frac{1}{2}$$

$$\langle i|d\rangle = (\langle u|\alpha^* + \langle d|\beta^*)|d\rangle =$$

$$\alpha^*\langle u|d\rangle + \beta^*\langle d|d\rangle =$$

$$\alpha^* \cdot 0 + \beta^* \cdot 1 = \beta^*$$

$$\langle d|i\rangle = \langle d|(\alpha|u\rangle + \beta|d\rangle) =$$

$$\alpha\langle d|u\rangle + \beta\langle d|d\rangle =$$

$$\alpha \cdot 0 + \beta \cdot 1 = \beta$$

$$\langle i|d\rangle\langle d|i\rangle = \beta^*\beta$$

According to equation 2.8 this gives $\beta^*\beta = \frac{1}{2}$.

b) Use the above result and Eqs. 2.9 to show that

$$\alpha^*\beta + \alpha\beta^* = 0$$

$$|i\rangle = \alpha|u\rangle + \beta|d\rangle \text{ and } \langle i| = \langle u|\alpha^* + \langle d|\beta^*$$

$$|r\rangle = \tfrac{1}{\sqrt{2}}|u\rangle + \tfrac{1}{\sqrt{2}}|d\rangle \text{ and } \langle r| = \langle u|\tfrac{1}{\sqrt{2}} + \langle d|\tfrac{1}{\sqrt{2}}$$

$$\langle i|r\rangle = (\langle u|\alpha^* + \langle d|\beta^*)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right) =$$

$$\frac{\alpha^*}{\sqrt{2}}\langle u|u\rangle + \frac{\alpha^*}{\sqrt{2}}\langle u|d\rangle + \frac{\beta^*}{\sqrt{2}}\langle d|u\rangle + \frac{\beta^*}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{\alpha^*}{\sqrt{2}}\cdot 1 + \frac{\alpha^*}{\sqrt{2}}\cdot 0 + \frac{\beta^*}{\sqrt{2}}\cdot 0 + \frac{\beta^*}{\sqrt{2}}\cdot 1 =$$

$$\frac{\alpha^*}{\sqrt{2}} + \frac{\beta^*}{\sqrt{2}}$$

$$\langle r|i\rangle = \left(\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right)(\langle u|\alpha + \langle d|\beta) =$$

$$\frac{\alpha}{\sqrt{2}}\langle u|u\rangle + \frac{\alpha}{\sqrt{2}}\langle u|d\rangle + \frac{\beta}{\sqrt{2}}\langle d|u\rangle + \frac{\beta}{\sqrt{2}}\langle d|d\rangle =$$

$$\frac{\alpha}{\sqrt{2}}\cdot 1 + \frac{\alpha}{\sqrt{2}}\cdot 0 + \frac{\beta}{\sqrt{2}}\cdot 0 + \frac{\beta}{\sqrt{2}}\cdot 1 =$$

$$\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}$$

$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}$$

$$\langle i|r\rangle\langle r|i\rangle = \left(\frac{\alpha^*}{\sqrt{2}} + \frac{\beta^*}{\sqrt{2}}\right)\left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right) =$$

$$\frac{1}{\sqrt{2}}(\alpha^* + \beta^*)\frac{1}{\sqrt{2}}(\alpha + \beta) =$$

$$\frac{1}{2}(\alpha^* + \beta^*)(\alpha + \beta) =$$

$$\frac{1}{2}(\alpha^*\alpha + \alpha^*\beta + \beta^*\alpha + \beta^*\beta) =$$

$$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}(\alpha^*\alpha + \beta^*\beta + \alpha^*\beta + \beta^*\alpha) = \frac{1}{2}$$

From part a) we know that $\alpha^*\alpha = \beta^*\beta = \frac{1}{2}$

$$\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + \alpha^*\beta + \beta^*\alpha\right) = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{2} + \alpha^*\beta + \beta^*\alpha = 1$$

$$1 + \alpha^*\beta + \beta^*\alpha = 1$$

$$\alpha^*\beta + \beta^*\alpha = 0$$

c) Show that $\alpha^*\beta$ must be pure imaginary.

Let $\alpha = r + is$, $\alpha^* = r - is$, $\beta = t + iu$, $\beta^* = t - iu$.

$$\alpha^*\beta + \beta^*\alpha = 0 \rightarrow (r - is)(t + iu) + (r + is)(t - iu) = 0$$

$$(rt + iru - ist + su) + (rt - iru + ist + su) = 0$$

$$2rt + 2su = 0$$

$$rt + su = 0$$

$$rt = -su$$

$$r = -\frac{su}{t}$$

$$\alpha^*\beta = (r - is)(t + iu) = \left(-\frac{su}{t} - is\right)(t + iu) =$$

$$-su - \frac{isu^2}{t} - ist + su =$$

$$-\frac{isu^2}{t} - ist = -i\left(\frac{su^2}{t} + st\right) = -i\left(\frac{su^2 + st^2}{t}\right)$$

Exercise 3.1

Prove: If a vector space is N-dimensional, an orthonormal basis of N vectors can be constructed from the eigenvectors of a Hermitian operator.

* * * * * * * * * *

The eigenvectors of a Hermitian operator form a complete set of linear independent vectors, out of these can be constructed an orthogonal basis and, by dividing through their length an orthonormal basis. This holds for finite dimensional vector spaces.

---

For the example of the $R^3$ we want to show how to construct the change of basis vectors.

Let $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ $and$ $\begin{pmatrix}0\\1\\1\end{pmatrix}$ be a set of vectors $B_1$, $B_2$ and $B_3$ that form a basis B of $R^3$.

They are linear independent:

$$a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

We transform this:

$$\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \rightarrow \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \rightarrow \begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \rightarrow \begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \rightarrow \begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}$$

which makes clear that the only solution to this is a = b = c = 0.

The linear independent vectors define a matrix P, a linear map:

$$P := \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}$$

Let $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be a vector with respect to this basis B: $\overline{x} = x_1B_1 + x_2B_2 + x_3B_3$.

Then $P\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ give the coordinates of $\overline{x}$ in the canonical basis $E_1$, $E_2$ and $E_3$: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ $and$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

$$P\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+x_2\\x_2+x_3\\x_1+x_3\end{pmatrix}$$

Check:

$$P\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$P\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$P\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

To get this the other way around we must find the inverse matrix $P^{-1}$.

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}|\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\qquad \text{line 3 minus line 1}$$

$$\begin{pmatrix}1&1&0\\0&1&1\\0&-1&1\end{pmatrix}|\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}\qquad \text{line 3 plus line 2}$$

$$\begin{pmatrix}1&1&0\\0&1&1\\0&0&2\end{pmatrix}|\begin{pmatrix}1&0&0\\0&1&0\\-1&1&1\end{pmatrix}\qquad \text{line 2 minus line 3/2}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}|\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\-1&1&1\end{pmatrix}\qquad \text{line 3 divided by 2}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}|\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

Our inverse matrix $P^{-1}$:

$$\frac{1}{2}\begin{pmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{pmatrix}$$

Applied to the linear independent vectors $B_1, B_2, B_3$ this must give the canonical basis $E_1, E_2, E_3$.

Check:

$$P^{-1}\begin{pmatrix}1\\0\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$P^{-1}\begin{pmatrix}1\\1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\2\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$P^{-1}\begin{pmatrix}0\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&1\\1&1&-1\\-1&1&1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Exercise 3.2

Prove that $\sigma_z$ of Eq. 3.16 is the unique solution to Eqs. 3.14 und 3.15.

Eq. 3.16

$$\sigma_z = \begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eq. 3.14

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eq. 3.15

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{21} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

* * * * * * * * * *

Eq. 3.14

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \to a = 1; c = 0$$

Eq. 3.15

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \to b = 0; d = -1$$

The result is the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

---

Proof by contradiction eq. 3.14:

let $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \neq 1$ and $b \neq 0$

and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$1 \cdot a + 0 \cdot b = 1 \to a = 1 \; contradiction$$

$$0 \cdot a - 1 \cdot b = 0 \to b = 0 \; contr.$$

The same holds for eq. 3.15:

let $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \neq 0$ and $b \neq 1$

and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$1 \cdot a + 0 \cdot b = 0 \to a = 0 \; contr.$$

$$0 \cdot a - 1 \cdot b = -1 \to b = 1 \; contr.$$

Exercise 3.3

Calculate the eigenvectors and eigenvalues of $\sigma_n$. Assume the eigenvector $\lambda_1$ has the form:

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$\alpha$ is an unknown parameter. Plug this vector into the eigenvalue equation and solve for $\alpha$ in terms of $\theta$. Why did we use a single parameter $\alpha$? Notice that our suggested column vector must have unit length.

$$\sigma_n = \begin{pmatrix} cos\ \theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

To show:

$$\begin{pmatrix} cos\ \theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

* * * * * * * * * *

$$\begin{pmatrix} cos\ \theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} cos\ \theta \cdot \cos\alpha + \sin\theta \cdot \sin\alpha \\ \sin\theta \cdot \cos\alpha - \cos\theta \cdot \sin\alpha \end{pmatrix}$$

$$\begin{pmatrix} cos\ \theta \cdot \cos\alpha + \sin\theta \cdot \sin\alpha \\ \sin\theta \cdot \cos\alpha - \cos\theta \cdot \sin\alpha \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

This must be valid for every coordinate, so we get two equations:

$$cos\ \theta \cdot \cos\alpha + \sin\theta \cdot \sin\alpha = \lambda_1 \cdot \cos\alpha$$

$$\sin\theta \cdot \cos\alpha - \cos\theta \cdot \sin\alpha = \lambda_1 \cdot \sin\alpha$$

Trigonometric identities:

$$cos\ \theta \cdot \cos\alpha + \sin\theta \cdot \sin\alpha = \cos(\theta - \alpha)$$

$$\sin\theta \cdot \cos\alpha - \cos\theta \cdot \sin\alpha = \sin(\theta - \alpha)$$

We have:

$$\cos(\theta - \alpha) = \lambda_1 \cdot \cos\alpha$$

$$\sin(\theta - \alpha) = \lambda_1 \cdot \sin\alpha$$

Sin and cos are nonlinear functions, the only possible solution is: $\lambda_1 = 1$ and $\alpha = \frac{\theta}{2}$:

$$\begin{pmatrix} cos\ \theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

For the second eigenvalue/eigenvector we use that both eigenvectors must be orthogonal.

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$a \cdot \cos\alpha + \mathrm{b} \cdot \sin\alpha = 0$$

Again, as sin and cos are nonlinear functions, possible solutions are

$$a = -\sin\alpha\, ; b = \cos\alpha$$

$$a = \sin\alpha\, ; b = -\cos\alpha$$

We try the first one: $a = -\sin\alpha\,; b = \cos\alpha$

$$\begin{pmatrix} cos\,\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} = \begin{pmatrix} -cos\,\theta\cdot\sin\alpha + \sin\theta\cdot\cos\alpha \\ -\sin\theta\cdot\sin\alpha - \cos\theta\cdot\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} -cos\,\theta\cdot\sin\alpha + \sin\theta\cdot\cos\alpha \\ -\sin\theta\cdot\sin\alpha - \cos\theta\cdot\cos\alpha \end{pmatrix} = \lambda_2\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

Using the trigonometric identities again we get:

$$\sin(\theta-\alpha) = \lambda_2\cdot(-\sin\alpha)$$

$$-\cos(\theta-\alpha) = \lambda_2\cdot\cos\alpha$$

with the solution: $\lambda_2 = -1$ and again $\alpha = \frac{\theta}{2}$

Check the second solution $a = \sin\alpha\,; b = -\cos\alpha$:

$$\begin{pmatrix} cos\,\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} = \begin{pmatrix} cos\,\theta\cdot\sin\alpha - \sin\theta\cdot\cos\alpha \\ \sin\theta\cdot\sin\alpha + \cos\theta\cdot\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} cos\,\theta\cdot\sin\alpha - \sin\theta\cdot\cos\alpha \\ \sin\theta\cdot\sin\alpha + \cos\theta\cdot\cos\alpha \end{pmatrix} = \lambda_2\begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}$$

Using the trigonometric identities again we get:

$$-\sin(\theta-\alpha) = \lambda_2\cdot\sin\alpha$$

$$\cos(\theta-\alpha) = \lambda_2\cdot(-\cos\alpha)$$

with the same solution: $\lambda_2 = -1$ and $\alpha = \frac{\theta}{2}$.

Why did we use a single parameter $\alpha$? Notice that our suggested column vector must have unit length.

Working with polar coordinates in a plane we need two parameters to determine a vector. One parameter is the length that is fixed to one, so we have as second parameter the angle.

Exercise 3.4

Let $n_z = cos\theta$, $n_x = sin\theta cos\phi$, and $n_y = sin\theta sin\phi$.

Angles $\theta$ and $\phi$ are defined according to the usual conventions for spherical coordinates.

Compute the eigenvalues and eigenvectors for the matrix $\sigma_n$:

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

* * * * * * * * * *

With the above definitions this transforms to:

$$\sigma_n = \begin{pmatrix} cos\ \theta & \sin\theta\ cos\phi - i\sin\theta\ sin\phi \\ \sin\theta\ cos\phi + \sin\theta\ sin\phi & -\cos\theta \end{pmatrix}$$

The determinant of the matrix

$$\begin{pmatrix} (cos\ \theta) - \lambda & \sin\theta\ cos\phi - i\sin\theta\ sin\phi \\ \sin\theta\ cos\phi + \sin\theta\ sin\phi & -(\cos\theta) - \lambda \end{pmatrix}$$

must be zero in order the matrix to have eigenvectors.

$$\left|\begin{pmatrix} (cos\ \theta) - \lambda & \sin\theta\ cos\phi - i\sin\theta\ sin\phi \\ \sin\theta\ cos\phi + \sin\theta\ sin\phi & -(\cos\theta) - \lambda \end{pmatrix}\right| = 0$$

$$((cos\ \theta) - \lambda)(-(\cos\theta) - \lambda) - (\sin\theta\ cos\phi - i\sin\theta\ sin\phi)(\sin\theta\ cos\phi + \sin\theta\ sin\phi) =$$

$$-cos^2\theta + \lambda^2 - (sin^2\theta cos^2\phi + sin^2\theta sin^2\phi) =$$

$$-cos^2\theta + \lambda^2 - sin^2\theta(cos^2\phi + sin^2\phi) =$$

$$-cos^2\theta + \lambda^2 - sin^2\theta =$$

$$\lambda^2 - (sin^2\theta + cos^2\theta) =$$

$$\lambda^2 - 1$$

The characteristic polynomial is $\lambda^2 - 1 = 0$ with the solutions $\lambda = \mp 1$.

As candidate for eigenvector we choose the same as in exercise 3.3: $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

Eigenvalue $\lambda = +1$

$$\begin{pmatrix} cos\ \theta & \sin\theta\ cos\phi - i\sin\theta\ sin\phi \\ \sin\theta\ cos\phi + \mathrm{i}\sin\theta\ sin\phi & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

leads to two equations:

$$cos\ \theta\cos\alpha + (\sin\theta\ cos\phi - i\sin\theta\ sin\phi)\sin\alpha = \cos\alpha$$

$$(\sin\theta\ cos\phi + \mathrm{i}\sin\theta\ sin\phi)\cos\alpha - \cos\theta\sin\alpha = \sin\alpha$$

First equation:

$$cos\ \theta\cos\alpha + \sin\alpha\sin\theta\ cos\phi - i\sin\alpha\sin\theta\ sin\phi = \cos\alpha$$

Second equation:

$$\sin\theta\, cos\phi \cos\alpha + i\sin\theta\, sin\phi \cos\alpha - \cos\theta \sin\alpha = \sin\alpha$$

We eliminate the imaginary part of the first equation:

$$\sin\alpha\sin\theta\, cos\phi - i\sin\alpha\sin\theta\, sin\phi =$$

$$\sin\alpha\sin\theta\,(cos\phi - i\, sin\phi) =$$

$$\sin\alpha\sin\theta$$

because

$$(cos\phi - i\, sin\phi) = e^{-i\phi}$$

and

$$\left|e^{-i\phi}\right| = 1$$

The same holds for the second equation:

$$\sin\theta\, cos\phi \cos\alpha + i\sin\theta\, sin\phi\cos\alpha =$$

$$\sin\theta\cos\alpha\,(cos\phi + i\, sin\phi) =$$

$$\sin\theta\cos\alpha$$

The two equations simplify:

First equation:

$$cos\ \theta\cos\alpha + \sin\alpha\sin\theta = \cos\alpha$$

Second equation:

$$\sin\theta\cos\alpha - \cos\theta\sin\alpha = \sin\alpha$$

Some more trigonometric identities

$$cos\ \theta\cos\alpha = \frac{1}{2}(\cos(\theta-\alpha) + cos(\theta+\alpha))$$

$$sin\ \theta\sin\alpha = \frac{1}{2}(\cos(\theta-\alpha) - cos(\theta+\alpha))$$

applied to the first equation

$$cos\ \theta\cos\alpha + \sin\alpha\sin\theta = \cos(\theta-\alpha)$$

we get

$$\cos(\theta-\alpha) = \cos\alpha$$

with the solution:

$$\alpha = \frac{\theta}{2}$$

More trigonometric identities:

$$cos\ \theta\ sin\ \theta = \frac{1}{2}(\sin(\theta-\alpha) + sin(\theta+\alpha))$$

$$cos\ \theta \sin\alpha = \frac{1}{2}(\sin(\alpha - \theta) + sin(\theta + \alpha))$$

With $\sin(x) = -\sin(-x)$ we write

$$cos\ \theta \sin\alpha = -\frac{1}{2}(\sin(\theta - \alpha) + sin(\theta + \alpha))$$

and apply to the second equation:

$$cos\ \theta\ sin\ \theta - cos\ \theta \sin\alpha = \ \sin(\theta - \alpha)$$

We get

$$\sin(\theta - \alpha) = \sin\alpha$$

with the solution:

$$\alpha = \frac{\theta}{2}$$

The computation for the eigenvalue $\lambda = +1$ is omitted (see exercise 3.3).

Exercise 3.5

Suppose that a spin is prepared so that $\sigma_m = +1$. The apparatus is then rotated to an arbitrary $\hat{n}$ direction and $\sigma_n$ is measured. What is the probability that the result is +1?

Note that $\sigma_m = \sigma \cdot \hat{m}$, using the same convention we used for $\sigma_n$.

* * * * * * * * * *

We rotate the coordinate system twice.

Rotation doesn't change the length of vectors and preserve the relative dependencies between vectors (see exercise 4.1. is a rotation matrix unitary?).

We rotate one time in a way that $\hat{m}$ will be the z-axis, a second time in a way that $\hat{n}$ will be in the x-z-plane.

This is the situation of exercise 3.3 with the solution

$$\lambda_1 = 1$$

$$|\lambda_1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{pmatrix}$$

$$\lambda_2 = -1$$

$$|\lambda_2\rangle = \begin{pmatrix} -sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix}$$

with the probability

$$P(+1) = |\langle u|\lambda_1\rangle|^2 = \left(cos\frac{\theta}{2}\right)^2$$

Exercise 4.1

Prove that if (the time operator) U is unitary, and if $|A\rangle$ and $|B\rangle$ are any two state-vectors, then the inner product of $U|A\rangle$ and $U|B\rangle$ is the same as the inner product of $|A\rangle$ and $|B\rangle$. One could call this the conservation of overlaps. It expresses the fact that the logical relation between states is preserved with time.

U is unitary:

$$U^\dagger U = I$$

$$U|A\rangle = \langle A|U^\dagger$$

$$* * * * * * * * * *$$

$$\langle A|B\rangle = \langle A|I|B\rangle = \left\langle A\middle|U^\dagger U\middle|B\right\rangle$$

with $\left\langle A\middle|U^\dagger U\middle|B\right\rangle$ being the inner product of $U|A\rangle$ and $U|B\rangle$.

Exercise 4.2
Prove that if $M$ and $L$ are both Hermitian, the (extended) commutator $i[M,L]$ is also Hermitian. Note that the $i$ is important. The commutator is, by itself, not Hermitian.

Hermitian: the diagonal is pure real and: $M = M^\dagger$

$$* * * * * * * * * *$$

Proof for a $3 \times 3$-matrix:

$M := \begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix}$ and $L := \begin{pmatrix} g & h & u \\ h^* & j & k \\ u^* & k^* & l \end{pmatrix}$ with $a, d, f, g, j, l$ being real.

$$ML = \begin{pmatrix} ag + bh^* + cu & ah + bj + ck^* & ai + bk + cl \\ b^*g + dh^* + eu^* & b^*h + dj + ek^* & b^*u + dk + el \\ c^*g + e^*h^* + fu^* & c^*h + e^*j + fk^* & c^*u + e^*k + fl \end{pmatrix}$$

$$LM = \begin{pmatrix} ga + hb^* + uc^* & gb + hd + ue^* & gc + he + uf \\ h^*a + jb^* + kc^* & h^*b + jd + ke^* & h^*c + je + kf \\ u^*a + k^*b^* + lc^* & u^*b + k^*d + le^* & u^*c + k^*e + lf \end{pmatrix}$$

$$ML - LM =$$

$$\begin{pmatrix} ag + bh^* + cu - ga - hb^* - uc^* & ah + bj + ck^* - gb - hd - ue^* & ai + bk + cl - gc - he - uf \\ b^*g + dh^* + eu^* - h^*a - jb^* - kc^* & b^*h + dj + ek^* - h^*b - jd - ke^* & b^*u + dk + el - h^*c - je - kf \\ c^*g + e^*h^* + fu^* - u^*a - k^*b^* - lc^* & c^*h + e^*j + fk^* - u^*b - k^*d - le^* & c^*u + e^*k + fl - u^*c - k^*e - lf \end{pmatrix}$$

… with some transformation work …

$$\begin{pmatrix} bh^* - hb^* + c^*u - uc^* & h(a-d) + b(j-g) + ck^* - ue^* & u(a-f) + c(l-g) + bk - he \\ -h^*(a-d) - b^*(j-g) - c^*k + u^*e & b^*h - h^*b + ek^* - ke^* & e(l-j) + k(d-f) + b^*u - h^*c \\ -u^*(a-f) - c^*(l-g) - b^*k^* + h^*e^* & e^*(l-j) - k^*(d-f) - bu^* + hc^* & c^*u - u^*c + e^*k - k^*e \end{pmatrix}$$

The entries $(1,1), (2,2), (3,3)$ are completely imaginary, because the differences $bh^* - hb^*$ etc. are imaginary and become thus become real if we multiplicate them by the imaginary unit $i$.

We check this with $z_1 := x + iy$, $z_2 := u + iv$, $x, y, u, v$ are real numbers.

$$z_1 \cdot {z_2}^* - {z_1}^* \cdot z_2 =$$

$$(x + iy)(u - iv) - (x - y)(u + iv) =$$

$$xu - ixv + iyu + yv - (xu + ixv - iyu + yv) =$$

$$xu - ixv + iyu + yv - xu - ixv + iyu - yv) =$$

$$-2ixv + 2iyu) =$$

$$2i(yu - xv)$$

This is a complete imaginary number.

We multiplicate the matrix with the imaginary unit $i$:

$$\begin{pmatrix} i(bh^* - hb^* + cu - uc^*) & ih(a-d) + ib(j-g) + ick^* - iue^* & iu(a-f) + ic(l-g) + ibk - ihe \\ -ih^*(a-d) - ib^*(j-g) - ic^*k + iu^*e & i(b^*h - h^*b + ek^* - ke^*) & ie(l-j) + ik(d-f) + ib^*u - ih^*c \\ -iu^*(a-f) - ic^*(l-g) - ib^*k^* + ih^*e^* & ie^*(l-j) - ik^*(d-f) - ibu^* + ihc^* & i(c^*u - u^*c + e^*k - k^*e) \end{pmatrix}$$

Multiplication of the whole matrix with the imaginary unit makes the entries on the diagonal real. We have to show that the entries away from the diagonal $(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)$ fulfil the Hermitian criterion: $(1,2)^* = (2,1)$ etc.

$(1,2)^* = (2,1)$:

$$(1,2) = ih(a-d) + ib(j-g) + ick^* - iue^*$$

Note that $a, d, j, g$ are real numbers.

$$(1,2)^* = -ih^*(a-d) - ib^*(j-g) - ic^*k + iu^*e$$

We compare this with $(2,1)$:

$$(2,1) = -ih^*(a-d) - ib^*(j-g) - ic^*k + iu^*e$$

This is correct. The same holds for $(1,3)^* = (3,1)$ and $(2,3)^* = (3,2)$.

Exercise 4.3

With the definition of Poisson brackets and check that the identification in Eq. 4.21 is dimensionally consistent. Show that without the factor $\hbar$, it would not be.

Eq. 4.21

$$[F,G] \leftrightarrow i\hbar\{F,G\}$$

$$[\hbar] = J \cdot s = \frac{kg \cdot m^2}{s^2} \cdot s$$

* * * * * * * * * *

(not a proof, more a reflection about …)

In Wikipedia we find

$$\dot{f} = \frac{df}{dt} = \sum_{k=1}^{s} \left( \frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} + \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial f}{\partial t}$$

$$\dot{f} = \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

In “Classical Mechanics, The Theoretical Minimum” we find:

$$\dot{F} = \sum_{i} \left( \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$\dot{F} = \{F, H\}$$

We can conclude that $F$ has no explicit time-dependency and in fact it is defined as $F(q,p)$ leading to $\frac{\partial F}{\partial t} = 0$.

Following calculus rules $\dot{F}$ must be something like $\frac{\triangle F}{\triangle t}$, so

$$[\dot{F}] \sim \frac{1}{s}$$

and accordingly, the unit of $\{F, H\}$.

On the other hand, we find in “Quantum Mechanics” (4.19):

$$\dot{L} = \frac{dL}{dt} = -\frac{i}{\hbar}[L, H]$$

H is an energy, so the unit of $H$ is $\frac{kg \cdot m^2}{s^2}$ (and I hope that the multiplication with L doesn’t change that). Divided by the unit of $\hbar$: $\frac{kg \cdot m^2}{s}$ this results in:

$$\frac{[H]}{[\hbar]} = \frac{\frac{kg \cdot m^2}{s^2}}{\frac{kg \cdot m^2}{s}} = \frac{kg \cdot m^2 \cdot s}{kg \cdot m^2 \cdot s^2} = \frac{1}{s}$$

giving the correct dimension for $\frac{dL}{dt} = \dot{L}$.

Exercise 4.4

Verify the commutation relations:

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$

$$[\sigma_z, \sigma_x] = 2i\sigma_y$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* * * * * * * * * *

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x =$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} =$$

$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\sigma_z$$

$$[\sigma_y, \sigma_z] = \sigma_y\sigma_z - \sigma_z\sigma_y =$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i\sigma_x$$

$$[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma_y$$

Exercise 4.5

Take any unit 3-vector $\vec{n}$ and form the operator

$$H = \frac{\hbar\omega}{2}\sigma \cdot \vec{n}$$

Find the energy eigenvalues and eigenvectors by solving the time-independent Schrödinger equation. Recall $\sigma \cdot \vec{n}$ in component form:

$$\sigma_n = \sigma \cdot \vec{n} = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

$$* * * * * * * * * *$$

$$H = \frac{\hbar\omega}{2}\sigma \cdot \vec{n} = \frac{\hbar\omega}{2}\begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

The time-independent Schrödinger equation (4.28)

$$H|E_j\rangle = E_j|E_j\rangle$$

$$\frac{\hbar\omega}{2}\begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}|E_j\rangle = E_j|E_j\rangle$$

The characteristic polynomial of the matrix $\sigma_n$ must be zero:

$$\left|\begin{pmatrix} n_z - \lambda & (n_x - in_y) \\ (n_x + in_y) & -n_z - \lambda \end{pmatrix}\right| =$$

$$(n_z - \lambda)(-n_z - \lambda) - (n_x + in_y)(n_x - in_y) =$$

$$\lambda^2 - (n_x^2 + n_y^2 + n_z^2) = 0$$

$$\lambda^2 = (n_x^2 + n_y^2 + n_z^2)$$

$$\lambda = \pm\sqrt{(n_x^2 + n_y^2 + n_z^2)}$$

Because n is a unit vector $\lambda = \pm 1$

The eigenvectors we get out of the equation

$$\begin{pmatrix} n_z - \lambda & (n_x - in_y) \\ (n_x + in_y) & -n_z - \lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvalue $\lambda = +1$

$$\begin{pmatrix} n_z - 1 & (n_x - in_y) \\ (n_x + in_y) & -n_z - 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get two independent equations:

$$I: (n_z - 1) \cdot a + \left(n_x - in_y\right) \cdot b = 0$$

$$II: \left(n_x + in_y\right) \cdot a - (n_z + 1) \cdot b = 0$$

Upper line:

$$(n_z - 1) \cdot a = -\left(n_x - in_y\right) \cdot b$$

$$-(n_z - 1) \cdot a = \left(n_x - in_y\right) \cdot b$$

$$(1 - n_z) \cdot a = \left(n_x - in_y\right) \cdot b$$

$$a = \frac{\left(n_x - in_y\right)}{(1 - n_z)} \cdot b$$

The result inserted in the lower line:

$$\left(n_x + in_y\right) \cdot \frac{\left(n_x - in_y\right)}{(1 - n_z)} \cdot b - (n_z + 1) \cdot b = 0$$

$$\frac{\left(n_x + in_y\right)\left(n_x - in_y\right)}{(1 - n_z)} \cdot b - (n_z + 1) \cdot b = 0$$

$$b\left(\frac{\left(n_x + in_y\right)\left(n_x - in_y\right)}{(1 - n_z)} - (n_z + 1)\right) = 0$$

$$b\left(\frac{\left(n_x + in_y\right)\left(n_x - in_y\right) - (n_z + 1)(1 - n_z)}{(1 - n_z)}\right) = 0$$

$$b\left(\left(n_x + in_y\right)\left(n_x - in_y\right) - (n_z + 1)(1 - n_z)\right) = 0$$

$$b\left(n_x^2 + n_y^2 + n_z^2 - 1\right) = 0$$

$$b \cdot 0 = 0$$

Valid for all b. Back to the upper line:

$$a = \frac{\left(n_x - in_y\right)}{(1 - n_z)} \cdot b$$

This defines the first eigenvector $|\lambda_1\rangle$:

$$|\lambda_1\rangle = \begin{pmatrix} \dfrac{\left(n_x - in_y\right)}{(1 - n_z)} \\ 1 \end{pmatrix}$$

eigenvalue $\lambda = -1$

$$\begin{pmatrix} n_z + 1 & (n_x - in_y) \\ (n_x + in_y) & -n_z + 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get two independent equations:

$$I: (n_z + 1) \cdot a + (n_x - in_y) \cdot b = 0$$

$$II: (n_x + in_y) \cdot a - (n_z - 1) \cdot b = 0$$

Upper line:

$$(n_z + 1) \cdot a = -(n_x - in_y) \cdot b$$

$$a = \frac{-(n_x - in_y)}{(n_z + 1)} \cdot b$$

The result inserted in the lower line:

$$(n_x + in_y) \cdot \frac{-(n_x - in_y)}{(n_z + 1)} \cdot b - (n_z - 1) \cdot b = 0$$

$$\frac{-(n_x + in_y)(n_x - in_y)}{(n_z + 1)} \cdot b - (n_z - 1) \cdot b = 0$$

$$b\left(\frac{-(n_x + in_y)(n_x - in_y)}{(n_z + 1)} - (n_z - 1)\right) = 0$$

$$b\left(\frac{-(n_x + in_y)(n_x - in_y) - (n_z - 1)(n_z + 1)}{(n_z + 1)}\right) = 0$$

$$b\left(-(n_x + in_y)(n_x - in_y) - (n_z - 1)(n_z + 1)\right) = 0$$

$$b\left(-n_x^2 - n_y^2 - n_z^2 + 1\right) = 0$$

$$b \cdot 0 = 0$$

Valid for all b. Back to the upper line:

$$a = \frac{-(n_x - in_y)}{(n_z + 1)} \cdot b$$

This defines the second eigenvector $|\lambda_2\rangle$:

$$|\lambda_2\rangle = \begin{pmatrix} \frac{-(n_x - in_y)}{(n_z + 1)} \\ 1 \end{pmatrix}$$

Additional we check whether the eigenvectors are orthogonal.

$$|\lambda_1\rangle = \begin{pmatrix} \frac{(n_x - in_y)}{(1 - n_z)} \\ 1 \end{pmatrix}$$

$$|\lambda_2\rangle = \begin{pmatrix} \frac{-(n_x - in_y)}{(n_z + 1)} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(in_y - n_x)}{(n_z + 1)} \\ 1 \end{pmatrix}$$

We need the bra $\langle\lambda_1|$ and conjugate it:

$$\langle\lambda_1| = \begin{pmatrix} \frac{(n_x + in_y)}{(1 - n_z)} & 1 \end{pmatrix}$$

$$|\lambda_2\rangle = \begin{pmatrix} \frac{(in_y - n_x)}{(n_z + 1)} \\ 1 \end{pmatrix}$$

The scalar product:

$$\langle\lambda_1|\lambda_2\rangle =$$

$$\begin{pmatrix} \frac{(n_x + in_y)}{(1 - n_z)} & 1 \end{pmatrix} \begin{pmatrix} \frac{(in_y - n_x)}{(n_z + 1)} \\ 1 \end{pmatrix} =$$

$$\frac{(n_x + in_y)(in_y - n_x)}{(1 - n_z)(n_z + 1)} + 1 =$$

$$\frac{-n_x^2 - n_y^2}{1 - n_z^2} + 1 =$$

$$\frac{-n_x^2 - n_y^2 + 1 - n_z^2}{1 - n_z^2} =$$

$$\frac{1 - (n_x^2 + n_y^2 + n_z^2)}{1 - n_z^2} = 0$$

Both vectors are orthogonal to each other.

Exercise 4.6

Carry out the Schrödinger Ket recipe for a single spin.

The Hamiltonian is $H = \frac{\hbar\omega}{2}\sigma_z$ and the final observable is $\sigma_x$.

The initial state is given as $|u\rangle$ (the state in which $\sigma_z = \pm 1$).

After time t, an experiment is done to measure $\sigma_y$.

What are the possible outcomes and what are the probabilities for those outcomes?

* * * * * * * * * *

For easier calculation we omit the factor $\frac{\hbar\omega}{2}$ and work with $H = \sigma_z$ only. We will correct this in the end.

**Recipe step 1**. Derive, look, guess, borrow or steal the Hamiltonian operator H

The Hamiltonian is given:

$$H = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**Recipe step 2**. Prepare an initial state $|\Psi(0)\rangle$

The initial state is given: $|u\rangle$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

**Recipe step 3**. Find the eigenvalues and eigenvectors of H by solving the time-independent Schrödinger equation:

$$H|E_j\rangle = E_j|E_j\rangle$$

The characteristic polynomial of the matrix $\sigma_z$ must be zero:

$$\left|\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}\right| =$$

$$(1-\lambda)(-1-\lambda) - 0 =$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

The eigenvectors we get out of the equation

$$\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvalue $\lambda_1 = +1$

$$\begin{pmatrix} 1-1 & 0 \\ 0 & -1-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get two independent equations:

$$I: 0 \cdot a + 0 \cdot b = 0$$

$$II: 0 \cdot a - 2 \cdot b = 0 \rightarrow b = 0$$

This defines the first eigenvector $|\lambda_1\rangle$:

$$|\lambda_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

eigenvalue $\lambda_2 = -1$

$$\begin{pmatrix}1+1 & 0\\ 0 & -1+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

We get two independent equations:

$$I: 2 \cdot a + 0 \cdot b = 0 \to a = 0$$

$$II: 0 \cdot a + 0 \cdot b = 0$$

This defines the second eigenvector $|\lambda_2\rangle$:

$$|\lambda_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Obviously both eigenvectors are orthogonal to each other:

$$\langle\lambda_1|\lambda_2\rangle = (1\ 0) \cdot \begin{pmatrix}0\\1\end{pmatrix} = 0$$

**Recipe step 4**. Use the initial state-vector $|\Psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3, to calculate the initial coefficients $a_j(0)$:

$$a_j(0) = \langle E_j|\Psi(0)\rangle$$

As both eigenvectors are real, we can change $|\lambda_1\rangle$ to $\langle\lambda_1|$ and $|\lambda_2\rangle$ to $\langle\lambda_2|$.

$a_1(0)$ for Eigenvector $|\lambda_1\rangle$

$$a_1(0) = \langle E_1|\Psi(0)\rangle = \langle\lambda_1|\mathrm{u}\rangle =$$

$$(1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$a_2(0)$ for Eigenvector $|\lambda_2\rangle$

$$a_2(0) = \langle E_2|\Psi(0)\rangle = \langle\lambda_2|\mathrm{u}\rangle =$$

$$(0\ 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$$

**Recipe step 5**. Rewrite $|\Psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $a_j(0)$:

$$|\Psi(0)\rangle = \sum_j a_j(0)|E_j\rangle$$

$$|\Psi(0)\rangle = 1 \cdot |\lambda_1\rangle + 0 \cdot |\lambda_2\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |u\rangle$$

**Recipe step 6**. In the above equation, replace each $a_j(0)$ with $a_j(t)$ to capture it's time-dependence. As a result, $|\Psi(0)\rangle$ becomes $|\Psi(t)\rangle$:

$$|\Psi(t)\rangle = \sum_j a_j(t)|E_j\rangle$$

$$|\Psi(t)\rangle = a_1(t) \cdot |\lambda_1\rangle + a_2(t) \cdot |\lambda_2\rangle = a_1(t)\begin{pmatrix}1\\0\end{pmatrix} + a_2(t)\begin{pmatrix}0\\1\end{pmatrix} = a_1(t)|u\rangle + a_1(t)|d\rangle$$

**Recipe step 7**. Using Eq. 4.30, replace each $a_j(t)$ with $a_j(0)e^{-\frac{i}{\hbar}E_j t}$:

$$|\Psi(t)\rangle = a_1(t)\cdot|\lambda_1\rangle + a_2(t)\cdot|\lambda_2\rangle = a_1(0)e^{-\frac{i}{\hbar}E_1 t}\begin{pmatrix}1\\0\end{pmatrix} + a_2(0)e^{-\frac{i}{\hbar}E_2 t}\begin{pmatrix}0\\1\end{pmatrix} =$$

$$1\cdot e^{-\frac{i}{\hbar}t}\begin{pmatrix}1\\0\end{pmatrix} + 0\cdot e^{-\frac{i}{\hbar}t}\begin{pmatrix}0\\1\end{pmatrix} = e^{-\frac{i}{\hbar}t}\begin{pmatrix}1\\0\end{pmatrix} = e^{-\frac{i}{\hbar}t}|u\rangle$$

Finally, we multiply the factor $\frac{\hbar\omega}{2}$ and get:

$$|\Psi(t)\rangle = e^{-\frac{i\hbar\omega}{2\hbar}t}|u\rangle = e^{-\frac{i\omega}{2}t}|u\rangle$$

Question: After time t, an experiment is done to measure $\sigma_y$. What are the possible outcomes and what are the probabilities for those outcomes?

We omit the time-changing phase and follow “3.7 Reaping the Results” modified in a way that:

$$n_z = cos\frac{\pi}{2}, \qquad n_y = sin\frac{\pi}{2}, \qquad n_x = 0$$

This gives

$$\sigma_n = \begin{pmatrix} cos\dfrac{\pi}{2} & -i\cdot sin\dfrac{\pi}{2} \\ i\cdot sin\dfrac{\pi}{2} & -cos\dfrac{\pi}{2} \end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} := \sigma_y$$

Calculating the eigenvectors of $\sigma_y$:

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \lambda\begin{pmatrix}a\\b\end{pmatrix}$$

$$I: -ib = \lambda a$$

$$II: ia = \lambda b$$

Solving this pair of equations leads to eigenvalues:

$$\lambda = \pm 1$$

The corresponding eigenvectors are:

$\lambda = +1$:

$$\begin{pmatrix}1\\i\end{pmatrix}$$

$\lambda = -1$:

$$\begin{pmatrix}1\\-i\end{pmatrix}$$

Normalization:

$$\left|\begin{pmatrix}1\\i\end{pmatrix}\right| = \sqrt{2}$$

$$\left|\begin{pmatrix}1\\-i\end{pmatrix}\right| = \sqrt{2}$$

The normalized eigenvectors are:

$$\frac{1}{\sqrt{2}}\binom{1}{i}$$

$$\frac{1}{\sqrt{2}}\binom{1}{-i}$$

We calculate the probability of observing $\sigma_y = +1$:

$$P(+1) = |\langle u|\lambda_1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left((1 \quad 0)\binom{1}{i}\right)\right|^2 =$$

$$\left|\frac{1}{\sqrt{2}} \cdot (1 + 0 \cdot i)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Analog the probability of observing $\sigma_y = -1$:

$$P(-1) = |\langle u|\lambda_2\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left((1 \quad 0)\binom{1}{-i}\right)\right|^2 =$$

$$\left|\frac{1}{\sqrt{2}} \cdot (1 - 0 \cdot i)\right|^2 = \frac{1}{2}$$

The possible outcomes of the measurement are $\pm 1$ with probability each 50%.

We ignore the phase-factor $e^{-\frac{i\omega}{2}t}$.

Exercise 5.1

Verify that any $2 \times 2$ Hermitian matrix L can be written as a sum of four terms,

$$L = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

where a, b, c and d are real numbers.

The four Pauli matrices:

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, I= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A general Hermitian matrix $(r, r')$ are real numbers:

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix}$$

* * * * * * * * * *

Verification:

$$L = a\sigma_x + b\sigma_y + c\sigma_z + dI =$$

$$a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} =$$

$$\begin{pmatrix} c + d & a - ib \\ a + ib & d - c \end{pmatrix}$$

Obviously, it's correct that $w^*$ and $w$ are complex conjugated: $a + ib$ and $a - ib$. Further $r$ and $r'$ are real numbers. As Hermitian matrices are not necessarily unitary, the coefficients a, b, c and d are free variables so we can express any $r$ and $r'$: $r = c + d, r' = d - c$.

Exercise 5.2

1) Show that $(\triangle A)^2 = \langle \bar{A}^2 \rangle$ and $(\triangle B)^2 = \langle \bar{B}^2 \rangle$
2) Show that $[\bar{A}, \bar{B}] = [A, B]$
3) Using these relations, show that $\triangle A \triangle B \geq \frac{1}{2} |\langle \Psi | [A, B] | \Psi \rangle|$

The square of uncertainty (or standard deviation) of A, $(\triangle A)^2$:

$$(\triangle A)^2 = \sum_a \bar{a}^2 P(a) = \sum_a (a - \langle A \rangle)^2 P(A)$$

$$* * * * * * * * * *$$

1) Show that $(\triangle A)^2 = \langle \bar{A}^2 \rangle$

First $(\triangle A)^2$:

$$(\triangle A)^2 = \sum_a (a - \langle A \rangle)^2 P(a) =$$

$$\sum_a (a^2 - 2a\langle A \rangle + \langle A \rangle^2) P(a) =$$

$$\sum_a a^2 P(a) - 2\langle A \rangle \sum_a a P(a) + \langle A \rangle^2 \sum_a P(a) =$$

$$\sum_a a^2 P(a) - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2 =$$

$$\sum_a a^2 P(a) - \langle A \rangle^2 =$$

$$\langle A^2 \rangle - \langle A \rangle^2$$

We get $(\triangle A)^2 = \langle A^2 \rangle - \langle A \rangle^2$.

On the other hand:

$$\langle \bar{A}^2 \rangle = \langle (A - \langle A \rangle)^2 \rangle =$$

$$\langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle =$$

$$\langle A^2 \rangle - 2\langle A \rangle \langle \langle A \rangle \rangle + \langle \langle A \rangle \langle A \rangle \rangle =$$

$$\langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle \langle A \rangle =$$

$$\langle A^2 \rangle - 2\langle A \rangle^2 + \langle A \rangle^2 =$$

$$\langle A^2 \rangle - \langle A \rangle^2$$

We get: $\langle \bar{A}^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$.

We can conclude $(\triangle A)^2 = \langle \bar{A}^2 \rangle$. The same holds for $(\triangle B)^2 = \langle \bar{B}^2 \rangle$.

2. Show that $[\bar{A}, \bar{B}] = [A, B]$

$$[\bar{A}, \bar{B}] = \bar{A}\bar{B} - \bar{B}\bar{A} = (A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle) =$$

$$AB - A\langle B \rangle - \langle A \rangle B + \langle A \rangle \langle B \rangle - (BA - B\langle A \rangle - \langle B \rangle A + \langle B \rangle \langle A \rangle) =$$

$$AB - A\langle B \rangle - \langle A \rangle B + \langle A \rangle \langle B \rangle + B\langle A \rangle + \langle B \rangle A - \langle B \rangle \langle A \rangle =$$

$$AB - BA - A\langle B \rangle + \langle B \rangle A - \langle A \rangle B + B\langle A \rangle + \langle A \rangle \langle B \rangle - \langle B \rangle \langle A \rangle =$$

$$AB - BA - A\langle B \rangle + A\langle B \rangle - B\langle A \rangle + B\langle A \rangle + \langle A \rangle \langle B \rangle - \langle A \rangle \langle B \rangle =$$

$$AB - BA = [A, B]$$

3. Using these relations, show that $\triangle A \triangle B \geq \frac{1}{2} |\langle \Psi | [A, B] | \Psi \rangle|$

The Cauchy-Schwarz inequality:

$$2|X||Y| \geq |\langle X|Y \rangle + \langle Y|X \rangle|$$

Let $|\Psi\rangle$ be any ket and let $A$ and $B$ be any two observables ($\in \mathbb{R}$). The associated variables $\bar{A}$ and $\bar{B}$ are built in a way that the expectation values of $\bar{A}$ and $\bar{B}$ are zero. We define $|\mathrm{X}\rangle$ and $|\mathrm{Y}\rangle$ as follows:

$$|\mathrm{X}\rangle = \bar{A}|\Psi\rangle$$

$$\langle \mathrm{X}| = \langle \Psi | \bar{\mathrm{A}}^* = \langle \Psi | \bar{A}$$

$$|\mathrm{Y}\rangle = i\bar{B}|\Psi\rangle$$

$$\langle \mathrm{Y}| = \langle \Psi |(-i\bar{B}^*)$$

Notice the $i$ in the definition of $Y$.

Now, substitute into the Cauchy-Schwarz inequality to get

$$2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} \geq |\langle \Psi | i\bar{A}\bar{B} | \Psi \rangle - \langle \Psi | i\bar{B}\bar{A} | \Psi \rangle| =$$

$$2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} \geq |i(\langle \Psi | \bar{A}\bar{B} | \Psi \rangle - \langle \Psi | \bar{B}\bar{A} | \Psi \rangle)| =$$

$$2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} \geq |\langle \Psi | \bar{A}\bar{B} | \Psi \rangle - \langle \Psi | \bar{B}\bar{A} | \Psi \rangle| =$$

The minus sign is due to the factor of $i$ in the definition of $|\mathrm{Y}\rangle$. Using the definition of a commutator, we find that

$$2\sqrt{\langle \bar{A}^2 \rangle \langle \bar{B}^2 \rangle} \geq |\langle \Psi | [\bar{A}, \bar{B}] | \Psi \rangle|$$

We use $(\triangle A)^2 = \langle \bar{A}^2 \rangle$ and $(\triangle B)^2 = \langle \bar{B}^2 \rangle$ and replace the left side of the inequality:

$$2\sqrt{(\triangle A)^2 (\triangle B)^2} \geq |\langle \Psi | [\bar{A}, \bar{B}] | \Psi \rangle|$$

We use $[\bar{A}, \bar{B}] = [A, B]$ and replace the right side of the inequality:

$$2\sqrt{(\triangle A)^2 (\triangle B)^2} \geq |\langle \Psi | [A, B] | \Psi \rangle|$$

We simplify the left side:

$$2(\triangle A)(\triangle B) \geq |\langle\Psi|[A,B]|\Psi\rangle|$$

We get the result:

$$(\triangle A)(\triangle B) \geq \frac{1}{2}|\langle\Psi|[A,B]|\Psi\rangle|$$

Exercise 6.1

Prove that if $P(a,b)$ factorizes: $P(a,b) = P_A(a)P_b(b)$

then the correlation between a and b is zero:

$$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle = 0$$

Average:

$$\langle\sigma_A\rangle = \sum_n a_n P(a_n)$$

$$\langle\sigma_B\rangle = \sum_n b_n P(b_n)$$

$$\langle\sigma_A\sigma_B\rangle = \sum_n \sum_n a_n b_n P(a_n b_n)$$

$$* * * * * * * * * *$$

Assuming we have n objects in A and b:

$$\langle\sigma_A\rangle\langle\sigma_B\rangle = \sum_n a_n P(a_n) \sum_n b_n P(b_n) =$$

$$a_1 P(a_1) \sum_n b_n P(b_n) + a_2 P(a_2) \sum_n b_n P(b_n) + \cdots + a_n P(a_n) \sum_n b_n P(b_n) =$$

$$\boxed{\begin{array}{c} a_1 P(a_1) b_1 P(b_1) + a_1 P(a_1) b_2 P(b_2) + \cdots + a_1 P(a_1) b_n P(b_n) + \\ a_2 P(a_2) b_1 P(b_1) + a_2 P(a_2) b_2 P(b_2) + \cdots + a_2 P(a_2) b_n P(b_n) + \\ \vdots \\ a_n P(a_n) b_1 P(b_1) + a_n P(a_n) b_2 P(b_2) + \cdots + a_n P(a_n) b_n P(b_n) = \end{array}}$$

$$\boxed{\begin{array}{c} a_1 b_1 P(a_1) P(b_1) + a_1 b_2 P(a_1) P(b_2) + \cdots + a_1 P b_n P(a_1) P(b_n) + \\ a_2 b_1 P(a_2) P(b_1) + a_2 b_2 P(a_2) P(b_2) + \cdots + a_2 b_n P(a_2) P(b_n) + \\ \vdots \\ a_n b_1 P(a_n) P(b_1) + a_n b_{2P(a_n)} P(b_2) + \cdots + a_n b_n P(a_n) P(b_n) = \end{array}}$$

$$\boxed{\begin{array}{c} a_1 b_1 P(a_1 b_1) + a_1 b_2 P(a_1 b_2) + \cdots + a_1 b_n P(a_1 b_n) + \\ a_2 b_1 P(a_2 b_1) + a_2 b_2 P(a_2 b_2) + \cdots + a_2 b_n P(a_2 b_n) + \\ \vdots \\ a_n b_1 P(a_n b_1) + a_n b_2 P(a_n b_2) + \cdots + a_n b_n P(a_n b_n) = \end{array}}$$

$$\sum_n \sum_n a_n\, b_n P(a_n b_n) = \langle\sigma_A\sigma_B\rangle$$

Exercise 6.2

Show that if the two normalization conditions are satisfied:

$$\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$$

$$\beta_u^* \beta_u + \beta_d^* \beta_d = 1$$

then the state-vector is automatically normalized as well:

$$\alpha_u \beta_u |uu\rangle + \alpha_u \beta_d |ud\rangle + \alpha_d \beta_u |du\rangle + \alpha_d \beta_d |dd\rangle$$

In other words, show that for this product state, normalizing the overall state-vector does not put any additional constraints on the $\alpha' s$ and $\beta' s$.

* * * * * * * * * *

Normalization means that

$$\alpha_u^* \beta_u^* \alpha_u \beta_u + \alpha_u^* \beta_d^* \alpha_u \beta_d + \alpha_d^* \beta_u^* \alpha_d \beta_u + \alpha_d^* \beta_d^* \alpha_d \beta_d = 1$$

From (6.4) we know that

$$(\alpha_u^* \alpha_u + \alpha_d^* \alpha_d)(\beta_u^* \beta_u + \beta_d^* \beta_d) = 1$$

We multiply the brackets:

$$(\alpha_u^* \alpha_u + \alpha_d^* \alpha_d)(\beta_u^* \beta_u + \beta_d^* \beta_d) =$$

$$\alpha_u^* \alpha_u \beta_u^* \beta_u + \alpha_u^* \alpha_u \beta_d^* \beta_d + \alpha_d^* \alpha_d \beta_u^* \beta_u + \alpha_d^* \alpha_d \beta_d^* \beta_d =$$

$$\alpha_u^* \beta_u^* \alpha_u \beta_u + \alpha_u^* \beta_d^* \alpha_u \beta_d + \alpha_d^* \beta_u^* \alpha_d \beta_u + \alpha_d^* \beta_d^* \alpha_d \beta_d = 1$$

This is exactly the normalization condition.

Exercise 6.3

Prove that the state $|sing\rangle$ cannot be written as a product state.

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

The shape of a product state:

$$\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

* * * * * * * * * *

If the singlet state can be composed out of a product state, the following equation should be valid:

$$\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

It follows that $\alpha_u\beta_u = 0$ and $\alpha_d\beta_d = 0$, as the vectors $|uu\rangle$ and $|dd\rangle$ do not appear in the singlet state.

$$\alpha_u\beta_u = 0 \to \alpha_u = 0 \; or \; \beta_u = 0 \text{ or both.}$$

$$\alpha_d\beta_d = 0 \to \alpha_d = 0 \; or \; \beta_d = 0 \text{ or both.}$$

From $\alpha_u\beta_d \neq 0$ it follows that $\alpha_u \neq 0$ and $\beta_d \neq 0$, from $\alpha_d\beta_u \neq 0$ it follows that $\alpha_d \neq 0$ and $\beta_u \neq 0$.

We get a contradiction and can conclude that it is not possible to combine the singlet state out of the parameters of a product state.

Exercise 6.4

Use the matrix forms of $\sigma_x, \sigma_y, \sigma_z$ and the column vectors for $|u\rangle$ and $|d\rangle$ to verify:

$$\sigma_z|u\rangle = |u\rangle \quad \text{and} \quad \sigma_z|d\rangle = -|d\rangle$$

$$\sigma_x|u\rangle = |d\rangle \quad \text{and} \quad \sigma_x|d\rangle = |u\rangle$$

$$\sigma_y|u\rangle = i|d\rangle \quad \text{and} \quad \sigma_y|d\rangle = -i|u\rangle$$

Then, use:

$$\tau_z|u\rangle = |u\rangle \quad \text{and} \quad \tau_z|d\rangle = -|d\rangle$$

$$\tau_x|u\rangle = |d\rangle \quad \text{and} \quad \tau_x|d\rangle = |u\rangle$$

$$\tau_y|u\rangle = i|d\rangle \quad \text{and} \quad \tau_y|d\rangle = -i|u\rangle$$

to write the equations for all possible combinations of the tensor product states $\sigma_z|uu\rangle = |uu\rangle$ etc.

* * * * * * * * * *

Written as matrices and column vectors:

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\sigma_z|u\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + (-1) \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle$$

$$\sigma_z|d\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 + (-1) \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|d\rangle$$

$$\sigma_x|u\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |d\rangle$$

$$\sigma_x|d\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 0 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle$$

$$\sigma_y|u\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + (-i) \cdot 0 \\ i \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|d\rangle$$

$$\sigma_y|d\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + (-i) \cdot 1 \\ i \cdot 0 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i|u\rangle$$

The complete list of tensor product states:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

Exercise 6.5

Prove the following theorem:

When any one of Alice's and Bob's spin operators acts on a product state, the result is still a product state.

Show that in a product state, the expectation value of any component of $\bar{\sigma}$ or $\bar{\tau}$ is the same as it would be in the individual single-spin states.

* * * * * * * * * *

This proof only performed for $\sigma_z$.

Equation (6.5) tells us the shape of a product state:

$$|P\rangle := \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

and

$$\langle P| := \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$$

A single state has the shape:

$$|A\rangle := \alpha_u|u\rangle + \alpha_d|d\rangle \text{ and } \langle A| := \langle u|\alpha_u^* + \langle d|\alpha_d^*$$

The operator $\sigma_z$:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We check the result of the expectation-value definition for both the generic product-state $P$ and the generic single state $A$.

The single state:

$$\langle\sigma_z\rangle = \langle A|\sigma_z|A\rangle =$$

$$\langle A|\sigma_z|(\alpha_u|u\rangle + \alpha_d|d\rangle)\rangle =$$

$$\langle A|(\alpha_u|u\rangle - \alpha_d|d\rangle)\rangle =$$

$$\langle(\langle u|\alpha_u^* + \langle d|\alpha_d^*)|(\alpha_u|u\rangle - \alpha_d|d\rangle)\rangle =$$

$$\langle u|\alpha_u^*\alpha_u|u\rangle - \langle u|\alpha_u^*\alpha_d|d\rangle + \langle d|\alpha_d^*\alpha_u|u\rangle - \langle d|\alpha_d^*\alpha_d|d\rangle =$$

$$\alpha_u^*\alpha_u - 0 + 0 - \alpha_d^*\alpha_d$$

$$\alpha_u^*\alpha_u - \alpha_d^*\alpha_d$$

The product state with the behavior that Alice's operator only acts on her half of the product state:

$$\langle\sigma_z\rangle = \langle P|\sigma_z|P\rangle =$$

$$\langle P|\sigma_z|(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle)\rangle =$$

$$\langle P|(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle)\rangle =$$

$$\big\langle\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*\big|\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle\big\rangle =$$

$$\begin{gathered}
\langle uu|\alpha_u^*\beta_u^*\alpha_u\beta_u|uu\rangle + \langle uu|\alpha_u^*\beta_u^*\alpha_u\beta_d|ud\rangle - \langle uu|\alpha_u^*\beta_u^*\alpha_d\beta_u|du\rangle - \langle uu|\alpha_u^*\beta_u^*\alpha_d\beta_d|dd\rangle + \\
\langle ud|\alpha_u^*\beta_d^*\alpha_u\beta_u|uu\rangle + \langle ud|\alpha_u^*\beta_d^*\alpha_u\beta_d|ud\rangle - \langle ud|\alpha_u^*\beta_d^*\alpha_d\beta_u|du\rangle - \langle ud|\alpha_u^*\beta_d^*\alpha_d\beta_d|dd\rangle + \\
\langle du|\alpha_d^*\beta_u^*\alpha_u\beta_u|uu\rangle + \langle du|\alpha_d^*\beta_u^*\alpha_u\beta_d|ud\rangle - \langle du|\alpha_d^*\beta_u^*\alpha_d\beta_u|du\rangle - \langle du|\alpha_d^*\beta_u^*\alpha_d\beta_d|dd\rangle + \\
\langle dd|\alpha_d^*\beta_d^*\alpha_u\beta_u|uu\rangle + \langle dd|\alpha_d^*\beta_d^*\alpha_u\beta_d|ud\rangle - \langle dd|\alpha_d^*\beta_d^*\alpha_d\beta_u|du\rangle - \langle dd|\alpha_d^*\beta_d^*\alpha_d\beta_d|dd\rangle =
\end{gathered}$$

$$\begin{gathered}
\langle uu|uu\rangle\alpha_u^*\beta_u^*\alpha_u\beta_u + \langle uu|ud\rangle\alpha_u^*\beta_u^*\alpha_u\beta_d - \langle uu|du\rangle\alpha_u^*\beta_u^*\alpha_d\beta_u - \langle uu|dd\rangle\alpha_u^*\beta_u^*\alpha_d\beta_d + \\
\langle ud|uu\rangle\alpha_u^*\beta_d^*\alpha_u\beta_u + \langle ud|ud\rangle\alpha_u^*\beta_d^*\alpha_u\beta_d - \langle ud|du\rangle\alpha_u^*\beta_d^*\alpha_d\beta_u - \langle ud|dd\rangle\alpha_u^*\beta_d^*\alpha_d\beta_d + \\
\langle du|uu\rangle\alpha_d^*\beta_u^*\alpha_u\beta_u + \langle du|ud\rangle\alpha_d^*\beta_u^*\alpha_u\beta_d - \langle du|du\rangle\alpha_d^*\beta_u^*\alpha_d\beta_u - \langle du|dd\rangle\alpha_d^*\beta_u^*\alpha_d\beta_d + \\
\langle dd|uu\rangle\alpha_d^*\beta_d^*\alpha_u\beta_u + \langle dd|ud\rangle\alpha_d^*\beta_d^*\alpha_u\beta_d - \langle dd|du\rangle\alpha_d^*\beta_d^*\alpha_d\beta_u - \langle dd|dd\rangle\alpha_d^*\beta_d^*\alpha_d\beta_d =
\end{gathered}$$

With orthogonality in $\langle uu|ud\rangle$ etc. we get

$$\begin{gathered}
\alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d - \alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d = \\
\alpha_u^*\alpha_u\beta_u^*\beta_u + \alpha_u^*\alpha_u\beta_d^*\beta_d - \alpha_d^*\alpha_d\beta_u^*\beta_u - \alpha_d^*\alpha_d\beta_d^*\beta_d = \\
\alpha_u^*\alpha_u(\beta_u^*\beta_u + \beta_d^*\beta_d) - \alpha_d^*\alpha_d(\beta_u^*\beta_u + \beta_d^*\beta_d) = \\
(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) = \\
(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)
\end{gathered}$$

This is the same result as in Alice's system.

Reflections:

$|uu\rangle$ is not simply the stacked vector $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, but instead $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$. It's built out of the tensor product:

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

The same holds for the other combinations.

The operator $\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ can't act on a state-vector with four components. Instead we need the tensor product of both Alice's and Bob's operator $\tau_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$: $\sigma_z \otimes \tau_z$.

We concentrate on Alice and let Bob alone, his operator will be the identity: $\tau_z = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

$$\sigma_{zi} \coloneqq \sigma_z \otimes I = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} & 0\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\0\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} & -1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We have the four state-vectors:

$$|uu\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|ud\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|du\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|dd\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now we can check the effect of this new $\sigma_{zi}$ on the combined vectors $|uu\rangle$ etc.

$$\sigma_{zi}|uu\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sigma_{zi}|ud\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\sigma_{zi}|du\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\sigma_{zi}|dd\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The results are as expected.

We will check one special case. As the operators $\sigma_z$ and $\tau_z$ both change the sign of the "d"-component, so $\sigma_z\tau_z \coloneqq \sigma_z \otimes \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ should leave $|dd\rangle$ intact.

$$\sigma_z\tau_z = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We check:

$$\sigma_z \tau_z |dd\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and get the correct result:

$$\sigma_z \tau_z |dd\rangle = |dd\rangle$$

Exercise 6.6

Assume Charlie has prepared the two spins in the singlet state. This time, Bob measures $\tau_y$ and Alice measures $\sigma_x$. What is the expectation value of $\sigma_x \tau_y$?

What does this say about the correlation between the two measurements?

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

$$\langle sing| = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)$$

* * * * * * * * * *

$$\langle \sigma_x \tau_y \rangle = \left\langle sing \middle| \sigma_x \tau_y \middle| sing \right\rangle =$$

$$\left\langle sing \middle| \sigma_x \tau_y \middle| \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \right\rangle =$$

$$\left\langle sing \middle| \sigma_x \middle| \frac{1}{\sqrt{2}}(-i|uu\rangle - i|dd\rangle) \right\rangle =$$

$$\left\langle sing \middle| \sigma_x \middle| \frac{-i}{\sqrt{2}}(|uu\rangle + |dd\rangle) \right\rangle =$$

$$\left\langle sing \middle| \frac{-i}{\sqrt{2}}(|du\rangle + |ud\rangle) \right\rangle =$$

$|sing\rangle$ is not an eigenvector of $\sigma_x \tau_y$.

$$\left\langle \frac{1}{\sqrt{2}}(\langle ud| - \langle du|) \middle| \frac{-i}{\sqrt{2}}(|du\rangle + |ud\rangle) \right\rangle =$$

$$\frac{-i}{2}(\langle ud|du\rangle + \langle ud|ud\rangle - \langle du|du\rangle - \langle du|ud\rangle) =$$

$$\frac{-i}{2}(0 + 1 - 1 - 0)) = 0$$

The expectation value $\langle \sigma_x \tau_y \rangle$ is 0.

What does this say about the correlation between the two measurements? The two measurements are not correlated, the results of the measurements are independent.

Exercise 6.7

Next *(after Charlie has had prepared the two spins in the singlet state*), Charlie prepares the spins in a different state, called $|T_1\rangle$, where

$$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$$

T stands for triplet. The triplet states are completely different from the states in the coin and die examples. What are the expectation values of the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$?

Notice what a difference a sign can make.

* * * * * * * * * *

$$\langle T_1| = \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)$$

The expectation value of $\langle\sigma_z\tau_z\rangle$:

$$\langle\sigma_z\tau_z\rangle = \langle T_1|\sigma_z\tau_z|T_1\rangle =$$

$$\left\langle T_1\middle|\sigma_z\tau_z\middle|\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)\right\rangle =$$

$$\left\langle T_1\middle|\sigma_z\middle|\frac{1}{\sqrt{2}}(-|ud\rangle + |du\rangle)\right\rangle =$$

$$\left\langle T_1\middle|\frac{1}{\sqrt{2}}(-|ud\rangle - |du\rangle)\right\rangle =$$

$|T_1\rangle$ is eigenvector of $\sigma_z\tau_z$ with eigenvalue of -1.

$$\left\langle\frac{1}{\sqrt{2}}(\langle ud| + \langle du|)\middle|\frac{1}{\sqrt{2}}(-|ud\rangle - |du\rangle)\right\rangle =$$

$$\frac{1}{2}(-\langle ud|ud\rangle - \langle ud|du\rangle - \langle du|ud\rangle - \langle du|du\rangle) =$$

$$\frac{1}{2}(-1 - 0 - 0 - 1)) = -1$$

The expectation value $\langle\sigma_z\tau_z\rangle$ is $-1$, correlation.

The expectation value of $\langle\sigma_x\tau_x\rangle$:

$$\langle\sigma_x\tau_x\rangle = \langle T_1|\sigma_x\tau_x|T_1\rangle =$$

$$\left\langle T_1\middle|\sigma_x\tau_x\middle|\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)\right\rangle =$$

$$\left\langle T_1\middle|\sigma_x\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$$\left\langle T_1\middle|\frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle)\right\rangle =$$

$|T_1\rangle$ is eigenvector of $\sigma_x\tau_x$ with eigenvalue 1.

$$\left\langle \frac{1}{\sqrt{2}}(\langle ud| + \langle du|) \middle| \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \right\rangle =$$

$$\frac{1}{2}(\langle ud|du\rangle + \langle ud|ud\rangle + \langle du|du\rangle + \langle du|ud\rangle) =$$

$$\frac{1}{2}(0 + 1 + 1 + 0) = 1$$

The expectation value $\langle \sigma_x \tau_x \rangle$ is 1, correlation.

The expectation value of $\langle \sigma_y \tau_y \rangle$:

$$\langle \sigma_y \tau_y \rangle = \langle T_1 | \sigma_y \tau_y | T_1 \rangle =$$

$$\left\langle T_1 \middle| \sigma_y \tau_y \middle| \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \right\rangle =$$

$$\left\langle T_1 \middle| \sigma_y \middle| \frac{i}{\sqrt{2}}(-|uu\rangle + |dd\rangle) \right\rangle =$$

$$\left\langle T_1 \middle| \frac{i}{\sqrt{2}}(-i|du\rangle - i|ud\rangle) \right\rangle =$$

$$\left\langle T_1 \middle| \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \right\rangle =$$

$|T_1\rangle$ is eigenvector of $\sigma_y \tau_y$ with eigenvalue 1.

$$\left\langle \frac{1}{\sqrt{2}}(\langle ud| + \langle du|) \middle| \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \right\rangle =$$

$$\frac{1}{2}(\langle ud|du\rangle + \langle ud|ud\rangle + \langle du|du\rangle + \langle du|ud\rangle) =$$

$$\frac{1}{2}(0 + 1 + 1 + 0) = 1$$

The expectation value $\langle \sigma_y \tau_y \rangle$ is 1, correlation.

What a difference a sign can make: if we change the singlet state to the triplet state, the behavior changes. With the singlet state, the correlation was always $-1$, now the expectation values changes: $\langle \sigma_z \tau_z \rangle$ would become $-1$, $\langle \sigma_x \tau_x \rangle = 1$ and $\langle \sigma_y \tau_y \rangle = 1$ .

Exercise 6.8

Calculate the expectation values of the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$ for the other two entangled triplet states:

$$|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$$

$$|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$$

*(Addendum: at the end of this exercise we check the expectation values for the singlet state too.)*

* * * * * * * * * *

First $|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$

The expectation value of $\langle\sigma_z\tau_z\rangle$:

$$\langle\sigma_z\tau_z\rangle = \langle T_2|\sigma_z\tau_z|T_2\rangle =$$

$$\left\langle T_2\middle|\sigma_z\tau_z\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$$\left\langle T_2\middle|\sigma_z\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\left\langle T_2\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$|T_2\rangle$ is eigenvector of $\sigma_z\tau_z$ with eigenvalue 1.

$$\left\langle\frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$$\frac{1}{2}(\langle uu|uu\rangle + \langle uu|dd\rangle + \langle dd|uu\rangle + \langle dd|dd\rangle) =$$

$$\frac{1}{2}(1 + 0 + 0 + 1)) = 1$$

The expectation value $\langle\sigma_z\tau_z\rangle$ is 1, correlation.

The expectation value of $\langle\sigma_x\tau_x\rangle$:

$$\langle\sigma_x\tau_x\rangle = \langle T_2|\sigma_x\tau_x|T_2\rangle =$$

$$\left\langle T_2\middle|\sigma_x\tau_x\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$$\left\langle T_2\middle|\sigma_x\middle|\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)\right\rangle =$$

$$\left\langle T_2\middle|\frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle)\right\rangle =$$

$|T_2\rangle$ is eigenvector of $\sigma_x\tau_x$.

$$\left\langle \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|) \middle| \frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) \right\rangle =$$

$$\frac{1}{2}(\langle uu|dd\rangle + \langle uu|uu\rangle + \langle dd|dd\rangle + \langle dd|uu\rangle) =$$

$$\frac{1}{2}(0 + 1 + 1 + 0) = 1$$

The expectation value $\langle \sigma_x \tau_x \rangle$ is 1, correlation.

The expectation value of $\langle \sigma_y \tau_y \rangle$:

$$\langle \sigma_y \tau_y \rangle = \langle T_2 | \sigma_y \tau_y | T_2 \rangle =$$

$$\left\langle T_2 \middle| \sigma_y \tau_y \middle| \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \right\rangle =$$

$$\left\langle T_2 \middle| \sigma_y \middle| \frac{i}{\sqrt{2}}(|ud\rangle - |du\rangle) \right\rangle =$$

$$\left\langle T_2 \middle| \frac{i}{\sqrt{2}}(i|dd\rangle + i|uu\rangle) \right\rangle =$$

$$\left\langle T_2 \middle| \frac{1}{\sqrt{2}}(-|dd\rangle - |uu\rangle) \right\rangle =$$

$|T_2\rangle$ is eigenvector of $\sigma_y \tau_y$ with eigenvalue $-1$.

$$\left\langle \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|) \middle| \frac{1}{\sqrt{2}}(-|dd\rangle - |uu\rangle) \right\rangle =$$

$$\frac{1}{2}(-\langle uu|dd\rangle - \langle uu|uu\rangle - \langle dd|dd\rangle - \langle dd|uu\rangle) =$$

$$\frac{1}{2}(-0 - 1 - 1 - 0) = -1$$

The expectation value $\langle \sigma_y \tau_y \rangle$ is $-1$, correlation.

Second $|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$

The expectation value of $\langle\sigma_z\tau_z\rangle$:

$$\langle\sigma_z\tau_z\rangle = \langle T_3|\sigma_z\tau_z|T_3\rangle =$$

$$\left\langle T_3\middle|\sigma_z\tau_z\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\sigma_z\middle|\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$|T_3\rangle$ is eigenvector of $\sigma_z\tau_z$ with eigenvalue 1.

$$\left\langle\frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\frac{1}{2}(\langle uu|uu\rangle - \langle uu|dd\rangle - \langle dd|uu\rangle + \langle dd|dd\rangle) =$$

$$\frac{1}{2}(1 - 0 - 0 + 1)) = 1$$

The expectation value $\langle\sigma_z\tau_z\rangle$ is 1, correlation.

The expectation value of $\langle\sigma_x\tau_x\rangle$:

$$\langle\sigma_x\tau_x\rangle = \langle T_3|\sigma_x\tau_x|T_3\rangle =$$

$$\left\langle T_3\middle|\sigma_x\tau_x\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\sigma_x\middle|\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle)\right\rangle =$$

$|T_3\rangle$ is eigenvector of $\sigma_x\tau_x$ with eigenvalue $-1$.

$$\left\langle\frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\middle|\frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle)\right\rangle =$$

$$\frac{1}{2}(\langle uu|dd\rangle - \langle uu|uu\rangle - \langle dd|dd\rangle + \langle dd|uu\rangle) =$$

$$\frac{1}{2}(0 - 1 - 1 + 0) = -1$$

The expectation value $\langle\sigma_x\tau_x\rangle$ is $-1$, correlation.

The expectation value of $\langle\sigma_y\tau_y\rangle$:

$$\langle\sigma_y\tau_y\rangle = \langle T_3|\sigma_y\tau_y|T_3\rangle =$$

$$\left\langle T_3\middle|\sigma_y\tau_y\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\sigma_y\middle|\frac{i}{\sqrt{2}}(|ud\rangle + |du\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\frac{i}{\sqrt{2}}(i|dd\rangle - i|uu\rangle)\right\rangle =$$

$$\left\langle T_3\middle|\frac{1}{\sqrt{2}}(-|dd\rangle + |uu\rangle)\right\rangle =$$

$|T_3\rangle$ is eigenvector of $\sigma_y\tau_y$ with eigenvalue 1.

$$\left\langle\frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\middle|\frac{1}{\sqrt{2}}(-|dd\rangle + |uu\rangle)\right\rangle =$$

$$\frac{1}{2}(-\langle uu|dd\rangle + \langle uu|uu\rangle + \langle dd|dd\rangle - \langle dd|uu\rangle) =$$

$$\frac{1}{2}(-0 + 1 + 1 - 0) = 1$$

The expectation value $\langle\sigma_y\tau_y\rangle$ is 1, correlation.

---

Addendum: for the singlet state we check these expectation values too.

Assume Charlie has prepared the two spins in the singlet state. What are the expectation values of operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$?

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

$$\langle sing| = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)$$

---

$$\langle\sigma_x\tau_x\rangle = \langle sing|\sigma_x\tau_x|sing\rangle =$$

$$\left\langle sing\middle|\sigma_x\tau_x\middle|\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)\right\rangle =$$

$$\left\langle sing\middle|\sigma_x\middle|\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)\right\rangle =$$

$$\left\langle sing\middle|\frac{1}{\sqrt{2}}(|du\rangle - |ud\rangle)\right\rangle =$$

$|sing\rangle$ is eigenvector of $\sigma_x\tau_x$ with eigenvalue -1, the expectation value of $\langle\sigma_x\tau_x\rangle$ is -1.

$$\langle \sigma_y \tau_y \rangle = \left\langle sing \middle| \sigma_y \tau_y \middle| sing \right\rangle =$$

$$\left\langle sing \middle| \sigma_y \tau_y \middle| \frac{1}{\sqrt{2}} (|ud\rangle - |du\rangle) \right\rangle =$$

$$\left\langle sing \middle| \sigma_y \middle| \frac{i}{\sqrt{2}} (-|uu\rangle - |dd\rangle) \right\rangle =$$

$$\left\langle sing \middle| \frac{i}{\sqrt{2}} (-i|du\rangle + i|ud\rangle) \right\rangle =$$

$$\left\langle sing \middle| \frac{1}{\sqrt{2}} (|du\rangle - |ud\rangle) \right\rangle =$$

$|sing\rangle$ is eigenvector of $\sigma_y \tau_y$ with eigenvalue -1, the expectation value of $\langle \sigma_x \tau_x \rangle$ is -1.

$$\langle \sigma_z \tau_z \rangle = \langle sing | \sigma_z \tau_z | sing \rangle =$$

$$\left\langle sing \middle| \sigma_z \tau_z \middle| \frac{1}{\sqrt{2}} (|ud\rangle - |du\rangle) \right\rangle =$$

$$\left\langle sing \middle| \sigma_z \middle| \frac{1}{\sqrt{2}} (-|ud\rangle - |du\rangle) \right\rangle =$$

$$\left\langle sing \middle| \frac{1}{\sqrt{2}} (-|ud\rangle + |du\rangle) \right\rangle =$$

$|sing\rangle$ is eigenvector of $\sigma_z \tau_z$ with eigenvalue -1, the expectation value of $\langle \sigma_x \tau_x \rangle$ is -1.

We get the following matrix:

$$\begin{array}{cccc} \langle\ \ \rangle & zz & xx & yy \\ T_1 & -1 & 1 & 1 \\ T_2 & 1 & 1 & -1 \\ T_3 & 1 & -1 & 1 \\ sing & -1 & -1 & -1 \end{array}$$

Exercise 6.9

Prove that the four vectors $|sing\rangle, |T_1\rangle, |T_2\rangle, |T_3\rangle$ are eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$:

$$\vec{\sigma} \cdot \vec{\tau} = \sigma_x \tau_x + \sigma_y \tau_y + \sigma_z \tau_z$$

What are their eigenvalues?

$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$

$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$ $\quad |T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$ $\quad |T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$

* * * * * * * * * *

The complete list of tensor product states:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

Applied to $|sing\rangle$:

$$\sigma_x\tau_x|sing\rangle = \sigma_x\tau_x\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = \sigma_x\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \frac{1}{\sqrt{2}}(|du\rangle - |ud\rangle) = -|sing\rangle$$

$$\sigma_y\tau_y|sing\rangle = \sigma_y\tau_y\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = \sigma_y\frac{1}{\sqrt{2}}(-i|uu\rangle - i|dd\rangle) = \frac{1}{\sqrt{2}}(|du\rangle - |ud\rangle) = -|sing\rangle$$

$$\sigma_z\tau_z|sing\rangle = \sigma_z\tau_z\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = \sigma_z\frac{1}{\sqrt{2}}(-|ud\rangle - |du\rangle) = \frac{1}{\sqrt{2}}(-|ud\rangle + |du\rangle) = -|sing\rangle$$

Result: $\vec{\sigma} \cdot \vec{\tau}|sing\rangle = -3|sing\rangle$

$|sing\rangle$ is eigenvector of $\vec{\sigma} \cdot \vec{\tau}$ with eigenvalue -3.

$\vec{\sigma} \cdot \vec{\tau}$ applied to $|T_1\rangle$ gives:

$$\sigma_x\tau_x|T_1\rangle = \sigma_x\tau_x\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = \sigma_x\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) = |T_1\rangle$$

$$\sigma_y\tau_y|T_1\rangle = \sigma_y\tau_y\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = \sigma_y\frac{1}{\sqrt{2}}(-i|uu\rangle + i|dd\rangle) = \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) = |T_1\rangle$$

$$\sigma_z\tau_z|T_1\rangle = \sigma_z\tau_z\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = \sigma_z\frac{1}{\sqrt{2}}(-|ud\rangle + |du\rangle) = \frac{1}{\sqrt{2}}(-|ud\rangle - |du\rangle) = -|T_1\rangle$$

Result: $\vec{\sigma} \cdot \vec{\tau}|T_1\rangle = |T_1\rangle$

$|T_1\rangle$ is eigenvector of $\vec{\sigma} \cdot \vec{\tau}$ with eigenvalue 1.

$\vec{\sigma} \cdot \vec{\tau}$ applied to $|T_2\rangle$ gives:

$$\sigma_x\tau_x|T_2\rangle = \sigma_x\tau_x\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \sigma_x\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = \frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) = |T_2\rangle$$

$$\sigma_y\tau_y|T_2\rangle = \sigma_y\tau_y\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \sigma_y\frac{1}{\sqrt{2}}(i|ud\rangle - i|du\rangle) = \frac{1}{\sqrt{2}}(-|dd\rangle - |uu\rangle) = -|T_2\rangle$$

$$\sigma_z\tau_z|T_2\rangle = \sigma_z\tau_z\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \sigma_z\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = |T_2\rangle$$

Result: $\vec{\sigma} \cdot \vec{\tau}|T_2\rangle = |T_2\rangle$

$|T_2\rangle$ is eigenvector of $\vec{\sigma} \cdot \vec{\tau}$ with eigenvalue 1.

$\vec{\sigma} \cdot \vec{\tau}$ applied to $|T_3\rangle$ gives:

$$\sigma_x\tau_x|T_3\rangle = \sigma_x\tau_x\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \sigma_x\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = \frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) = -|T_3\rangle$$

$$\sigma_y\tau_y|T_3\rangle = \sigma_y\tau_y\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \sigma_y\frac{1}{\sqrt{2}}(i|ud\rangle + i|du\rangle) = \frac{1}{\sqrt{2}}(-|dd\rangle + |uu\rangle) = |T_3\rangle$$

$$\sigma_z\tau_z|T_3\rangle = \sigma_z\tau_z\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \sigma_z\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = |T_3\rangle$$

Result: $\vec{\sigma} \cdot \vec{\tau}|T_3\rangle = |T_3\rangle$

$|T_3\rangle$ is eigenvector of $\vec{\sigma} \cdot \vec{\tau}$ with eigenvalue 1.

$|sing\rangle$ is eigenvector with eigenvalue -3,

$|T_1\rangle$, $|T_2\rangle$ und $|T_3\rangle$ are eigenvectors with eigenvalue 1, they are degenerated.

Exercise 6.10

A system of two spins has the Hamiltonian:

$$\mathcal{H} = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}$$

Question 1)

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian?

Question 2)

Suppose the system starts in the state $|uu\rangle$.

What is the state at any later time?

Answer the same question for initial states of $|ud\rangle$, $|du\rangle$, $|dd$.

* * * * * * * * * *

Question 1)

The Hamilton-operator for the 2-spin-system:

$$\mathcal{H} = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}$$

From exercise 6.9 we know the eigenvectors and eigenvalues:

$|sing\rangle$ with eigenvalue -3, $|T_1\rangle$, $|T_2\rangle$ und $|T_3\rangle$ with eigenvalue 1.

This gives the energies to $-\frac{3\omega\hbar}{2}$ for $|sing\rangle$ and each $\frac{\omega\hbar}{2}$ for $|T_1\rangle$, $|T_2\rangle$ and $|T_3\rangle$.

Question 2)

$$\mathcal{H}|uu\rangle = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}|uu\rangle$$

We use $\vec{\sigma}\cdot\vec{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z$ and apply this to $|uu\rangle$ (we omit the factor $\frac{\omega\hbar}{2}$ and add it in the end):

$\sigma_x\tau_x|uu\rangle = \sigma_x|ud\rangle = |dd\rangle$, gives $\frac{\omega\hbar}{2}\sigma_x\tau_x|uu\rangle = \frac{\omega\hbar}{2}|dd\rangle$

$\sigma_y\tau_y|uu\rangle = \sigma_y i|ud\rangle = -|dd\rangle$, gives $\frac{\omega\hbar}{2}\sigma_y\tau_y|uu\rangle = -\frac{\omega\hbar}{2}|dd\rangle$

$\sigma_z\tau_z|uu\rangle = \sigma_z|uu\rangle = |uu\rangle$, gives $\frac{\omega\hbar}{2}\sigma_z\tau_z|uu\rangle = \frac{\omega\hbar}{2}|uu\rangle$

Building the sum, we get:

$$\mathcal{H}|uu\rangle = \frac{\omega\hbar}{2}|uu\rangle$$

The other combinations analog:

$$\mathcal{H}|ud\rangle = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}|ud\rangle$$

$\sigma_x\tau_x|ud\rangle = \sigma_x|uu\rangle = |du\rangle$, gives $\frac{\omega\hbar}{2}\sigma_x\tau_x|ud\rangle = \frac{\omega\hbar}{2}|du\rangle$

$\sigma_y \tau_y \big|ud\rangle = \sigma_y - i\big|uu\rangle = |du\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_y \tau_y |ud\rangle = \frac{\omega\hbar}{2}|du\rangle$

$\sigma_z \tau_z |ud\rangle = \sigma_z - |ud\rangle = -|ud\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_z \tau_z |ud\rangle = -\frac{\omega\hbar}{2}|ud\rangle$

Building the sum, we get:

$$\mathcal{H}|ud\rangle = \omega\hbar|du\rangle - \frac{\omega\hbar}{2}|ud\rangle$$

$$\mathcal{H}|du\rangle = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}|du\rangle$$

$\sigma_x \tau_x |du\rangle = \sigma_x |dd\rangle = |ud\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_x \tau_x |du\rangle = \frac{\omega\hbar}{2}|ud\rangle$

$\sigma_y \tau_y \big|du\rangle = \sigma_y i\big|dd\rangle = |ud\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_y \tau_y |du\rangle = \frac{\omega\hbar}{2}|ud\rangle$

$\sigma_z \tau_z |du\rangle = \sigma_z |du\rangle = -|du\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_z \tau_z |du\rangle = -\frac{\omega\hbar}{2}|du\rangle$

Building the sum, we get:

$$\mathcal{H}|du\rangle = \omega\hbar|ud\rangle - \frac{\omega\hbar}{2}|du\rangle$$

$$\mathcal{H}|dd\rangle = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}|dd\rangle$$

$\sigma_x \tau_x |dd\rangle = \sigma_x |du\rangle = |uu\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_x \tau_x |dd\rangle = \frac{\omega\hbar}{2}|uu\rangle$

$\sigma_y \tau_y \big|dd\rangle = \sigma_y - i\big|du\rangle = |uu\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_y \tau_y |dd\rangle = \frac{\omega\hbar}{2}|uu\rangle$

$\sigma_z \tau_z |dd\rangle = \sigma_z - |dd\rangle = |dd\rangle,$ gives $\frac{\omega\hbar}{2}\sigma_z \tau_z |dd\rangle = \frac{\omega\hbar}{2}|dd\rangle$

Building the sum, we get:

$$\mathcal{H}|dd\rangle = \omega\hbar|ud\rangle + \frac{\omega\hbar}{2}|dd\rangle$$

Question 2 targets to chapter 4 with the aim of developing the time depend Schrödinger-equation. We use “4.13 Recipe for a Schrödinger Ket” to perform this.

Recipe:

1. The Hamilton-operator:

$$\mathcal{H} = \frac{\omega\hbar}{2}\vec{\sigma}\cdot\vec{\tau}$$

2. We prepare an initial state $|\psi(0)\rangle$:

$$|\psi(0)\rangle = |uu\rangle$$

3. We calculate eigenvectors and eigenvalues of H:

$E_1 \coloneqq |sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$ with eigenvalue -3

$E_2 \coloneqq |T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$ with eigenvalue 1

$E_3 \coloneqq |T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$ with eigenvalue 1

$E_4 \coloneqq |T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$ with eigenvalue 1

4. We calculate the initial coefficients $\alpha_j(0) = \langle E_j|\psi(0)\rangle$

$\alpha_1(0) = 0$:

$$\alpha_1(0) = \langle E_1|\psi(0)\rangle = \langle sing|uu\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)|uu\rangle =$$

$$= \frac{1}{\sqrt{2}}(\langle ud - \langle du)|uu\rangle = \frac{1}{\sqrt{2}}(\langle ud|uu\rangle - \langle du|uu\rangle) = \frac{1}{\sqrt{2}}(0-0) = 0$$

$\alpha_2(0) = 0$:

$$\alpha_2(0) = \langle E_2|\psi(0)\rangle = \langle T_1|uu\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)|uu\rangle =$$

$$= \frac{1}{\sqrt{2}}(\langle ud + \langle du)|uu\rangle = \frac{1}{\sqrt{2}}(\langle ud|uu\rangle + \langle du|uu\rangle) = \frac{1}{\sqrt{2}}(0+0) = 0$$

$\alpha_3(0) = \frac{1}{\sqrt{2}}$:

$$\alpha_3(0) = \langle E_3|\psi(0)\rangle = \langle T_2|uu\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)|uu\rangle =$$

$$= \frac{1}{\sqrt{2}}(\langle uu + \langle dd)|uu\rangle = \frac{1}{\sqrt{2}}(\langle uu|uu\rangle + \langle dd|uu\rangle) = \frac{1}{\sqrt{2}}(1+0) = \frac{1}{\sqrt{2}}$$

$\alpha_4(0) = \frac{1}{\sqrt{2}}$:

$$\alpha_4(0) = \langle E_4|\psi(0)\rangle = \langle T_3|uu\rangle =$$

$$= \frac{1}{\sqrt{2}}(\langle uu| - \langle dd)|uu\rangle = \frac{1}{\sqrt{2}}(\langle uu|uu\rangle - \langle dd|uu\rangle) = \frac{1}{\sqrt{2}}(1-0) = \frac{1}{\sqrt{2}}$$

5. We rewrite $|\psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $\alpha_j(0)$:

$$|\psi(0)\rangle = \sum_{j=1}^{4} \alpha_j(0)|E_j\rangle = 0|sing\rangle + 0|T_1\rangle + \frac{1}{\sqrt{2}}|T_2\rangle + \frac{1}{\sqrt{2}}|T_3\rangle$$

6. We replace each $\alpha_j(0)$ with $\alpha_j(t)$, as a result $|\psi(0)\rangle$ becomes $|\psi(t)\rangle$.

$$|\psi(t)\rangle = \sum_{j=1}^{4} \alpha_j(t)|E_j\rangle$$

7. We use equation 4.30: $\alpha_j(t) = \alpha_j(0)e^{-\frac{i}{\hbar}E_j(t)}$ and replace each $\alpha_j(t)$ with equation 4.30, using the basic vectors (eigenvectors):

$$|\psi(t)\rangle = \sum_{j=1}^{4} \alpha_j(0)e^{-\frac{i}{\hbar}E_j(t)}|E_j\rangle$$

We use: $\alpha_1(0) = \alpha_2(0) = 0$, $\alpha_3(0) = \alpha_4(0) = \frac{1}{\sqrt{2}}$, $E_1 = -3, E_2, E_3, E_4 = 1$

$$|\psi(t)\rangle = 0|sing\rangle + 0|T_1\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}t}|T_2\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}t}|T_3\rangle =$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}t}|T_2\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}t}|T_3\rangle =$$

$$= \frac{1}{2}e^{-\frac{i}{\hbar}t}(|uu\rangle + |dd\rangle + |uu\rangle - |dd\rangle) =$$

$$= \frac{2}{2}e^{-\frac{i}{\hbar}t}(|uu\rangle)$$

Summed up:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}t}(|uu\rangle)$$

In the same way we calculate the other initial states $|ud\rangle, |du\rangle\ und\ |dd\rangle$.

**Initial state $|ud\rangle$:**

$\alpha_1(0) = \frac{1}{\sqrt{2}} = \alpha_2(0)$ and $\alpha_3(0) = 0 = \alpha_4(0)$, because of the products $(\langle uu| - \langle dd|)|ud\rangle$ etc. vanishing resp. resulting in 1.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{3i}{\hbar}t} \Big|sing\rangle + \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t} \Big|T_1\rangle + 0|T_2\rangle + 0|T_3\rangle =$$

$$= \frac{1}{2} e^{-\frac{3i}{\hbar}t} \Big|sing\rangle + \frac{1}{2} e^{-\frac{i}{\hbar}t} \Big|T_1\rangle =$$

$$= \frac{1}{2}\left( e^{-\frac{3i}{\hbar}t}(|ud\rangle - |du\rangle) + e^{-\frac{i}{\hbar}t}(|ud\rangle + |du\rangle)\right) =$$

$$= \frac{1}{2} e^{-\frac{i}{\hbar}t}\left( e^{-\frac{2i}{\hbar}t}(|ud\rangle - |du\rangle) + (|ud\rangle + |du\rangle)\right)$$

**Initial state $|du\rangle$:**

$\alpha_1(0) = -\frac{1}{\sqrt{2}}$, $\alpha_2(0) = \frac{1}{\sqrt{2}}$ and $\alpha_3(0) = 0 = \alpha_4(0)$, because of the products $(\langle uu| - \langle dd|)|du\rangle$ etc. vanishing resp. resulting in 1.

$$|\psi(t)\rangle = -\frac{1}{\sqrt{2}} e^{-\frac{3i}{\hbar}t} \Big|sing\rangle + \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t} \Big|T_1\rangle + 0|T_2\rangle + 0|T_3\rangle =$$

$$= -\frac{1}{2} e^{-\frac{3i}{\hbar}t} \Big|sing\rangle + \frac{1}{2} e^{-\frac{i}{\hbar}t} \Big|T_1\rangle =$$

$$= \frac{1}{2}\left( e^{-\frac{3i}{\hbar}t}(-|ud\rangle + |du\rangle) + e^{-\frac{i}{\hbar}t}(|ud\rangle + |du\rangle)\right) =$$

$$= \frac{1}{2} e^{-\frac{i}{\hbar}t}\left( e^{-\frac{2i}{\hbar}t}(-|ud\rangle + |du\rangle) + (|ud\rangle + |du\rangle)\right)$$

**Initial state $|dd\rangle$:**

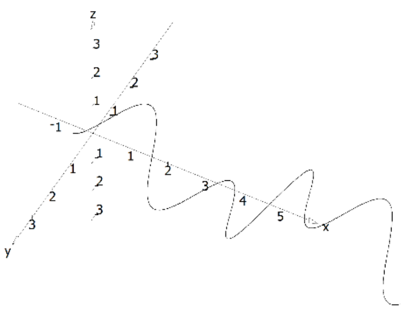
$\alpha_3(0) = \frac{1}{\sqrt{2}}$, $\alpha_4(0) = -\frac{1}{\sqrt{2}}$ und $\alpha_1(0) = 0 = \alpha_2(0)$, because of the products $(\langle uu| - \langle dd|)|dd\rangle$ etc. vanishing resp. resulting in 1.

$$|\psi(t)\rangle = 0|sing\rangle + 0|T_1\rangle + \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t}|T_2\rangle - \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t}|T_3\rangle =$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t}|T_2\rangle - \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}t}|T_3\rangle =$$

$$= \frac{1}{2} e^{-\frac{i}{\hbar}t}(|uu\rangle + |dd\rangle - |uu\rangle + |dd\rangle) =$$

$$= e^{-\frac{i}{\hbar}t}(|dd\rangle)$$



Note: for the states |uu⟩ and |dd⟩ $e^{-\frac{i}{\hbar}t}$ will give them the shape of a wave: $e^{-\frac{i}{\hbar}t}(|uu\rangle) \to Re\left(e^{-\frac{i}{\hbar}t}\right)(|uu\rangle) \sim \cos(t)|uu\rangle$.

For the states $(|ud\rangle)$ and $(|du\rangle)$ we get another behavior.

$$\frac{1}{2} e^{-\frac{i}{\hbar}t}\left( e^{-\frac{2i}{\hbar}t}(|ud\rangle - |du\rangle) + (|ud\rangle + |du\rangle)\right)$$

starts at time $t = 0$ in the state $|ud\rangle$ but then begins oscillating between $|ud\rangle$ and $|du\rangle$. The picture might illustrate this.

Exercise 7.1

Write the tensor product $I \otimes \tau_x$ as a matrix, and apply that matrix to each of the $|uu\rangle$, $|ud\rangle$, $|du\rangle$ and $|dd\rangle$ column vectors. Show that Alice's half of the state-vector is unchanged in each case. Recall that $I$ is the $2 \times 2$ unit matrix.

* * * * * * * * * *

$$\mathrm{I} \otimes \tau_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

applied to: $|uu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(\mathrm{I} \otimes \tau_x)|uu\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |ud\rangle$$

applied to: $|ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$(\mathrm{I} \otimes \tau_x)|ud\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |uu\rangle$$

applied to: $|du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$(\mathrm{I} \otimes \tau_x)|du\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |dd\rangle$$

applied to: $|dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$(\mathrm{I} \otimes \tau_x)|dd\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |du\rangle$$

Exercise 7.2

Calculate the matrix elements of $\sigma_z \otimes \tau_x$ by forming the inner product analog to:

$$\sigma_z \otimes I = \begin{pmatrix} \langle uu|\sigma_z I|uu\rangle & \langle uu|\sigma_z I|ud\rangle & \langle uu|\sigma_z I|du\rangle & \langle uu|\sigma_z I|dd\rangle \\ \langle ud|\sigma_z I|uu\rangle & \langle ud|\sigma_z I|ud\rangle & \langle ud|\sigma_z I|du\rangle & \langle ud|\sigma_z I|dd\rangle \\ \langle du|\sigma_z I|uu\rangle & \langle du|\sigma_z I|ud\rangle & \langle du|\sigma_z I|du\rangle & \langle du|\sigma_z I|dd\rangle \\ \langle dd|\sigma_z I|uu\rangle & \langle dd|\sigma_z I|ud\rangle & \langle dd|\sigma_z I|du\rangle & \langle dd|\sigma_z I|dd\rangle \end{pmatrix}$$

$\sigma_z$ operates to the left, $I$ to the right.

* * * * * * * * * *

$\sigma_z \otimes \tau_x$ according to the rules of the tensor product:

$$\sigma_z \otimes \tau_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\sigma_z \otimes \tau_x$:

$$(\sigma_z \otimes \tau_x) = \begin{pmatrix} \langle uu|\sigma_z\tau_x|uu\rangle & \langle uu|\sigma_z\tau_x|ud\rangle & \langle uu|\sigma_z\tau_x|du\rangle & \langle uu|\sigma_z\tau_x|dd\rangle \\ \langle ud|\sigma_z\tau_x|uu\rangle & \langle ud|\sigma_z\tau_x|ud\rangle & \langle ud|\sigma_z\tau_x|du\rangle & \langle ud|\sigma_z\tau_x|dd\rangle \\ \langle du|\sigma_z\tau_x|uu\rangle & \langle du|\sigma_z\tau_x|ud\rangle & \langle du|\sigma_z\tau_x|du\rangle & \langle du|\sigma_z\tau_x|dd\rangle \\ \langle dd|\sigma_z\tau_x|uu\rangle & \langle dd|\sigma_z\tau_x|ud\rangle & \langle dd|\sigma_z\tau_x|du\rangle & \langle dd|\sigma_z\tau_x|dd\rangle \end{pmatrix} =$$

$\sigma_z$ applied to the left, $\tau_x$ applied to the right:

$$= \begin{pmatrix} \langle uu|ud\rangle & \langle uu|uu\rangle & \langle uu|dd\rangle & \langle uu|du\rangle \\ \langle ud|ud\rangle & \langle ud|uu\rangle & \langle ud|dd\rangle & \langle ud|du\rangle \\ \langle -du|ud\rangle & \langle -du|uu\rangle & \langle -du|dd\rangle & \langle -du|du\rangle \\ \langle -dd|ud\rangle & \langle -dd|uu\rangle & \langle -dd|dd\rangle & \langle -dd|du\rangle \end{pmatrix}$$

Because of u and d being orthonormal vectors, this condenses to:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

This gives the same matrix as the tensor-product above.

Exercise 7.3

Consider the equation:

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

$A$ and $B$ represent $2 \times 2$ matrices (or operators), and $A$ and $b$ represent $2 \times 1$ column vectors.

a) Rewrite the equation

$$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$$

in component form, replacing the symbols $A, B, a, b$ with the following matrices and column vectors:

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}$$

b) Perform the matrix multiplication $Aa$ and $Bb$ on the right-hand side. Verify that each result is a $4 \times 1$ matrix.

c) Expand all three Kronecker products.

d) Verify the row and column sizes of each Kronecker product:

- $A \otimes B : 4 \times 4$
- $a \otimes b : 4 \times 1$
- $Aa \otimes Bb : 4 \times 1$

e) Perform the matrix multiplication on the left-hand side, resulting in a $4 \times 1$ column vector. Each row should be the sum of four separate terms

f) Finally, verify that the resulting column vectors on the left and right sides are identical.

* * * * * * * * * *

a) Rewrite $(\mathrm{A} \otimes \mathrm{B})(a \otimes \mathrm{b}) = (\mathrm{Aa} \otimes \mathrm{Bb})$ in component form:

$$A \otimes \mathrm{B} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{12}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ A_{21}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{22}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

$$(a \otimes \mathrm{b}) = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \otimes \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}$$

b) Perform the matrix multiplication:

$$\mathrm{Aa} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix}$$

$$\mathrm{Bb} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix}$$

$$(\mathrm{Aa} \otimes \mathrm{Bb}) = \begin{pmatrix} (A_{11}a_{11} + A_{12}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{11}a_{11} + A_{12}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{11}b_{11} + B_{12}b_{21}) \\ (A_{21}a_{11} + A_{22}a_{21})(B_{21}b_{11} + B_{22}b_{21}) \end{pmatrix} =$$

$$= \begin{pmatrix} A_{11}a_{11}B_{11}b_{11} + A_{11}a_{11}B_{12}b_{21} + A_{12}a_{21}B_{11}b_{11} + A_{12}a_{21}B_{12}b_{21} \\ A_{11}a_{11}B_{21}b_{11} + A_{11}a_{11}B_{22}b_{21} + A_{12}a_{21}B_{21}b_{11} + A_{12}a_{21}B_{22}b_{21} \\ A_{21}a_{11}B_{11}b_{11} + A_{21}a_{11}B_{12}b_{21} + A_{22}a_{21}B_{11}b_{11} + A_{22}a_{21}B_{12}b_{21} \\ A_{21}a_{11}B_{21}b_{11} + A_{21}a_{11}B_{22}b_{21} + A_{22}a_{21}B_{21}b_{11} + A_{22}a_{21}B_{22}b_{21} \end{pmatrix}$$

c) Expand all three Kronecker products: $(\mathrm{A} \otimes \mathrm{B}), (a \otimes \mathrm{b})$ *and* $(\mathrm{Aa} \otimes \mathrm{Bb})$

… already done …

d) Verify:

$A \otimes \mathrm{B}$ = 4 x 4, $(a \otimes \mathrm{b})$ = 4 x 1, $(\mathrm{Aa} \otimes \mathrm{Bb})$ = 4 x 1

e) Perform the matrix multiplication $(\mathrm{A} \otimes \mathrm{B})(a \otimes \mathrm{b})$:

$$= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}\begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix} =$$

$$= \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix}$$

f) Finally, verify the identity:

Left side:

$$= \begin{pmatrix} A_{11}a_{11}B_{11}b_{11} + A_{11}a_{11}B_{12}b_{21} + A_{12}a_{21}B_{11}b_{11} + A_{12}a_{21}B_{12}b_{21} \\ A_{11}a_{11}B_{21}b_{11} + A_{11}a_{11}B_{22}b_{21} + A_{12}a_{21}B_{21}b_{11} + A_{12}a_{21}B_{22}b_{21} \\ A_{21}a_{11}B_{11}b_{11} + A_{21}a_{11}B_{12}b_{21} + A_{22}a_{21}B_{11}b_{11} + A_{22}a_{21}B_{12}b_{21} \\ A_{21}a_{11}B_{21}b_{11} + A_{21}a_{11}B_{22}b_{21} + A_{22}a_{21}B_{21}b_{11} + A_{22}a_{21}B_{22}b_{21} \end{pmatrix}$$

Right side:

$$= \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix}$$

Exercise 7.4

Calculate the density matrix for $|\Psi\rangle = \alpha|uu\rangle + \beta|uu\rangle$.

Answer:

$$\psi(u) = \alpha;\ \psi^*(u) = \alpha^*$$

$$\psi(d) = \beta;\ \psi^*(d) = \beta^*$$

$$\rho_{a\prime a} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{pmatrix}$$

Now try plugging some numbers for $\alpha$ and $\beta$.

Make sure they are normalized to 1. For example, $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}}$

$$* * * * * * * * * *$$

The density matrix is defined as

$$\rho_{aa\prime} = \langle a|\Psi\rangle\langle\Psi|a'\rangle$$

with the values of a and a' being u and d (4 possible combinations).

We calculate the density matrix for:

$$|\Psi\rangle = \alpha|u\rangle + \beta|d\rangle$$

We need the bra too:

$$\langle\Psi| = \langle u|\alpha^* + \langle u|\beta^*$$

Then we calculate the elements of the density matrix.

$\rho_{uu}$

$$\rho_{uu} = \langle u|\Psi\rangle\langle\Psi|u\rangle =$$

$$\rho_{uu} = \langle u(\alpha|u\rangle + \beta|d\rangle)\rangle\langle(\langle u|\alpha^* + \langle d|\beta^*)|u\rangle =$$

$$(\alpha\langle u|u\rangle + \beta\langle u|d\rangle)(\langle u|u\rangle\alpha^* + \langle d|u\rangle\beta^*) =$$

$$(\alpha + 0)(\alpha^* + 0) = \alpha\alpha^*$$

$\rho_{ud}$

$$\rho_{ud} = \langle u|\Psi\rangle\langle\Psi|d\rangle =$$

$$\rho_{ud} = \langle u(\alpha|u\rangle + \beta|d\rangle)\rangle\langle(\langle u|\alpha^* + \langle d|\beta^*)|d\rangle =$$

$$(\alpha\langle u|u\rangle + \beta\langle u|d\rangle)(\langle u|d\rangle\alpha^* + \langle d|d\rangle\beta^*) =$$

$$(\alpha + 0)(0 + \beta^*) = \alpha\beta^*$$

$\rho_{du}$

$$\rho_{du} = \langle d|\Psi\rangle\langle\Psi|u\rangle =$$

$$\rho_{du} = \langle d(\alpha|u\rangle + \beta|d\rangle)\rangle\langle(\langle u|\alpha^* + \langle d|\beta^*)|u\rangle =$$

$$(\alpha\langle d|u\rangle + \beta\langle d|d\rangle)(\langle u|u\rangle\alpha^* + \langle d|u\rangle\beta^*) =$$

$$(0 + \beta)(\alpha^* + 0) = \beta\alpha^*$$

$\rho_{dd}$

$$\rho_{dd} = \langle d|\Psi\rangle\langle\Psi|d\rangle =$$

$$\rho_{dd} = \langle d(\alpha|u\rangle + \beta|d\rangle)\rangle\langle(\langle u|\alpha^* + \langle d|\beta^*)|d\rangle =$$

$$(\alpha\langle d|u\rangle + \beta\langle d|d\rangle)(\langle u|d\rangle\alpha^* + \langle d|d\rangle\beta^*) =$$

$$(0 + \beta)(0 + \beta^*) = \beta\beta^*$$

The density matrix goes like this:

$$\rho_{aa\prime} = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

We plug in the example $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{2}}$ and get:

$$\rho_{aa\prime} = \begin{pmatrix} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Choosing the values $\alpha = \frac{1}{8}$ and $\beta = \frac{\sqrt{63}}{8}$ (designed to fit $\alpha^2 + \beta^2 = 1$):

$$\rho_{aa\prime} = \begin{pmatrix} \frac{1}{8}\frac{1}{8} & \frac{1}{8}\frac{\sqrt{63}}{8} \\ \frac{\sqrt{63}}{8}\frac{1}{8} & \frac{\sqrt{63}}{8}\frac{\sqrt{63}}{8} \end{pmatrix} = \begin{pmatrix} \frac{1}{64} & \frac{\sqrt{63}}{64} \\ \frac{\sqrt{63}}{64} & \frac{63}{64} \end{pmatrix}$$

Exercise 7.5

a) Show:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

b) Now, suppose:

$$\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Calculate:

$$\rho^2, \qquad Tr(\rho), \qquad Tr(\rho^2)$$

c) If $\rho$ is a density matrix, does it represent a pure state or a mixed state?

* * * * * * * * * *

a) Show that:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a \cdot a + 0 \cdot 0 & a \cdot 0 + 0 \cdot b \\ 0 \cdot a + b \cdot a & 0 \cdot 0 + b \cdot b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

b) Now, suppose

$$\rho = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

Calculate $\rho^2, Tr(\rho), Tr(\rho^2)$

$$\rho^2 = \rho \cdot \rho = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{4}{9} \end{pmatrix}$$

$$Tr(\rho) = Tr\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} = \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$$

$$Tr(\rho^2) = Tr\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{4}{9} \end{pmatrix} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9} \neq 1$$

c) If $\rho$ ist a density matrix, does it represent a pure stat or a mixed state?

A pure state is defined as a state in which $\rho$ is nonzero at only one point. This corresponds to a matrix where only one entry on the diagonal is nonzero. As the sum of the diagonals need to be one for the matrix to be a density matrix, this one entry must have the value 1 with the rest of all entries being zero. So, the matrix $\rho$ presents a mixed state.

Exercise 7.6

By the standard rules of probability, the probability for a:

$$P(a) = \sum_b \psi^*(a,b)\psi(a,b)$$

This is just a diagonal entry in the density matrix:

$$P(a) = \rho_{aa}$$

Use $P(a) = \rho_{aa}$ to show that if $\rho$ is a density matrix, then:

$$Tr(\rho) = 1$$

* * * * * * * * * *

Say that Alice's system has more than one state but the states $a_1, a_2, \ldots, a_n$. The probability for each one calculated according to $P(a) = \rho_{aa}$ is

$$P(a_i) = \sum_b \psi^*(a_i,b)\psi(a_i,b)$$

As Alice's system must have a state (it couldn't vanish), the sum over all possible states must be 1:

$$\sum_i P(a_i) = 1$$

Every probability $P(a_i)$ corresponds to a diagonal entry $\rho_{a_i a_i}$, so the sum over all diagonal elements in the matrix must be 1. For a pure state with only one entry $\rho_{a_i a_i} \neq 0$ this means that this entry must be 1 too.

Exercise 7.7

We have the density matrix:

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Calculate $\rho^2$. How does this result confirm that $\rho$ represents an entangled state?

* * * * * * * * * *

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2}\right)^2 & 0 \\ 0 & \left(\frac{1}{2}\right)^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

In a mixed state more than one entry on the diagonal will be nonzero with the sum giving 1.

Only for pure states the equation $\rho^2 = \rho$ holds, because the one and only "1" replicates itself.

Exercise 7.8

Consider the following states:

$$|\psi_1\rangle = \frac{1}{2}(|uu\rangle + |ud\rangle + |du\rangle + |dd\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$$

$$|\psi_3\rangle = \frac{1}{5}(3|uu\rangle + 4|dd\rangle)$$

For each one, calculate Alice's density matrix and Bob's density matrix. Check their properties.

* * * * * * * * * *

Alice's density matrix: calculate the sum $\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b)$ (7.23)

First, we expand the states $|\psi_2\rangle$ and $|\psi_3\rangle$ to the full set and get:

$$|\psi_1\rangle = \frac{1}{2}|uu\rangle + \frac{1}{2}|ud\rangle + \frac{1}{2}|du\rangle + \frac{1}{2}|dd\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|uu\rangle + 0|ud\rangle + 0|du\rangle + \frac{1}{\sqrt{2}}|dd\rangle$$

$$|\psi_3\rangle = \frac{3}{5}|uu\rangle + \frac{4}{5}|ud\rangle + 0|du\rangle + 0|dd\rangle$$

**The values of $\boldsymbol{\psi(a,b)}$ for the first case $\boldsymbol{|\psi_1\rangle}$:**

$$\psi(u,u) = \frac{1}{2} = \psi(u,d) = \psi(d,u) = \psi(d,d)$$

$$\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b)$$

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{ud} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{du} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

The density matrix $\rho$ of Alice goes like this:

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Check: a density matrix must be Hermitian and the trace of a density matrix must be 1, both conditions are fulfilled for this result.

For a pure state holds:

$$\rho^2 = \rho \; and \; Tr(\rho^2) = 1$$

We check:

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$Tr(\rho^2) = Tr\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = 1$$

Alice's matrix represents a pure state.

Bob's density matrix: calculate the sum $\rho_{b'b} = \sum_a \psi^*(a,b)\psi(a,b')$ (7.23')

The values of $\psi(a,b)$ for the first case $|\psi_1\rangle$:

$$\psi(u,u) = \frac{1}{2} = \psi(u,d) = \psi(d,u) = \psi(d,d)$$

$$\rho_{b'b} = \sum_a \psi^*(a,b)\psi(a,b')$$

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(d,u)\psi(d,u) = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{ud} = \psi^*(u,d)\psi(u,u) + \psi^*(d,d)\psi(d,u) = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{du} = \psi^*(u,u)\psi(u,d) + \psi^*(d,u)\psi(d,d) = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{dd} = \psi^*(u,d)\psi(u,d) + \psi^*(d,d)\psi(d,d) = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

The density matrix $\rho$ of Bob goes like this:

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Check: a density matrix must be Hermitian and the trace of a density matrix must be 1, both conditions are fulfilled for this result. Bob's matrix represents (like Alice's matrix) a pure state.

**The values of $\boldsymbol{\psi(a,b)}$ for the second case $\boldsymbol{|\psi_2\rangle}$:**

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|uu\rangle + 0|ud\rangle + 0|du\rangle + \frac{1}{\sqrt{2}}|dd\rangle$$

We calculate Alice's density matrix:

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\rho_{ud} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = 0 + 0 = 0$$

$$\rho_{du} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0 + 0 = 0$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

The density matrix $\rho$ of Alice goes like this:

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Check: a density matrix must be Hermitian and the trace of a density matrix must be 1, both conditions are fulfilled for this result.

For a pure state holds:

$$\rho^2 = \rho \; and \; Tr(\rho^2) = 1$$

We check:

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ & \frac{1}{4} \end{pmatrix} \neq \rho$$

$$Tr(\rho^2) = Tr \begin{pmatrix} \frac{1}{4} & 0 \\ & \frac{1}{4} \end{pmatrix} = \frac{1}{2} \neq 1$$

The matrix of Alice represents a mixed state.

We calculate Bob's density matrix:

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(d,u)\psi(d,u) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\rho_{ud} = \psi^*(u,d)\psi(u,u) + \psi^*(d,d)\psi(d,u) = 0 + 0 = 0$$

$$\rho_{du} = \psi^*(u,u)\psi(u,d) + \psi^*(d,u)\psi(d,d) = 0 + 0 = 0$$

$$\rho_{dd} = \psi^*(u,d)\psi(u,d) + \psi^*(d,d)\psi(d,d) = 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

The density matrix $\rho$ of Bob goes like this:

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

As with the matrix of Alice, Bob's matrix represents a mixed state.

**The values of $\boldsymbol{\psi(a,b)}$ for the third case $\boldsymbol{|\psi_3\rangle}$:**

$$|\psi_3\rangle = \frac{3}{5}\Big|uu\rangle + \frac{4}{5}\Big|ud\rangle + 0|du\rangle + 0|dd\rangle$$

We calculate Alice's density matrix:

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = \frac{3}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{4}{5} = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$

$$\rho_{ud} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = 0\cdot\frac{3}{5} + 0\cdot\frac{4}{5} = 0$$

$$\rho_{du} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = 0$$

The density matrix $\rho$ of Alice goes like this:

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Check: a density matrix must be Hermitian and the trace of a density matrix must be 1, both conditions are fulfilled for this result.

For a pure state holds:

$$\rho^2 = \rho \; and \; Tr(\rho^2) = 1$$

This is clearly fulfilled for the matrix of Alice. Alice's matrix represents a pure state.

We calculate Bob's density matrix:

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(d,u)\psi(d,u) = \frac{3}{5}\cdot\frac{3}{5} + 0 = \frac{9}{25}$$

$$\rho_{ud} = \psi^*(u,d)\psi(u,u) + \psi^*(d,d)\psi(d,u) = \frac{4}{5}\cdot\frac{3}{5} + 0\cdot 0 = \frac{12}{25}$$

$$\rho_{du} = \psi^*(u,u)\psi(u,d) + \psi^*(d,u)\psi(d,d) = \frac{3}{5}\cdot\frac{4}{5} + 0\cdot 0 = \frac{12}{25}$$

$$\rho_{dd} = \psi^*(u,d)\psi(u,d) + \psi^*(d,d)\psi(d,d) = \frac{4}{5}\cdot\frac{4}{5} + 0 = \frac{16}{25}$$

The density matrix $\rho$ of Bob goes like this:

$$\rho = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

Check: a density matrix must be Hermitian and the trace of a density matrix must be 1, both conditions are fulfilled for this result.

For a pure state holds:

$$\rho^2 = \rho \; and \; Tr(\rho^2) = 1$$

We check:

$$\rho^2 = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}\begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

$$Tr(\rho^2) = Tr\begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} = 1$$

The matrix of Bob represents a pure state.

Exercise 7.9

Given any Alice observable $A$ and Bo observable $B$, show that for a product state, the correlation $C(A, B)$ is zero.

* * * * * * * * * *

Alice prepares her spin in state $\langle A \rangle = \alpha_u |u\rangle + \alpha_d |d\rangle$.

Bob prepares his spin in state $\langle B \rangle = \beta_u |u\rangle + \beta_d |d\rangle$.

The product of these two states $\langle A \rangle \cdot \langle B \rangle$ gives:

$$(\alpha_u |u\rangle + \alpha_d |d\rangle) \cdot (\beta_u |u\rangle + \beta_d |d\rangle) =$$

$$(\alpha_u |u\rangle \cdot \beta_u |u\rangle) + (\alpha_u |u\rangle \cdot \beta_d |d\rangle) + (\alpha_d |d\rangle \cdot \beta_u |u\rangle) + (\alpha_d |d\rangle \cdot \beta_d |d\rangle) =$$

$$(\alpha_u \beta_u |uu\rangle) + (\alpha_u \beta_d |ud\rangle) + (\alpha_d \beta_u |du\rangle) + (\alpha_d \beta_d |dd\rangle)$$

The product state $\langle AB \rangle$ is the tensor product of both states, written as

$$A \otimes B =$$

$$\{\alpha_u |u\rangle + \alpha_d |d\rangle\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\} =$$

$$(\alpha_u |u\rangle)(\beta_u |u\rangle + \beta_d |d\rangle) + (\alpha_d |d\rangle)(\beta_u |u\rangle + \beta_d |d\rangle) =$$

$$(\alpha_u \beta_u |uu\rangle) + (\alpha_u \beta_d |ud\rangle) + (\alpha_d \beta_u |du\rangle) + (\alpha_d \beta_d |dd\rangle)$$

As the result for the “normal” multiplication and the tensor product is the same, the difference equals zero.

Exercise 7.10

Given a measuring apparatus with the states $|b\rangle$ for initial blank state, $|+1\rangle$ for "result of spin measurement is up" and $|-1\rangle$ for "result of spin measurement is down".

Verify that the state-vector

$$\alpha_u|u, b\rangle + \alpha_d|d, b\rangle$$

represents a completely unentangled state.

* * * * * * * * * *

We have two states, the state of the spin and the state of the apparatus:

$(\alpha_u|u\rangle + \alpha_d|d\rangle)$ and $(1|b\rangle + 0|(+1)\rangle + 0|(-1)\rangle)$, because in the beginning the apparatus is certainly in the $|b\rangle$-state.

We build the product state according to p. 164:

$$|\text{product state}\rangle = \{\alpha_u|u\rangle + \alpha_d|d\rangle\}\otimes\{1|b\rangle + 0|(+1)\rangle + 0|(-1)\rangle\} =$$

$$\{\alpha_u|u\rangle + \alpha_d|d\rangle\}\otimes\{1|b\rangle\} =$$

$$\alpha_u|u\rangle 1|b\rangle + \alpha_d|d\rangle 1|b\rangle =$$

$$\alpha_u|u, b\rangle + \alpha_d|d, b\rangle$$

Note: product states are not entangled.

Exercise 7.11

Calculate Alice's density matrix for $\sigma_z$ for the "near-singlet" state.

* * * * * * * * * *

Spin Operator Multiplication Table for the Up-Down-Basis:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

The state-vector leads to the following wave-function:

$\psi_{uu} = 0|uu\rangle$ $\quad\psi_{ud} = \sqrt{0.6}|ud\rangle$ $\quad\psi_{du} = -\sqrt{0.4}|du\rangle$ $\quad\psi_{dd} = 0|dd\rangle$

As the values are all real, the complex conjugated are identical: $\psi_{uu} = \psi^*{}_{uu}$ etc.

Obviously, the wave function is normalized: $0^2 + \sqrt{0.6}^2 + (-\sqrt{0.4})^2 + 0^2 = 1$

The density matrix:

$\psi(a,b)$ takes the form $\quad\psi(a,b) = \psi_{ud} + \psi_{du} = \sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$

and results in: $\quad\psi_{uu} = 0,\ \psi_{ud} = \sqrt{0.6},\ \psi_{du} = -\sqrt{0.4},\ \psi_{dd} = 0$

The density matrix of Alice: $\quad\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b) \quad (7.23)$

expanded a, a' (with $\psi^* = \psi$ due to all coefficients being real):

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = 0.6$$

$$\rho_{ud} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0$$

$$\rho_{du} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = 0$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = 0.4$$

gives Alice density matrix:

$$\rho := \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix};\ \rho^2 := \begin{pmatrix} 0.36 & 0 \\ 0 & 0.16 \end{pmatrix}$$

For Alice's subsystem holds:

$\rho^2 \neq \rho$ and Trace $(\rho^2) < 1$

Exercise 7.12

Verify the numerical values in each rap sheet.

This is a very specific exercise that refers to so called “rap sheets”. The solution here is for reasons of completeness.

* * * * * * * * * *

State-Vector Rap Sheet 1

Spin Operator Multiplication Table for the Up-Down-Basis:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

The state-vector leads to the following wave-function:

$\psi_{uu} = \alpha_u\beta_u|uu\rangle$ $\quad\psi_{ud} = \alpha_u\beta_d|ud\rangle$ $\quad\psi_{du} = \alpha_d\beta_u|du\rangle$ $\quad\psi_{dd} = \alpha_d\beta_d|dd\rangle$

with the normalization

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$$

$$\beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

The expectation values:

$$\langle\boldsymbol{\sigma_z}\rangle = \langle\boldsymbol{\psi}|\boldsymbol{\sigma_z}|\boldsymbol{\psi}\rangle =$$

First the part $\sigma_z|\psi\rangle$:

$$\sigma_z|\psi\rangle = \sigma_z(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle - \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$\alpha_u^*\beta_u^*\alpha_u\beta_u + \alpha_u^*\beta_d^*\alpha_u\beta_d - \alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d =$$

$$\alpha_u^*\alpha_u\beta_u^*\beta_u + \alpha_u^*\alpha_u\beta_d^*\beta_d - \alpha_d^*\alpha_d\beta_u^*\beta_u - \alpha_d^*\alpha_d\beta_d^*\beta_d =$$

$$\alpha_u^*\alpha_u(\beta_u^*\beta_u + \beta_d^*\beta_d) - \alpha_d^*\alpha_d(\beta_u^*\beta_u + \beta_d^*\beta_d) =$$

$$(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) =$$

$$(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(1) =$$

$$(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)$$

The expectation values:

$$\langle \boldsymbol{\sigma_y} \rangle = \langle \boldsymbol{\psi} | \boldsymbol{\sigma_y} | \boldsymbol{\psi} \rangle =$$

First the part $\sigma_y|\psi\rangle$:

$$\sigma_y|\psi\rangle = \sigma_y(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(i\alpha_u\beta_u|du\rangle + i\alpha_u\beta_d|dd\rangle - i\alpha_d\beta_u|uu\rangle - i\alpha_d\beta_d|ud\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(i\alpha_u\beta_u|du\rangle + i\alpha_u\beta_d|dd\rangle - i\alpha_d\beta_u|uu\rangle - i\alpha_d\beta_d|ud\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$-i\alpha_u^*\beta_u^*\alpha_d\beta_u - i\alpha_u^*\beta_d^*\alpha_d\beta_d + i\alpha_d^*\beta_u^*\alpha_u\beta_u + i\alpha_d^*\beta_d^*\alpha_u\beta_d =$$

$$-i\alpha_u^*\alpha_d\beta_u^*\beta_u - i\alpha_u^*\alpha_d\beta_d^*\beta_d + i\alpha_d^*\alpha_u\beta_u^*\beta_u + i\alpha_d^*\alpha_u\beta_d^*\beta_d =$$

$$i\alpha_d^*\alpha_u\beta_u^*\beta_u - i\alpha_u^*\alpha_d\beta_u^*\beta_u + i\alpha_d^*\alpha_u\beta_d^*\beta_d - i\alpha_u^*\alpha_d\beta_d^*\beta_d =$$

$$i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)\beta_u^*\beta_u + i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)\beta_d^*\beta_d =$$

$$i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) =$$

$$i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)(1) =$$

$$i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)$$

$$\langle \boldsymbol{\sigma_x} \rangle = \langle \boldsymbol{\psi} | \boldsymbol{\sigma_x} | \boldsymbol{\psi} \rangle =$$

First the part $\sigma_x|\psi\rangle$:

$$\sigma_x|\psi\rangle = \sigma_x(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(\alpha_u\beta_u|du\rangle + \alpha_u\beta_d|dd\rangle + \alpha_d\beta_u|uu\rangle + \alpha_d\beta_d|ud\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(\alpha_u\beta_u|du\rangle + \alpha_u\beta_d|dd\rangle + \alpha_d\beta_u|uu\rangle + \alpha_d\beta_d|ud\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$\alpha_u^*\beta_u^*\alpha_d\beta_u + \alpha_u^*\beta_d^*\alpha_d\beta_d + \alpha_d^*\beta_u^*\alpha_u\beta_u + \alpha_d^*\beta_d^*\alpha_u\beta_d =$$

$$\alpha_u^*\alpha_d\beta_u^*\beta_u + \alpha_u^*\alpha_d\beta_d^*\beta_d + \alpha_d^*\alpha_u\beta_u^*\beta_u + \alpha_d^*\alpha_u\beta_d^*\beta_d =$$

$$\alpha_u^*\alpha_d(\beta_u^*\beta_u + \beta_d^*\beta_d) + \alpha_d^*\alpha_u(\beta_u^*\beta_u + \beta_d^*\beta_d) =$$

$$(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)(\beta_u^*\beta_u + \beta_d^*\beta_d) =$$

$$(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)(1) =$$

$$(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)$$

Now we must calculate

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 =$$

$$(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)^2 + \left(i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)\right)^2 + (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)^2 =$$

$$(\alpha_u^*\alpha_d)^2 + 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + (\alpha_d^*\alpha_u)^2 - (\alpha_d^*\alpha_u)^2 + 2\alpha_d^*\alpha_u\alpha_u^*\alpha_d - (\alpha_u^*\alpha_d)^2 + (\alpha_u^*\alpha_u)^2 - 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_d^*\alpha_d)^2 =$$

$$2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + 2\alpha_d^*\alpha_u\alpha_u^*\alpha_d - 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_u^*\alpha_u)^2 + (\alpha_d^*\alpha_d)^2 =$$

$$2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u - 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + (\alpha_u^*\alpha_u)^2 + (\alpha_d^*\alpha_d)^2 =$$

$$(\alpha_u^*\alpha_u)^2 + 2\alpha_u^*\alpha_d\alpha_d^*\alpha_u + (\alpha_d^*\alpha_d)^2 =$$

$$(\alpha_u^*\alpha_u)^2 + 2\alpha_u^*\alpha_u\alpha_d^*\alpha_d + (\alpha_d^*\alpha_d)^2 =$$

$$(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)^2 = 1$$

The expectation values: $\langle \boldsymbol{\tau_z} \rangle = \langle \boldsymbol{\psi} | \boldsymbol{\tau_z} | \boldsymbol{\psi} \rangle =$

First the part $\tau_z|\psi\rangle$:

$$\tau_z|\psi\rangle = \tau_z(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(\alpha_u\beta_u|uu\rangle - \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(\alpha_u\beta_u|uu\rangle - \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle - \alpha_d\beta_d|dd\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$\alpha_u^*\beta_u^*\alpha_u\beta_u - \alpha_u^*\beta_d^*\alpha_u\beta_d + \alpha_d^*\beta_u^*\alpha_d\beta_u - \alpha_d^*\beta_d^*\alpha_d\beta_d =$$

$$\alpha_u^*\alpha_u\beta_u^*\beta_u - \alpha_u^*\alpha_u\beta_d^*\beta_d + \alpha_d^*\alpha_d\beta_u^*\beta_u - \alpha_d^*\alpha_d\beta_d^*\beta_d =$$

$$\alpha_u^*\alpha_u(\beta_u^*\beta_u - \beta_d^*\beta_d) + \alpha_d^*\alpha_d(\beta_u^*\beta_u - \beta_d^*\beta_d) =$$

$$(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u - \beta_d^*\beta_d) =$$

$$(1)(\beta_u^*\beta_u - \beta_d^*\beta_d) =$$

$$(\beta_u^*\beta_u - \beta_d^*\beta_d)$$

The expectation values: $\langle \boldsymbol{\tau_y} \rangle = \langle \boldsymbol{\psi} | \boldsymbol{\tau_y} | \boldsymbol{\psi} \rangle =$

First the part $\tau_y|\psi\rangle$:

$$\tau_y|\psi\rangle = \tau_y(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(i\alpha_u\beta_u|ud\rangle - i\alpha_u\beta_d|uu\rangle + i\alpha_d\beta_u|dd\rangle - i\alpha_d\beta_d|du\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(i\alpha_u\beta_u|ud\rangle - i\alpha_u\beta_d|uu\rangle + i\alpha_d\beta_u|dd\rangle - i\alpha_d\beta_d|du\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$-i\alpha_u^*\beta_u^*\alpha_u\beta_d + i\alpha_u^*\beta_d^*\alpha_u\beta_u - i\alpha_d^*\beta_u^*\alpha_d\beta_d + i\alpha_d^*\beta_d^*\alpha_d\beta_u =$$

$$-i\alpha_u^*\alpha_u\beta_u^*\beta_d + i\alpha_u^*\alpha_u\beta_d^*\beta_u - i\alpha_d^*\alpha_d\beta_u^*\beta_d + i\alpha_d^*\alpha_d\beta_d^*\beta_u =$$

$$-i\alpha_u^*\alpha_u\beta_u^*\beta_d - i\alpha_d^*\alpha_d\beta_u^*\beta_d + i\alpha_u^*\alpha_u\beta_d^*\beta_u + i\alpha_d^*\alpha_d\beta_d^*\beta_u =$$

$$-i\beta_u^*\beta_d(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d) + i\beta_d^*\beta_u(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d) =$$

$$(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(-i\beta_u^*\beta_d + i\beta_d^*\beta_u) =$$

$$i(\beta_d^*\beta_u - \beta_u^*\beta_d)$$

The expectation values: $\langle \boldsymbol{\tau_x}\rangle = \langle \boldsymbol{\psi}|\boldsymbol{\tau_x}|\boldsymbol{\psi}\rangle =$

First the part $\sigma_x|\psi\rangle$:

$$\tau_x|\psi\rangle = \tau_x(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle) =$$

$$(\alpha_u\beta_u|ud\rangle + \alpha_u\beta_d|uu\rangle + \alpha_d\beta_u|dd\rangle + \alpha_d\beta_d|du\rangle)$$

The left part $\langle\psi|$ added: $\langle\psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*$

$$(\langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*)(\alpha_u\beta_u|ud\rangle + \alpha_u\beta_d|uu\rangle + \alpha_d\beta_u|dd\rangle + \alpha_d\beta_d|du\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$\alpha_u^*\beta_u^*\alpha_u\beta_d + \alpha_u^*\beta_d^*\alpha_u\beta_u + \alpha_d^*\beta_u^*\alpha_d\beta_d + \alpha_d^*\beta_d^*\alpha_d\beta_u =$$

$$\alpha_u^*\alpha_u\beta_u^*\beta_d + \alpha_u^*\alpha_u\beta_d^*\beta_u + \alpha_d^*\alpha_d\beta_u^*\beta_d + \alpha_d^*\alpha_d\beta_d^*\beta_u =$$

$$\alpha_u^*\alpha_u(\beta_u^*\beta_d + \beta_d^*\beta_u) + \alpha_d^*\alpha_d(\beta_u^*\beta_d + \beta_d^*\beta_u) =$$

$$(\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_d + \beta_d^*\beta_u) =$$

$$(1)(\beta_u^*\beta_d + \beta_d^*\beta_u) =$$

$$(\beta_u^*\beta_d + \beta_d^*\beta_u)$$

Now we must calculate

$$\langle\tau_x\rangle^2 + \langle\tau_y\rangle^2 + \langle\tau_z\rangle^2 =$$

$$(\beta_u^*\beta_d + \beta_d^*\beta_u)^2 + (i(\beta_d^*\beta_u - \beta_u^*\beta_d))^2 + (\beta_u^*\beta_u - \beta_d^*\beta_d)^2 =$$

$$(\beta_u^*\beta_d + \beta_d^*\beta_u)^2 - (\beta_d^*\beta_u - \beta_u^*\beta_d)^2 + (\beta_u^*\beta_u - \beta_d^*\beta_d)^2 =$$

$$(\beta_u^*\beta_d)^2 + 2\beta_u^*\beta_d\beta_d^*\beta_u + (\beta_d^*\beta_u)^2 - ((\beta_d^*\beta_u)^2 - 2\beta_d^*\beta_u\beta_u^*\beta_d + (\beta_u^*\beta_d)^2) + (\beta_u^*\beta_u)^2 - 2\beta_u^*\beta_u\beta_d^*\beta_d + (\beta_d^*\beta_d)^2 =$$

$$(\beta_u^*\beta_d)^2 + 2\beta_u^*\beta_d\beta_d^*\beta_u + (\beta_d^*\beta_u)^2 - (\beta_d^*\beta_u)^2 + 2\beta_d^*\beta_u\beta_u^*\beta_d - (\beta_u^*\beta_d)^2) + (\beta_u^*\beta_u)^2 - 2\beta_u^*\beta_u\beta_d^*\beta_d + (\beta_d^*\beta_d)^2 =$$

$$(\beta_u^*\beta_u)^2 + 2\beta_u^*\beta_u\beta_d^*\beta_d + (\beta_d^*\beta_d)^2 =$$

$$(\beta_u^*\beta_u + \beta_d^*\beta_d)^2 = 1$$

The composite expectation value: $\langle \tau_z \sigma_z \rangle = \langle \psi | \tau_z \sigma_z | \psi \rangle$

$$\sigma_z |\psi\rangle = (\alpha_u \beta_u |uu\rangle + \alpha_u \beta_d |ud\rangle - \alpha_d \beta_u |du\rangle - \alpha_d \beta_d |dd\rangle)$$

$$\tau_z \sigma_z |\psi\rangle =$$

$$\tau_z(\alpha_u \beta_u |uu\rangle + \alpha_u \beta_d |ud\rangle - \alpha_d \beta_u |du\rangle - \alpha_d \beta_d |dd\rangle) =$$

$$(\alpha_u \beta_u |uu\rangle - \alpha_u \beta_d |ud\rangle - \alpha_d \beta_u |du\rangle + \alpha_d \beta_d |dd\rangle)$$

The left part $\langle \psi |$ added: $\langle \psi | = \langle uu|\alpha_u^* \beta_u^* + \langle ud|\alpha_u^* \beta_d^* + \langle du|\alpha_d^* \beta_u^* + \langle dd|\alpha_d^* \beta_d^*$

$$(\langle uu|\alpha_u^* \beta_u^* + \langle ud|\alpha_u^* \beta_d^* + \langle du|\alpha_d^* \beta_u^* + \langle dd|\alpha_d^* \beta_d^*)(\alpha_u \beta_u |uu\rangle - \alpha_u \beta_d |ud\rangle - \alpha_d \beta_u |du\rangle + \alpha_d \beta_d |dd\rangle)$$

As only products of the form $\langle uu|uu\rangle$ contribute, the scalar product results in:

$$\alpha_u^* \beta_u^* \alpha_u \beta_u - \alpha_u^* \beta_d^* \alpha_u \beta_d - \alpha_d^* \beta_u^* \alpha_d \beta_u + \alpha_d^* \beta_d^* \alpha_d \beta_d =$$

$$\alpha_u^* \alpha_u \beta_u^* \beta_u - \alpha_u^* \alpha_u \beta_d^* \beta_d - \alpha_d^* \alpha_d \beta_u^* \beta_u + \alpha_d^* \alpha_d \beta_d^* \beta_d =$$

$$\alpha_u^* \alpha_u (\beta_u^* \beta_u - \beta_d^* \beta_d) - \alpha_d^* \alpha_d (\beta_u^* \beta_u - \beta_d^* \beta_d)$$

The product $\langle \sigma_z \rangle \langle \tau_z \rangle$:

$$\langle \sigma_z \rangle = (\alpha_u^* \alpha_u - \alpha_d^* \alpha_d)$$

$$\langle \tau_z \rangle = (\beta_u^* \beta_u - \beta_d^* \beta_d)$$

$$\langle \sigma_z \rangle \langle \tau_z \rangle = (\alpha_u^* \alpha_u - \alpha_d^* \alpha_d)(\beta_u^* \beta_u - \beta_d^* \beta_d) =$$

$$\alpha_u^* \alpha_u (\beta_u^* \beta_u - \beta_d^* \beta_d) - \alpha_d^* \alpha_d (\beta_u^* \beta_u - \beta_d^* \beta_d)$$

$\langle \tau_z \sigma_z \rangle$:

$$\langle \tau_z \sigma_z \rangle = \alpha_u^* \alpha_u (\beta_u^* \beta_u - \beta_d^* \beta_d) - \alpha_d^* \alpha_d (\beta_u^* \beta_u - \beta_d^* \beta_d)$$

The correlation:

$$\langle \tau_z \sigma_z \rangle = \langle \tau_z \rangle \langle \sigma_z \rangle \rightarrow \langle \tau_z \sigma_z \rangle - \langle \tau_z \rangle \langle \sigma_z \rangle = 0$$

State-Vector Rap Sheet 2

Spin Operator Multiplication Table for the Up-Down-Basis:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

The state-vector leads to the following wave-function:

$\psi_{uu} = 0|uu\rangle$ $\qquad \psi_{ud} = \frac{1}{\sqrt{2}}|ud\rangle$ $\qquad \psi_{du} = -\frac{1}{\sqrt{2}}|du\rangle$ $\qquad \psi_{dd} = 0|dd\rangle$

As the values are all real, the complex conjugated are identical: $\psi_{uu} = \psi^*{}_{uu}$ etc.

Obviously, the wave function is normalized: $0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (-\frac{1}{\sqrt{2}})^2 + 0^2 = 1$

The density matrix:

$\psi(a, b)$ takes the form $\qquad \psi(a, b) = \psi_{ud} + \psi_{du} = \frac{1}{\sqrt{2}}|ud\rangle - \frac{1}{\sqrt{2}}|du\rangle$

and results in: $\qquad \psi_{uu} = 0,\ \psi_{ud} = \frac{1}{\sqrt{2}},\ \psi_{du} = -\frac{1}{\sqrt{2}},\ \psi_{dd} = 0$

The density matrix of Alice: $\qquad \rho_{a'a} = \sum_b \psi^*(a, b)\psi(a', b) \quad (7.23)$

expanding a, a' to uu, ud, du and dd (with $\psi^* = \psi$ due to all coefficients being real):

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\rho_{ud} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0 \cdot \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot 0 = 0$$

$$\rho_{du} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = \frac{-1}{\sqrt{2}} \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} = 0$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = \frac{-1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} + 0 \cdot 0 = \frac{1}{2}$$

gives Alice density matrix:

$$\rho \coloneqq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix};\ \rho^2 \coloneqq \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

For Alice's subsystem holds:

$\rho^2 \neq \rho$ and Trace $(\rho^2) < 1$

The expectation values:

$$\langle\sigma_z\rangle = \langle\psi|\sigma_z|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\sigma_z\middle|\left(\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\middle|ud\rangle + \frac{1}{\sqrt{2}}\middle|du\rangle\right) =$$

$$\frac{1}{2}\langle ud|ud\rangle + \frac{1}{2}\langle ud|du\rangle + \frac{1}{2}\langle du|ud\rangle - \frac{1}{2}\langle du|du\rangle =$$

$$\frac{1}{2} + 0 + 0 - \frac{1}{2} = 0$$

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\sigma_x\middle|\left((\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\middle|dd\rangle - \frac{1}{\sqrt{2}}\middle|uu\rangle\right) =$$

$$\frac{1}{2}\langle ud|dd\rangle - \frac{1}{2}\langle ud|uu\rangle - \frac{1}{2}\langle du|dd\rangle + \frac{1}{2}\langle du|uu\rangle =$$

$$0 - 0 - 0 + 0 = 0$$

$$\langle\sigma_y\rangle = \langle\psi|\sigma_y|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\sigma_y\middle|\left((\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(i\frac{1}{\sqrt{2}}\middle|dd\rangle + i\frac{1}{\sqrt{2}}\middle|uu\rangle\right) =$$

$$i\frac{1}{2}\langle ud|dd\rangle + i\frac{1}{2}\langle ud|uu\rangle - i\frac{1}{2}\langle du|dd\rangle - i\frac{1}{2}\langle du|uu\rangle =$$

$$0 + 0 - 0 - 0 = 0$$

$$\langle\tau_z\rangle = \langle\psi|\tau_z|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\tau_z\middle|\left((\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right) =$$

$$-\frac{1}{2}\langle ud|ud\rangle - \frac{1}{2}\langle ud|du\rangle + \frac{1}{2}\langle du|ud\rangle + \frac{1}{2}\langle du|du\rangle =$$

$$-\frac{1}{2} + 0 + 0 + \frac{1}{2} = 0$$

$$\langle\tau_x\rangle = \langle\psi|\tau_x|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\tau_x\middle|\left(\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\middle|uu\rangle - \frac{1}{\sqrt{2}}\middle|dd\rangle\right) =$$

$$\frac{1}{2}\langle ud|uu\rangle - \frac{1}{2}\langle ud|dd\rangle - \frac{1}{2}\langle du|uu\rangle + \frac{1}{2}\langle du|dd\rangle =$$

$$0 - 0 - 0 + 0 = 0$$

$$\langle\tau_y\rangle = \langle\psi|\tau_y|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\tau_y\middle|\left(\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(-i\frac{1}{\sqrt{2}}\middle|uu\rangle - i\frac{1}{\sqrt{2}}\middle|dd\rangle\right) =$$

$$-i\frac{1}{2}\langle ud|uu\rangle - i\frac{1}{2}\langle ud|dd\rangle + i\frac{1}{2}\langle du|uu\rangle + i\frac{1}{2}\langle du|dd\rangle =$$

$$-0 - 0 + 0 + 0 = 0$$

The composite expectation values:

$$\langle\tau_z\sigma_z\rangle = \langle\psi|\tau_z\sigma_z|\psi\rangle =$$

$$\left\langle\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\middle|\tau_z\sigma_z\middle|\left(\frac{1}{\sqrt{2}}\middle|ud\rangle - \frac{1}{\sqrt{2}}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)|\tau_z|\left(\frac{1}{\sqrt{2}}\middle|ud\rangle + \frac{1}{\sqrt{2}}\middle|du\rangle\right) =$$

$$\left(\langle ud|\frac{1}{\sqrt{2}} - \langle du|\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\middle|ud\rangle + \frac{1}{\sqrt{2}}\middle|du\rangle\right) =$$

$$-\frac{1}{2}\langle ud|ud\rangle + \frac{1}{2}\langle ud|du\rangle + \frac{1}{2}\langle du|ud\rangle - \frac{1}{2}\langle du|du\rangle =$$

$$-\frac{1}{2} + 0 + 0 - \frac{1}{2} = -1$$

$$\langle \tau_y \sigma_y \rangle = \langle \psi | \tau_y \sigma_y | \psi \rangle =$$

$$\left\langle \left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) \middle| \tau_y \sigma_y \middle| \left( \frac{1}{\sqrt{2}} \middle| ud \rangle - \frac{1}{\sqrt{2}} \middle| du \rangle \right) \right\rangle =$$

$$\left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) |\tau_y| \left( \frac{i}{\sqrt{2}} \middle| dd \rangle + \frac{i}{\sqrt{2}} \middle| uu \rangle \right) =$$

$$\left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) \left( \frac{-i^2}{\sqrt{2}} \middle| du \rangle - \frac{i^2}{\sqrt{2}} \middle| ud \rangle \right) =$$

$$\left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \middle| du \rangle + \frac{1}{\sqrt{2}} \middle| ud \rangle \right) =$$

$$\frac{1}{2} \langle ud|du \rangle - \frac{1}{2} \langle ud|ud \rangle - \frac{1}{2} \langle du|du \rangle - \frac{1}{2} \langle du|ud \rangle =$$

$$0 - \frac{1}{2} - \frac{1}{2} + 0 = -1$$

$$\langle \tau_x \sigma_x \rangle = \langle \psi | \tau_x \sigma_x | \psi \rangle =$$

$$\left\langle \left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) \middle| \tau_x \sigma_x \middle| \left( \frac{1}{\sqrt{2}} \middle| ud \rangle - \frac{1}{\sqrt{2}} \middle| du \rangle \right) \right\rangle =$$

$$\left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) |\tau_x| \left( \frac{1}{\sqrt{2}} \middle| dd \rangle - \frac{1}{\sqrt{2}} \middle| uu \rangle \right) =$$

$$\left( \langle ud| \frac{1}{\sqrt{2}} - \langle du| \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \middle| du \rangle - \frac{1}{\sqrt{2}} \middle| ud \rangle \right) =$$

$$\frac{1}{2} \langle ud|du \rangle - \frac{1}{2} \langle ud|ud \rangle - \frac{1}{2} \langle du|du \rangle + \frac{1}{2} \langle du|ud \rangle =$$

$$0 - \frac{1}{2} - \frac{1}{2} + 0 = -1$$

The correlation:

$$\langle \tau_z \sigma_z \rangle - \langle \tau_z \rangle \langle \sigma_z \rangle = -1 - 0 = -1$$

Vector Rap Sheet 3

Spin Operator Multiplication Table for the Up-Down-Basis:

| | | | |
|---|---|---|---|
| $\sigma_z\|uu\rangle = \|uu\rangle$ | $\sigma_z\|ud\rangle = \|ud\rangle$ | $\sigma_z\|du\rangle = -\|du\rangle$ | $\sigma_z\|dd\rangle = -\|dd\rangle$ |
| $\sigma_x\|uu\rangle = \|du\rangle$ | $\sigma_x\|ud\rangle = \|dd\rangle$ | $\sigma_x\|du\rangle = \|uu\rangle$ | $\sigma_x\|dd\rangle = \|ud\rangle$ |
| $\sigma_y\|uu\rangle = i\|du\rangle$ | $\sigma_y\|ud\rangle = i\|dd\rangle$ | $\sigma_y\|du\rangle = -i\|uu\rangle$ | $\sigma_y\|dd\rangle = -i\|ud\rangle$ |
| $\tau_z\|uu\rangle = \|uu\rangle$ | $\tau_z\|ud\rangle = -\|ud\rangle$ | $\tau_z\|du\rangle = \|du\rangle$ | $\tau_z\|dd\rangle = -\|dd\rangle$ |
| $\tau_x\|uu\rangle = \|ud\rangle$ | $\tau_x\|ud\rangle = \|uu\rangle$ | $\tau_x\|du\rangle = \|dd\rangle$ | $\tau_x\|dd\rangle = \|du\rangle$ |
| $\tau_y\|uu\rangle = i\|ud\rangle$ | $\tau_y\|ud\rangle = -i\|uu\rangle$ | $\tau_y\|du\rangle = i\|dd\rangle$ | $\tau_y\|dd\rangle = -i\|du\rangle$ |

The state-vector leads to the following wave-function:

$\psi_{uu} = 0|uu\rangle$ $\quad$ $\psi_{ud} = \sqrt{0.6}|ud\rangle$ $\quad$ $\psi_{du} = -\sqrt{0.4}|du\rangle$ $\quad$ $\psi_{dd} = 0|dd\rangle$

As the values are all real, the complex conjugated are identical: $\psi_{uu} = \psi^*{}_{uu}$ etc.

Obviously, the wave function is normalized: $0^2 + \sqrt{0.6}^2 + (-\sqrt{0.4})^2 + 0^2 = 1$

The density matrix:

$\psi(a,b)$ takes the form $\quad$ $\psi(a,b) = \psi_{ud} + \psi_{du} = \sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$

and results in: $\quad$ $\psi_{uu} = 0,\ \psi_{ud} = \sqrt{0.6},\ \psi_{du} = -\sqrt{0.4},\ \psi_{dd} = 0$

The density matrix of Alice: $\quad$ $\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b)$ $\quad$ (7.23)

expanded a, a' (with $\psi^* = \psi$ due to all coefficients being real):

$$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = 0.6$$

$$\rho_{ud} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0$$

$$\rho_{du} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = 0$$

$$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = 0.4$$

gives Alice density matrix:

$$\rho \coloneqq \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix};\ \rho^2 \coloneqq \begin{pmatrix} 0.36 & 0 \\ 0 & 0.16 \end{pmatrix}$$

For Alice's subsystem holds:

$\rho^2 \neq \rho$ and Trace $(\rho^2) < 1$

The expectation values:

$$\langle\sigma_z\rangle = \langle\psi|\sigma_z|\psi\rangle =$$

$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\sigma_z\middle|\left(\sqrt{0{,}6}\middle|ud\rangle - \sqrt{0{,}4}\middle|du\rangle\right)\right\rangle =$$

$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(\sqrt{0{,}6}|ud\rangle + \sqrt{0{,}4}|du\rangle\right) =$$

$$0.6\langle ud|ud\rangle + \sqrt{0.24}\langle ud|du\rangle + \sqrt{0.24}\langle du|ud\rangle - 0.4\langle du|du\rangle =$$

$$0.6 + 0 + 0 - 0.4 = 0.2$$

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle =$$
$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\sigma_x\middle|\left(\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right)\right\rangle =$$
$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(\sqrt{0{,}6}|dd\rangle - \sqrt{0{,}4}|uu\rangle\right) =$$
$$0.6\langle ud|dd\rangle - \sqrt{0.24}\langle ud|uu\rangle - \sqrt{0.24}\langle du|dd\rangle + 0.4\langle du|uu\rangle =$$
$$0 - 0 - 0 + 0 = 0$$

$$\langle\sigma_y\rangle = \langle\psi|\sigma_y|\psi\rangle =$$
$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\sigma_y\middle|\left(\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right)\right\rangle =$$
$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(i\sqrt{0{,}6}|dd\rangle + i\sqrt{0{,}4}|uu\rangle\right) =$$
$$i0.6\langle ud|dd\rangle + i\sqrt{0.24}\langle ud|uu\rangle - i\sqrt{0.24}\langle du|dd\rangle - i0.4\langle du|uu\rangle =$$
$$0 + 0 - 0 - 0 = 0$$

$$\langle\tau_z\rangle = \langle\psi|\tau_z|\psi\rangle =$$
$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\tau_z\middle|\left(\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right)\right\rangle =$$
$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(-\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right) =$$
$$-0.6\langle ud|ud\rangle - \sqrt{0.24}\langle ud|du\rangle + \sqrt{0.24}\langle du|ud\rangle + 0.4\langle du|du\rangle =$$
$$-0.6 + 0 + 0 + 0.4 = -0.2$$

$$\langle\tau_x\rangle = \langle\psi|\tau_x|\psi\rangle =$$
$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\tau_x\middle|\left(\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right)\right\rangle =$$
$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(\sqrt{0{,}6}|uu\rangle - \sqrt{0{,}4}|dd\rangle\right) =$$
$$0.6\langle ud|uu\rangle - \sqrt{0.24}\langle ud|dd\rangle - \sqrt{0.24}\langle du|uu\rangle + 0.4\langle du|dd\rangle =$$
$$0 - 0 - 0 + 0 = 0$$
$$\langle\tau_y\rangle = \langle\psi|\tau_y|\psi\rangle =$$
$$\left\langle\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\middle|\tau_y\middle|\left(\sqrt{0{,}6}|ud\rangle - \sqrt{0{,}4}|du\rangle\right)\right\rangle =$$
$$\left(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\right)\left(-i\sqrt{0{,}6}|uu\rangle - i\sqrt{0{,}4}|dd\rangle\right) =$$
$$-i0.6\langle ud|uu\rangle - i\sqrt{0.24}\langle ud|dd\rangle + i\sqrt{0.24}\langle du|uu\rangle + i0.4\langle du|dd\rangle =$$
$$-0 - 0 + 0 + 0 = 0$$

The composite expectation values:

$$\langle\tau_z\sigma_z\rangle = \langle\psi|\tau_z\sigma_z|\psi\rangle =$$

$$\Big\langle\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)\Big|\tau_z\sigma_z\Big|\big(\sqrt{0{,}6}\big|ud\rangle - \sqrt{0{,}4}\big|du\rangle\big)\Big\rangle =$$

$$\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)|\tau_z|\big(\sqrt{0{,}6}\big|ud\rangle + \sqrt{0{,}4}\big|du\rangle\big) =$$

$$\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)\big(-\sqrt{0{,}6}\big|ud\rangle + \sqrt{0{,}4}\big|du\rangle\big) =$$

$$-0.6\langle ud|ud\rangle + \sqrt{0.24}\langle ud|du\rangle + \sqrt{0.24}\langle du|ud\rangle - 0.4\langle du|du\rangle =$$

$$-0.6 + 0 + 0 - 0.4 = -1$$

$$\langle\tau_x\sigma_x\rangle = \langle\psi|\tau_x\sigma_x|\psi\rangle =$$

$$\Big\langle\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)\Big|\tau_x\sigma_x\Big|\big(\sqrt{0{,}6}\big|ud\rangle - \sqrt{0{,}4}\big|du\rangle\big)\Big\rangle =$$

$$\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)|\tau_x|\big(\sqrt{0{,}6}\big|dd\rangle - \sqrt{0{,}4}\big|uu\rangle\big) =$$

$$\big(\langle ud|\sqrt{0{,}6} - \langle du|\sqrt{0{,}4}\big)\big(\sqrt{0{,}6}\big|du\rangle - \sqrt{0{,}4}\big|ud\rangle\big) =$$

$$0.6\langle ud|du\rangle - \sqrt{0.24}\langle ud|ud\rangle - \sqrt{0.24}\langle du|du\rangle + 0.4\langle du|ud\rangle =$$

$$0 - \sqrt{0.24} - \sqrt{0.24} + 0 = -2\cdot\sqrt{0.24} = -\sqrt{0.96}$$

The correlation:

$$\langle\tau_z\sigma_z\rangle - \langle\tau_z\rangle\langle\sigma_z\rangle = -1 - 0.2\cdot(-0.2) = -1 + 0.04 = -0.96$$

Exercise 8.1

Prove that the position operator $X$ and the momentum operator $D$ are linear operators.

$$\boldsymbol{X}: \boldsymbol{X}\varphi(x) = x\varphi(x)$$

$$\boldsymbol{D}: \boldsymbol{D}\varphi(x) = \frac{d\varphi(x)}{dx}$$

* * * * * * * * * *

Properties of a linear operator $\boldsymbol{L}$:

1) $\boldsymbol{L}(f+g) = \boldsymbol{L}f + \boldsymbol{L}g$
2) $\boldsymbol{L}(tf) = t\boldsymbol{L}f$

For $\boldsymbol{X}$

1)

$$\boldsymbol{X}(\varphi(x) + \theta(x)) = x(\varphi(x) + \theta(x)) =$$

$$x\varphi(x) + x\theta(x) = \boldsymbol{X}\varphi(x) + \boldsymbol{X}\theta(x)$$

2)

$$\boldsymbol{X}\big(z\varphi(x)\big) = xz\varphi(x) = zx\varphi(x) = z\boldsymbol{X}\big(\varphi(x)\big)$$

For $\boldsymbol{D}$

1)

$$\boldsymbol{D}(\varphi(x) + \theta(x)) = \frac{d(\varphi(x) + \theta(x))}{dx} =$$

$$\frac{d\varphi(x)}{dx} + \frac{d\theta(x)}{dx} = \boldsymbol{D}\varphi(x) + \theta\boldsymbol{D}(x)$$

2)

$$\boldsymbol{D}\big(z\varphi(x)\big) = \frac{d(z\varphi(x))}{dx} = \frac{z(d\varphi(x))}{dx} = z\boldsymbol{D}\big(\varphi(x)\big)$$

Exercise 9.1

Applicate the Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$$

to the wave function:

$$\psi(x) = e^{\frac{ip}{\hbar}x}$$

Show that this wave function is a solution, if we set:

$$E = \frac{p^2}{2m}$$

* * * * * * * * * *

We start with the Hamiltonian:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$$

We insert the wave function:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 e^{\frac{ipx}{\hbar}}}{\partial x^2} =;$$

We derive partial twice:

Once:

$$\frac{\partial e^{\frac{ipx}{\hbar}}}{\partial x} = \frac{ip}{\hbar}e^{\frac{ipx}{\hbar}}$$

Twice:

$$\frac{\partial\left(\frac{ip}{\hbar}e^{\frac{ipx}{\hbar}}\right)}{\partial x} = \frac{ip}{\hbar}\frac{ip}{\hbar}e^{\frac{ipx}{\hbar}} = -\frac{p^2}{\hbar^2}e^{\frac{ipx}{\hbar}}$$

… fill in the result:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 e^{\frac{ipx}{\hbar}}}{\partial x^2} = \left(-\frac{\hbar^2}{2m}\right)\left(-\frac{p^2}{\hbar^2}e^{\frac{ipx}{\hbar}}\right) =$$

$$\left(\frac{p^2}{2m}e^{\frac{ipx}{\hbar}}\right) = \frac{p^2}{2m}\,\psi(x)$$

We get:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = \frac{p^2}{2m}\,\psi(x) := E\psi(x)$$

We can conclude:

$$\frac{p^2}{2m} = E$$

Exercise 9.2

Prove the following equation by expanding each side and comparing the results:

$$[P^2, X] = P[P, X] + [P, X]P$$

$P$ is the momentum operator, $X$ is the position operator – both are matrices.

$[P, X]$ is the commutator relation: $[P, X] = PX - XP$

$$* * * * * * * * * *$$

$$[P^2, X] = P[P, X] + [P, X]P$$

Left side:

$$[P^2, X] = PPX - XPP$$

Right side:

$$P[P, X] + [P, X]P =$$

$$P(PX - XP) + (PX - XP)P =$$

$$PPX - PXP + PXP - XPP =$$

$$PPX - XPP$$

Obviously both sides of the equation are the same.

Exercise 9.3

Show that the right-hand side of

$$[V(x), P]\psi(x) = V(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) - \left(-i\hbar\frac{d}{dx}\right)V(x)\psi(x)$$

simplifies to the right-hand side of:

$$[V(x), P]\psi(x) = i\hbar\frac{dV(x)}{dx}$$

Hint: First expand the second term by taking the derivative of the product.

Then look for cancellations.

* * * * * * * * * *

$$[V(x), P]\varphi(x) = V(x)\left(-i\hbar\frac{d}{dx}\right)\varphi(x) - \left(-i\hbar\frac{d}{dx}\right)V(x)\varphi(x)$$

We calculate the derivations:

$$V(x)\left(-i\hbar\frac{d}{dx}\right)\varphi(x) - \left(-i\hbar\frac{d}{dx}\right)V(x)\varphi(x) =$$

$$V(x)\left(-i\hbar\frac{d\varphi(x)}{dx}\right) + i\hbar\left(\frac{dV(x)}{dx}\cdot\varphi(x) + \frac{d\varphi(x)}{dx}\cdot V(x)\right) =$$

$$-i\hbar\frac{d\varphi(x)}{dx}\cdot V(x) + i\hbar\frac{dV(x)}{dx}\cdot\varphi(x) + i\hbar\frac{d\varphi(x)}{dx}\cdot V(x) =$$

$$i\hbar\frac{dV(x)}{dx}\cdot\varphi(x)$$

We get

$$[V(x), P]\varphi(x) = i\hbar\frac{dV(x)}{dx}\cdot\varphi(x)$$

or

$$[V(x), P] = i\hbar\frac{dV(x)}{dx}$$

Exercise 10.1

Find the second derivative of $x$:

$$x = A \cdot cos(\omega t) + B \cdot sin(\omega t)$$

Show thereby that it solves:

$$-\omega^2 x = \ddot{x}$$

* * * * * * * * * *

We calculate:

$$x = A \cdot cos(\omega t) + B \cdot sin(\omega t)$$

$$\dot{x} = -A \cdot sin(\omega t) \cdot \omega + B \cdot cos(\omega t) \cdot \omega =$$

$$\omega\big(B \cdot cos(\omega t) - A \cdot sin(\omega t)\big)$$

$$\ddot{x} = \omega(-A \cdot cos(\omega t) \cdot \omega - B \cdot sin(\omega t) \cdot \omega) =$$

$$-\omega^2\big(A \cdot cos(\omega t) + B \cdot sin(\omega t)\big) =$$

$$-\omega^2 x$$

The same calculation with complex numbers:

$$x = e^{i\omega t}$$

$$\dot{x} = i\omega e^{i\omega t}$$

$$\ddot{x} = i^2\omega^2 i e^{i\omega t} = -\omega^2 e^{i\omega t} = -\omega^2 x$$