These worked examples follow "Rechenmethoden der Quantentheorie, Siegfried Flügge, Springer, Berlin, 1965". Library of Congress Catalog Card Number 65-24546, title-nr. 7288

Hope I can help you with learning quantum mechanics.

Solve the time-dependent Schrödinger equation (no potential).

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi$$

Note: $\Psi(r,t)$

Show that there exist solutions representing plane waves and discuss their physical meaning.

We separate variables:

$$\Psi(\vec{r},t) = u(\vec{r})\cdot g(t)$$

We get:

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi \rightarrow$$

$$-\frac{\hbar}{i}\cdot\frac{\partial\big(u(\vec{r})\cdot g(t)\big)}{\partial t} = -\frac{\hbar^2}{2m}\cdot\Delta\big(u(\vec{r})\cdot g(t)\big)$$

$$-\frac{\hbar}{i}\cdot\left(\frac{\partial u(\vec{r})}{\partial t}\cdot g(t) + u(\vec{r})\cdot\frac{d}{dt}g(t)\right) = -\frac{\hbar^2}{2m}\cdot\big(\Delta u(\vec{r})\cdot g(t) + u(\vec{r})\cdot\Delta g(t)\big)$$

$$-\frac{\hbar}{i}\cdot u(\vec{r})\cdot\frac{d}{dt}g(t) = -\frac{\hbar^2}{2m}\cdot\Delta u(\vec{r})\cdot g(t)$$

$$-\frac{\hbar}{i}\cdot\frac{1}{g(t)}\cdot\frac{d}{dt}g(t) = -\frac{\hbar^2}{2m}\cdot\frac{\Delta u(\vec{r})}{u(\vec{r})}$$

We rewrite:

$$-\frac{\hbar}{2m}\cdot\frac{\Delta u(\vec{r})}{u(\vec{r})} := \omega \qquad (1)$$

We get:

$$-\frac{\hbar}{i}\cdot\frac{1}{g(t)}\cdot\frac{d}{dt}g(t) = \hbar\omega$$

We can now express

$$\frac{d}{dt}g(t) = -i\omega g(t)$$

This is a differential equation with solution:

$$g(t) = e^{-i\omega t}$$

We recycle (1):

$$-\frac{\hbar}{2m}\cdot\frac{\Delta u(\vec{r})}{u(\vec{r})} := \omega$$

$$\Delta u(\vec{r}) = -\frac{2m\omega}{\hbar}\cdot u(\vec{r})$$

$$\Delta u(\vec{r}) + \frac{2m\omega}{\hbar}\cdot u(\vec{r}) = 0$$

We rename:

$$\frac{2m\omega}{\hbar} := k^2$$

Note: $[k] = \frac{1}{meter}$, the unit of k is a reciprocal length. As k is a property of a plane wave we could write it as the value of the vector $|\vec{k}|$, pointing into the direction the wave propagates.

We get:

$$\Delta u(\vec{r}) + k^2 \cdot u(\vec{r}) = 0$$

This is a differential equation for $u(r)$ with the solution:

$$u(\vec{r}) = C \cdot e^{i\vec{k}\vec{r}}$$

$\vec{k}$ is an arbitrary constant vector in direction of the plane wave of length k.

We write the solution to the time dependent Schrödinger equation $-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi$:

$$\Psi_k(\vec{r},t) = u(\vec{r}) \cdot g(t) = C \cdot e^{i\vec{k}\vec{r}} \cdot e^{-i\omega t} \tag{2}$$

This is a running wave with phase velocity $\frac{\omega}{k}$ in direction $\vec{k}$.

We need the probability density ρ:

$$\rho = \Psi^*\Psi = |C|^2$$

The probability density states that the density is constant everywhere. The probability for localizing the “particle” at any point in space is the same.

We need the probability current $\vec{j}$:

$$\vec{j} = \frac{\hbar}{2mi}(\Psi^* grad\Psi - \Psi \mathrm{grad}\Psi^*) = |C|^2 \frac{\hbar}{m}\vec{k}$$

The probability current for the “particle”:

$$\vec{j} = \rho \cdot \vec{v}$$

We get the “particle-speed”:

$$\vec{v} = \frac{\hbar}{m}\vec{k}$$

Using $k = |\vec{k}|$ we define the wave-length λ:

$$\lambda = \frac{2\pi}{k}$$

We define the momentum $\vec{p}$:

$$\vec{p} = m \cdot \vec{v}$$

With these definitions we can write:

$$\vec{p} = \hbar \cdot \vec{k}$$

$$p = |\vec{p}| = \frac{2\pi \cdot \hbar}{\lambda}$$

These are de Broglie's equations.

We use kinetic energy:

$$E_{kin} = \frac{p^2}{2 \cdot m} = \frac{\hbar^2 \cdot k^2}{2 \cdot m}$$

We remember:

$$\frac{2 \cdot m \cdot \omega}{\hbar} \coloneqq k^2$$

We get:

$$E_{kin} = \frac{p^2}{2 \cdot m} = \frac{\hbar^2 \cdot k^2}{2 \cdot m} = \frac{\hbar^2 \cdot 2 \cdot m \cdot \omega}{2 \cdot m \cdot \hbar} = \hbar \cdot \omega$$

The phase velocity:

$$\frac{\omega}{k}$$

The phase velocity of the "particle":

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar \cdot \omega}{\hbar \cdot k} = \frac{E_{kin}}{p} = \frac{m \cdot v^2}{2 \cdot m \cdot v} = \frac{v}{2}$$