These worked examples follow “Rechenmethoden der Quantentheorie, Siegfried Flügge, Springer, Berlin, 1965”. Library of Congress Catalog Card Number 65-24546, title-nr. 7288

Hope I can help you with learning quantum mechanics.

Solve a wave packet in **one** dimension without potential by superposition of plane waves.

In FAQ 1 we found a particular solution $\Psi_k(\vec{r},t)$ of the time dependent Schrödinger equation:

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi$$

The solution was:

$$\Psi_k(\vec{r},t) = C \cdot e^{i(\vec{k}\vec{r}-\omega t)}$$

Note:

$$\omega = \frac{k^2 \cdot \hbar}{2 \cdot m}, k \in \mathbb{R}$$

We reduce to one dimension x and get the complete solution by integrating over all possible k:

$$\Psi(x,t) = \int_{-\infty}^{\infty} C(k) \cdot e^{i\left(k \cdot x - \frac{t \cdot \hbar}{2 \cdot m} \cdot k^2\right)} dk$$

The function $C(k)$ is a free parameter with the property needed that for $|k| \to \infty$ the function approaches zero at least like $1/_k$.

By help of $C(k)$ we can select one specific solution.

We set $t = 0$:

$$\Psi(x,0) = \int_{-\infty}^{\infty} C(k) \cdot e^{ikx} dk$$

This is a wave packet concentrated at position $x = 0$

We want the wave packet to have momentum $p_0 = \hbar \cdot k_0$.

A wave function fulfilling this condition:

$$\Psi(x,0) = A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)$$

We check:

The probability density ρ:

$$\rho = \Psi^*\Psi =$$

$$\left(A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\right)\left(A^* \cdot exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\right) =$$

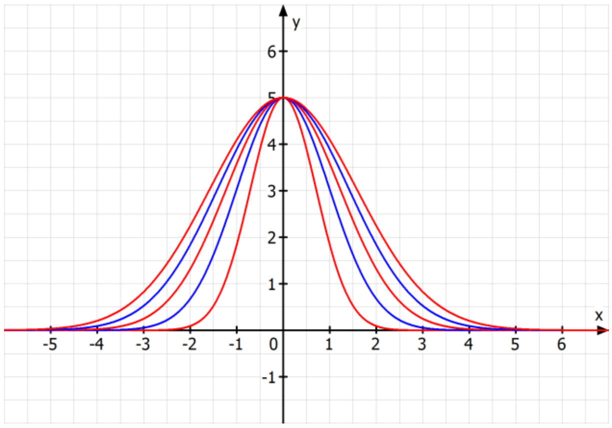
$$|A|^2 exp\left(-\frac{x^2}{a^2}\right)$$

Result:

$$\rho = |A|^2 exp\left(-\frac{x^2}{a^2}\right)$$

Note: A might be a complex number.

We plot with the real parameters $A = 5, a \in \{1,2,3,4,5\}$:



We see the gaussian with variance $\sqrt{a}$ centered around zero. The smaller a, the sharper the localization.

We check the probability current j:

$$j = \frac{\hbar}{2mi}\left(\Psi^* grad\Psi - \Psi \mathrm{grad}\Psi^*\right) =$$

$$\frac{\hbar}{2mi}\left(\Psi^* \frac{d}{dx}\Psi - \Psi \frac{d}{dx}\Psi^*\right) =$$

$$\frac{\hbar}{2mi}\left(A^* \cdot exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\frac{d}{dx}\left(A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\right) - A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\frac{d}{dx}\left(A^* \cdot exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\right)\right) =$$

$$\frac{\hbar}{2mi}\left(A^* \cdot exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\left(A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\left(-\frac{x}{a^2} + i \cdot k_0\right)\right) - A \cdot exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\left(A^* \cdot exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\left(-\frac{x}{a^2} - i \cdot k_0\right)\right)\right) =$$

$$\frac{\hbar}{2mi}|A|^2\left(exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\left(exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\left(-\frac{x}{a^2} + i \cdot k_0\right)\right) - exp\left(-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x\right)\left(exp\left(-\frac{x^2}{2a^2} - i \cdot k_0 \cdot x\right)\left(-\frac{x}{a^2} - i \cdot k_0\right)\right)\right) =$$

$$\frac{\hbar}{2mi}|A|^2\left(exp\left(-\frac{x^2}{a^2}\right)\left(\left(-\frac{x}{a^2} + i \cdot k_0\right)\right) - exp\left(-\frac{x^2}{a^2}\right)\left(\left(-\frac{x}{a^2} - i \cdot k_0\right)\right)\right) =$$

$$\frac{\hbar}{2mi}|A|^2\left(-\frac{x}{a^2} \cdot exp\left(-\frac{x^2}{a^2}\right) + i \cdot k_0 \cdot exp\left(-\frac{x^2}{a^2}\right) + \frac{x}{a^2} \cdot exp\left(-\frac{x^2}{a^2}\right) + i \cdot k_0 \cdot exp\left(-\frac{x^2}{a^2}\right)\right) =$$

$$\frac{\hbar}{mi} \cdot i \cdot k_0 \cdot |A|^2 \cdot exp\left(-\frac{x^2}{a^2}\right) =$$

$$\frac{\hbar}{m} \cdot k_0 \cdot |A|^2 \cdot exp\left(-\frac{x^2}{a^2}\right)$$

From the result for the probability density ρ we got:

$$\rho = |A|^2 exp\left(-\frac{x^2}{a^2}\right)$$

For the probability current we get:

$$j = \rho \cdot \frac{\hbar}{m} \cdot k_0$$

The velocity v of the wave packet:

$$v = \frac{\hbar}{m} \cdot k_0 = \frac{j}{\rho}$$

The momentum p of the wave packet:

$$p = \hbar \cdot k_0 = \frac{v}{m}$$

The wave packet represents one particle, so we need:

$$\int_{-\infty}^{\infty} \rho dx = 1$$

We calculate:

$$\int_{-\infty}^{\infty} \rho dx = \int_{-\infty}^{\infty} |A|^2 exp\left(-\frac{x^2}{a^2}\right) dx = |A|^2 \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{a^2}\right) dx =$$

$$|A|^2 \cdot a \cdot \sqrt{\pi} = 1$$

From this we calculate $|A|^2$:

$$|A|^2 = \frac{1}{a \cdot \sqrt{\pi}}$$

We know that:

$$\Psi(x,0) = \int_{-\infty}^{\infty} C(k) e^{i \cdot k \cdot x} dk$$

is a Fourier transform of $C(k)$.

We build the inverse Fourier transform:

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(x,0) e^{-i \cdot k \cdot x} dx$$

We use the function:

$$\Psi(x,0) = A \cdot e^{-\frac{x^2}{2a^2} + i \cdot k_0 \cdot x} = A \cdot e^{-\frac{x^2}{2a^2}} \cdot e^{i \cdot k_0 \cdot x}$$

We calculate the integral:

$$C(k) = \frac{1}{2\pi} \cdot A \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2}+i\cdot k_0\cdot x} e^{-i\cdot k\cdot x} dx =$$

$$\frac{1}{2\pi} \cdot A \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2}} e^{i\cdot x(k_0-k)} dx$$

We take a look at Wikipedia: https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions:

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)} dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

We use this and get the spectral function:

$$C(k) = \frac{1}{2\pi} \cdot A \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2}} e^{-i\cdot x(k_0-k)} dx =$$

$$\frac{1}{2\pi} \cdot A \cdot \sqrt{2a^2\pi} \cdot e^{-(k_0-k)^2 \cdot \frac{2a^2}{4}} =$$

$$\frac{A \cdot a}{\sqrt{2\pi}} \cdot e^{-(k_0-k)^2 \cdot \frac{a^2}{2}}$$

We interpret:

$$\Psi(x,0) = A \cdot e^{-\frac{x^2}{2a^2}} \cdot e^{i\cdot k_0\cdot x}$$

For $x \gg a$ the expression $e^{-\frac{x^2}{2a^2}}$ will tend to zero. The function exists in the surrounding of $x = a$ like a particle at position a.

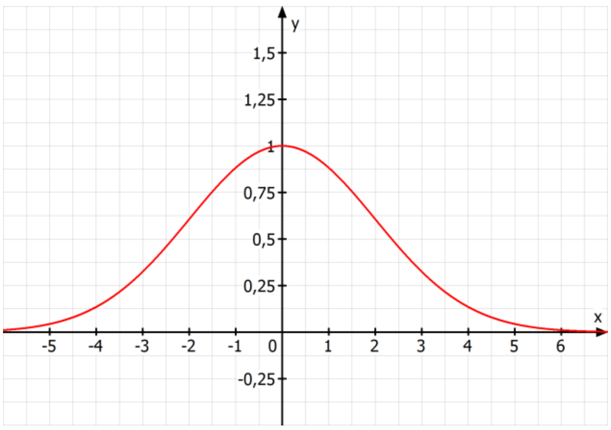
We have an uncertainty of about a:

$$\Delta x \cong a$$

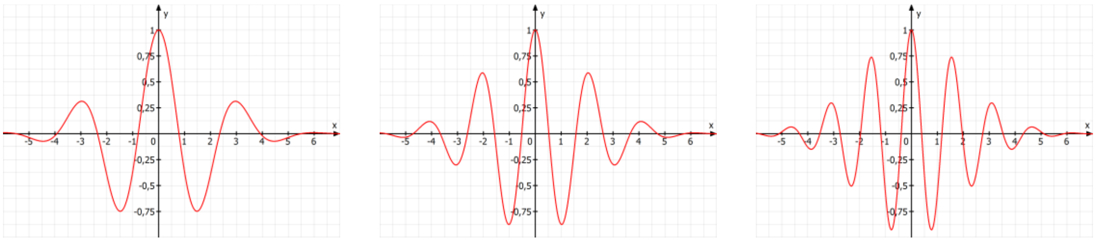
Note: a gives the inflection point of the Gaussian.

The expression $e^{i\cdot k_0\cdot x}$ gives an additional modulation.

We plot for $A = 1, a = 2, k_0 = 0$, replacing $e^{i\cdot k_0\cdot x}$ by $cos(i \cdot k_0 \cdot x)$:



The same for $A = 1, a = 2, k_0 = 2,3,4$:

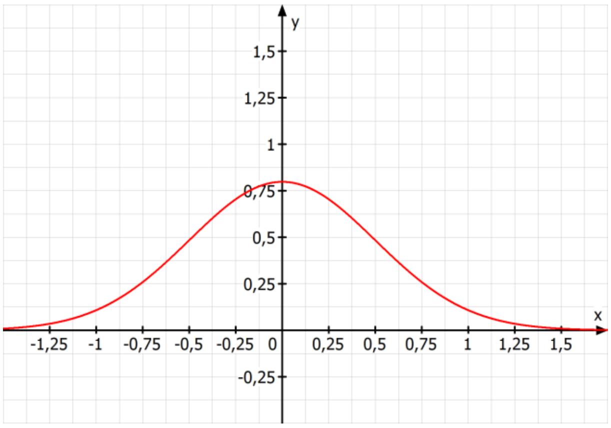


We interpret:

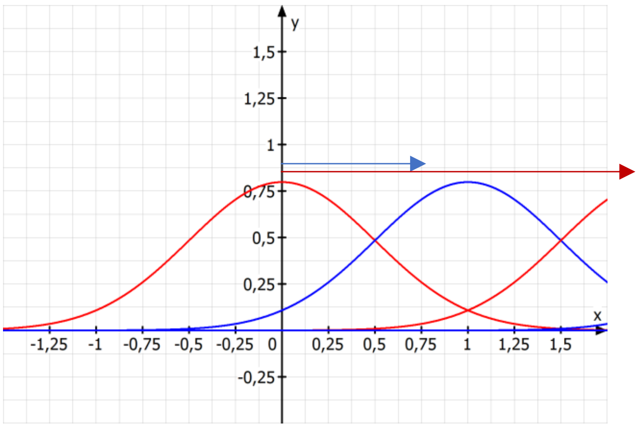
$$\frac{A \cdot a}{\sqrt{2\pi}} \cdot e^{-(k_0-k)^2 \cdot \frac{a^2}{2}}$$

This expression exists in the surrounding of $k = k_0$ with an uncertainty $\Delta k = {}^{1}/_{a}$ resp. $\Delta p = {}^{\hbar}/_{a}$.

We plot for $k_0 = 0, A = 1, a = 2$:



Modifying k_0 results in a shift along the x-axis:



Independently from a we get the uncertainty relation:

$$\Delta x \cdot \Delta k = 1$$

$$\Delta x \cdot \Delta p = \hbar$$

We work with $\lim_{a\to\infty} \Psi(x,0)$:

$$\Psi(x,0) = A \cdot e^{-\frac{x^2}{2a^2}} \cdot e^{i\cdot k_0 \cdot x}$$

We get the particular solution:

$$\Psi(x,0) = A \cdot e^{i\cdot k_0 \cdot x}$$

This is an oscillating wave through the whole space. In this case we have $a \sim \Delta x \to \infty$ and $\Delta k \to 0$, the spectrum becomes infinitely sharp.

Taking the limit of the spectral function, $\lim_{a\to\infty} C(k)$ we get:

$$\lim_{a\to\infty} C(k) = \lim_{a\to\infty} \frac{A \cdot a}{\sqrt{2\pi}} \cdot e^{-(k_0-k)^2 \cdot \frac{a^2}{2}} =$$

$$\delta(k - k_0)$$

We remember:

$$\delta(k-k_0) = \begin{cases} \infty & for\ k = k_0 \\ 0 & else \end{cases}$$

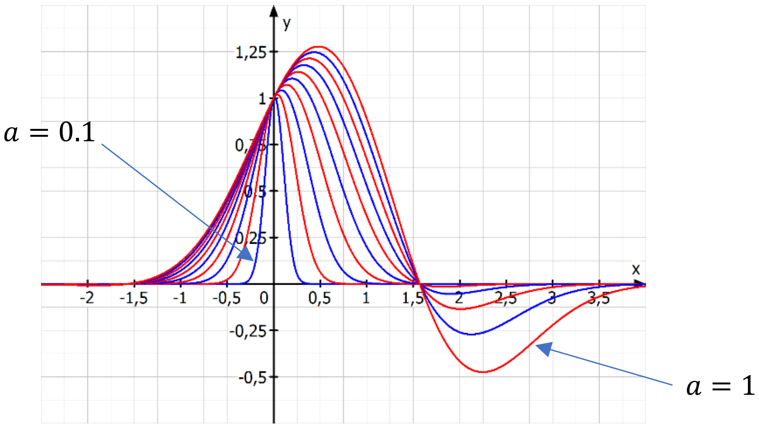
This is the Dirac function holding:

$$\int_{-\infty}^{\infty} \delta(k)dk = 1$$

We work with $\lim_{a\to 0} \Psi(x,0)$:

$$\Psi(x,0) = A \cdot e^{-\frac{x^2}{2a^2}} \cdot e^{i\cdot k_0 \cdot x}$$

We check this graphically and plot the exponentials $f(x,a) = \cos(x) \cdot exp(x)/exp(x^2/2a^2)$ for $a = 0.1, 0.2, \ldots, 1.0$:



We see the function converging to a Dirac-style function with:

$$\lim_{a\to 0} A \cdot e^{-\frac{x^2}{2a^2}} \cdot e^{i\cdot k_0 \cdot x} = \begin{cases} A & for\ x = 0 \\ 0 & else \end{cases}$$

We get:

$$\Psi(x,0) = \begin{cases} A & for\ x = 0 \\ 0 & else \end{cases}$$

We insert this into the integral:

$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(x,0) \cdot e^{-i\cdot k\cdot x} dx$$

We get:

$$C(k) = \frac{A}{2\pi}$$

This the uncertainty of momentum, an arbitrary constant.

We calculate $\Psi(x,t)$:

$$\Psi(x,t) = \int_{-\infty}^{\infty} C(k) \cdot e^{i\left(k\cdot x - \frac{t\cdot\hbar}{2\cdot m}\cdot k^2\right)} dk$$

We use $C(k)$:

$$C(k) = \frac{A\cdot a}{\sqrt{2\pi}} \cdot e^{-(k_0-k)^2\cdot\frac{a^2}{2}}$$

We calculate:

$$\Psi(x,t) = \frac{A\cdot a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-(k_0-k)^2\cdot\frac{a^2}{2} + i\cdot k\cdot x - i\cdot\frac{t\cdot\hbar}{2\cdot m}\cdot k^2\right) dk$$

We inspect the exponent:

$$-(k_0-k)^2\cdot\frac{a^2}{2} + i\cdot k\cdot x - i\cdot\frac{t\cdot\hbar}{2\cdot m}\cdot k^2 =$$

$$-\frac{{k_0}^2a^2}{2} + k_0ka^2 - \frac{k^2a^2}{2} + i\cdot k\cdot x - i\cdot\frac{t\cdot\hbar}{2\cdot m}\cdot k^2 =$$

$$-\frac{{k_0}^2a^2}{2} + k(k_0\cdot a^2 + i\cdot x) - k^2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m} + \frac{a^2}{2}\right) =$$

$$-\left(k^2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m} + \frac{a^2}{2}\right) - k(k_0\cdot a^2 + i\cdot x) + \frac{{k_0}^2a^2}{2}\right) \to$$

This looks like a quadratic expression in k: $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

We try completing the square:

$$k^2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m} + \frac{a^2}{2}\right) - k(k_0\cdot a^2 + i\cdot x) + \frac{{k_0}^2a^2}{2}$$

We find:

$$2\alpha\beta \coloneqq k(k_0\cdot a^2 + i\cdot x)$$

$$\alpha^2 \coloneqq k^2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m} + \frac{a^2}{2}\right) \to \alpha = k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m} + \frac{a^2}{2}\right)^{\frac{1}{2}}$$

From this we can calculate the "β" needed:

$$2\alpha\beta = 2k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}\cdot\beta = k(k_0\cdot a^2+i\cdot x)\rightarrow$$

$$\beta = \frac{k(k_0\cdot a^2+i\cdot x)}{2k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}} = \frac{k_0\cdot a^2+i\cdot x}{2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}$$

We rewrite the exponent:

$$-\left(k^2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)-k(k_0\cdot a^2+i\cdot x)+\frac{{k_0}^2a^2}{2}\right)\rightarrow$$

$$-\left(\left(k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}-\frac{k_0\cdot a^2+i\cdot x}{2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}\right)^2-\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}+\frac{{k_0}^2a^2}{2}\right)$$

We get $\Psi(x,t)$:

$$\Psi(x,t) = \frac{A\cdot a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-(k_0-k)^2\cdot\frac{a^2}{2}+i\cdot k\cdot x-i\cdot\frac{t\cdot\hbar}{2\cdot m}\cdot k^2\right)dk\rightarrow$$

$$\frac{A\cdot a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-\left(\left(k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}-\frac{k_0\cdot a^2+i\cdot x}{2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}\right)^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)dk$$

We now have the exponent in quadratic form – we solve it by help of the error function.

We substitute:

$$z := k\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}-\frac{k_0\cdot a^2+i\cdot x}{2\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}$$

We get:

$$\Psi(x,t) = \frac{A\cdot a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-\left(z^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)dk$$

We notice that the exponent is an expression like z^2-c

We get $\frac{dz}{dk}$:

$$\frac{dz}{dk} = \left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}\rightarrow dk = \frac{dz}{\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}$$

We insert dk into the integral:

$$\Psi(x,t)=\frac{A\cdot a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}exp\left(-\left(z^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)\cdot\frac{1}{\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}dz=$$

$$\frac{A\cdot a}{\sqrt{2\pi}}\cdot\frac{1}{\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}\cdot\int_{-\infty}^{\infty}exp\left(-\left(z^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)dz$$

We calculate the integral:

$$\int_{-\infty}^{\infty}exp\left(-\left(z^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)dz=;$$

We take a look at Wikipedia: *https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions*:

$$\int_{-\infty}^{\infty}exp\left(-(z^2+c)\right)dz=\sqrt{\pi}\cdot e^{-c}$$

Note: z vanishes.

We work with our function:

$$\int_{-\infty}^{\infty}exp\left(-\left(z^2-\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)\right)\right)dz=$$

$$\sqrt{\pi}\cdot exp\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)$$

We get the full solution:

$$\Psi(x,t)=\frac{A\cdot a}{\sqrt{2\pi}}\cdot\frac{1}{\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)^{\frac{1}{2}}}\cdot\sqrt{\pi}\cdot exp\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)=$$

$$\frac{A\cdot a}{\left(i\cdot\frac{t\cdot\hbar}{m}+a^2\right)^{\frac{1}{2}}}\cdot exp\left(\frac{(k_0\cdot a^2+i\cdot x)^2}{4\left(i\cdot\frac{t\cdot\hbar}{2\cdot m}+\frac{a^2}{2}\right)}-\frac{{k_0}^2a^2}{2}\right)=;$$

We work with the exponent:

$$\left(\frac{(k_0 \cdot a^2 + i \cdot x)^2}{4\left(i \cdot \frac{t \cdot \hbar}{2 \cdot m} + \frac{a^2}{2}\right)} - \frac{{k_0}^2 a^2}{2}\right) =$$

$$\left(\frac{{k_0}^2 \cdot a^4 + 2 \cdot k_0 \cdot a^2 \cdot i \cdot x - x^2}{4\left(i \cdot \frac{t \cdot \hbar}{2 \cdot m} + \frac{a^2}{2}\right)} - \frac{{k_0}^2 a^2}{2}\right) =$$

$$\left(\frac{{k_0}^2 \cdot a^4 + 2 \cdot k_0 \cdot a^2 \cdot i \cdot x - x^2 - 2 \cdot {k_0}^2 \cdot a^2 \cdot \left(i \cdot \frac{t \cdot \hbar}{2 \cdot m} + \frac{a^2}{2}\right)}{4\left(i \cdot \frac{t \cdot \hbar}{2 \cdot m} + \frac{a^2}{2}\right)}\right) =$$

$$\left(\frac{{k_0}^2 \cdot a^4 + 2 \cdot k_0 \cdot a^2 \cdot i \cdot x - x^2 - {k_0}^2 \cdot a^2 \cdot i \cdot \frac{t \cdot \hbar}{m} - {k_0}^2 \cdot a^4}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) =$$

$$\left(\frac{2 \cdot k_0 \cdot a^2 \cdot i \cdot x - x^2 - {k_0}^2 \cdot a^2 \cdot i \cdot \frac{t \cdot \hbar}{m}}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) =$$

$$\left(\frac{-x^2 + 2 \cdot k_0 \cdot a^2 \cdot i \cdot x - {k_0}^2 \cdot a^2 \cdot i \cdot \frac{t \cdot \hbar}{m}}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) =$$

$$-\left(\frac{x^2 - 2 \cdot k_0 \cdot a^2 \cdot i \cdot x + {k_0}^2 \cdot a^2 \cdot i \cdot \frac{t \cdot \hbar}{m}}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) =$$

$$-\left(\frac{x^2 - k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right)$$

We get the result:

$$\Psi(x,t) = \frac{A \cdot a}{\left(i \cdot \frac{t \cdot \hbar}{m} + a^2\right)^{\frac{1}{2}}} \cdot exp - \left(\frac{x^2 - k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right)$$

We interpret the result.

We calculate the probability density $\rho = |\Psi(x,t)|^2$:

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) =$$

$$\frac{A \cdot a}{\left(-i \cdot \frac{t \cdot \hbar}{m} + a^2\right)^{\frac{1}{2}}} \cdot exp\left(-\frac{x^2 + k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{-2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) \cdot \frac{A \cdot a}{\left(i \cdot \frac{t \cdot \hbar}{m} + a^2\right)^{\frac{1}{2}}} \cdot exp\left(-\frac{x^2 - k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right) =$$

$$\frac{(A \cdot a)^2}{\left(\left(-i \cdot \frac{t \cdot \hbar}{m} + a^2\right)\left(i \cdot \frac{t \cdot \hbar}{m} + a^2\right)\right)^{\frac{1}{2}}} \cdot exp\left(-\frac{x^2 + k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{-2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2} - \frac{x^2 - k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right)$$

We do this in parts.

First the factor in front:

$$\frac{(A\cdot a)^2}{\left(\left(-i\cdot\frac{t\cdot\hbar}{m}+a^2\right)\left(i\cdot\frac{t\cdot\hbar}{m}+a^2\right)\right)^{\frac{1}{2}}}=\frac{(A\cdot a)^2}{\left(a^4+\left(\frac{t\cdot\hbar}{m}\right)^2\right)^{\frac{1}{2}}}=$$

$$\frac{A^2}{\left(1+\left(\frac{t\cdot\hbar}{a^2m}\right)^2\right)^{\frac{1}{2}}}$$

Then the exponent:

$$-\left(\frac{x^2+k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)}{-2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2}\right)-\left(\frac{x^2-k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)}{2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2}\right)=$$

$$-\left(\frac{x^2+k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)}{-2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2}+\frac{x^2-k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)}{2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2}\right)=$$

$$-\frac{\left(x^2+k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)\right)\left(2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2\right)+\left(x^2-k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)\right)\left(-2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2\right)}{\left(-2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2\right)\left(2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2\right)}=$$

$$-\frac{x^2 2i\frac{t\cdot\hbar}{m}+x^2 2a^2+k_0a^2i\left(2x-k_0\frac{t\cdot\hbar}{m}\right)2i\frac{t\cdot\hbar}{m}+2k_0a^4i\left(2x+k_0\frac{t\cdot\hbar}{m}\right)-x^2 2i\frac{t\cdot\hbar}{m}+x^2 2a^2+k_0a^2i\left(2x+k_0\frac{t\cdot\hbar}{m}\right)2i\frac{t\cdot\hbar}{m}-2k_0a^4i\left(2x-k_0\frac{t\cdot\hbar}{m}\right)}{\left(4\cdot\left(\frac{t\cdot\hbar}{m}\right)^2+4a^4\right)}=$$

$$-\frac{4x^2a^2-4k_0a^2\left(2x-k_0\frac{t\cdot\hbar}{m}\right)\frac{t\cdot\hbar}{m}}{\left(4\cdot\left(\frac{t\cdot\hbar}{m}\right)^2+4a^4\right)}=$$

$$-\frac{x^2-k_0\left(2x-k_0\frac{t\cdot\hbar}{m}\right)\frac{t\cdot\hbar}{m}}{\left(\left(\frac{t\cdot\hbar}{ma}\right)^2+a^2\right)}=$$

$$-\frac{\left(x-k_0\frac{t\cdot\hbar}{m}\right)^2}{\left(\left(\frac{t\cdot\hbar}{ma^2}\right)^2+1\right)a^2}$$

We combine the result:

$$\rho=\frac{A^2}{\left(1+\left(\frac{t\cdot\hbar}{a^2m}\right)^2\right)^{\frac{1}{2}}}\cdot exp\left(-\frac{\left(x-\frac{k_0\cdot\hbar}{m}\cdot t\right)^2}{\left(\left(\frac{t\cdot\hbar}{ma^2}\right)^2+1\right)a^2}\right)$$

Note. In case A is complex valued we must replace A^2 by $|A|^2$.

We interpret the probability density ρ.

The exponential part gives the center of the wave packet resp. the particle. It is located at time $t = 0$ at position x.	$exp\left(-\frac{\left(x-\frac{k_0\cdot\hbar}{m}\cdot t\right)^2}{\left(\left(\frac{t\cdot\hbar}{ma^2}\right)^2+1\right)a^2}\right)$
The particle is moving in direction x with group velocity v_0.	$v_0 = \frac{k_0\cdot\hbar}{m}$
The wave packet broadens over time. Reason for this behavior is the denominator:	$\frac{A^2}{\left(1+\left(\frac{t\cdot\hbar}{a^2m}\right)^2\right)^{\frac{1}{2}}}$
For small values of t the denominator has value ~ 1.	$\lim_{t\to 0}\left(1+\left(\frac{t\cdot\hbar}{a^2m}\right)^2\right)^{\frac{1}{2}} = 1$
For large times t this becomes proportional to t.	$\lim_{t\to\infty}\left(1+\left(\frac{t\cdot\hbar}{a^2m}\right)^2\right)^{\frac{1}{2}} = t$
We remember:	$\Psi(x,0) = A\cdot e^{-\frac{x^2}{2a^2}}\cdot e^{i\cdot k_0\cdot x}$
From this we get the uncertainty in position:	$\Delta x \sim a$
We remember:	$C(k) = \frac{A\cdot a}{\sqrt{2\pi}}\cdot e^{-(k_0-k)^2\cdot\frac{a^2}{2}}$
From this we get the uncertainty in the wave number k:	$\Delta k \sim \frac{1}{a}$
We combine this with the group velocity v_0:	$v_0 = \frac{k_0\cdot\hbar}{m}$
We get Δv_0:	$\Delta v_0 = \frac{\Delta k\cdot\hbar}{m}\sim\frac{\hbar}{m\cdot a}$
The wave packet broadens after time t:	$\Delta x(t)\sim t\cdot\Delta v_0\sim\frac{\hbar\cdot t}{m\cdot a}$

We interpret the probability current j.

We remember:	$j = \frac{\hbar}{2mi}\left(\Psi^*\frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right)$
We use:	$\Psi(x,t) = \frac{A\cdot a}{\left(i\cdot\frac{t\cdot\hbar}{m}+a^2\right)^{\frac{1}{2}}}\cdot exp\left(-\left(\frac{x^2-k_0\cdot a^2\cdot i\cdot\left(2x-k_0\cdot\frac{t\cdot\hbar}{m}\right)}{2\cdot i\cdot\frac{t\cdot\hbar}{m}+2a^2}\right)\right)$

	$\Psi^*(x,t) = \frac{A \cdot a}{\left(-i \cdot \frac{t \cdot \hbar}{m} + a^2\right)^{\frac{1}{2}}} \cdot exp\left(-\left(\frac{x^2 + k_0 \cdot a^2 \cdot i \cdot \left(2x - k_0 \cdot \frac{t \cdot \hbar}{m}\right)}{-2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right)\right)$
We calculate $\frac{\partial}{\partial x}\Psi$:	$\frac{\partial}{\partial x}\Psi = \left(\frac{-x + k_0 \cdot a^2 \cdot i}{i \cdot \frac{t \cdot \hbar}{m} + a^2}\right) \cdot \frac{A \cdot a}{\left(i \cdot \frac{t \cdot \hbar}{m} + a^2\right)^{\frac{1}{2}}} \cdot exp\left(-\left(\frac{x^2 - k_0 \cdot a^2 \cdot i \cdot 2x + {k_0}^2 \cdot a^2 \cdot i \cdot \frac{t \cdot \hbar}{m}}{2 \cdot i \cdot \frac{t \cdot \hbar}{m} + 2a^2}\right)\right) =$ $\left(\frac{-x + k_0 \cdot a^2 \cdot i}{i \cdot \frac{t \cdot \hbar}{m} + a^2}\right) \cdot \Psi(x,t)$
We rewrite the first factor:	$\frac{-x + k_0 \cdot a^2 \cdot i}{i \cdot \frac{t \cdot \hbar}{m} + a^2} = i \cdot k_0 \frac{\frac{ix}{k_0} + a^2}{i \cdot \frac{t \cdot \hbar}{m} + a^2} = i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right)$
We get $\frac{\partial}{\partial x}\Psi$:	$\frac{\partial}{\partial x}\Psi = i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi(x,t)$
The same way we get $\frac{\partial}{\partial x}\Psi^*$:	$\frac{\partial}{\partial x}\Psi^* = -i \cdot k_0 \left(\frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi^*(x,t)$
We calculate $\Psi^* \frac{\partial}{\partial x}\Psi$:	$\Psi^* \cdot i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi = i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi^* \cdot \Psi =;$
We use:	$\rho = \Psi^* \cdot \Psi$
We get:	$\Psi^* \frac{\partial}{\partial x}\Psi = i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \rho$
We calculate $\Psi \frac{\partial}{\partial x}\Psi^*$:	$\Psi \cdot -i \cdot k_0 \left(\frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi^* = -i \cdot k_0 \left(\frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \Psi \cdot \Psi^* =$ $-i \cdot k_0 \left(\frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}\right) \cdot \rho$
We want to calculate the probability current j:	$j = \frac{\hbar}{2mi}\left(\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^*\right)$

We calculate $\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^*$:	$i \cdot k_0 \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} \right) \cdot \rho + i \cdot k_0 \left(\frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} \right) \cdot \rho =$ $i \cdot k_0 \cdot \rho \left(\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} + \frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} \right) =;$
We calculate the parentheses:	$\frac{\frac{i \cdot x}{a^2 \cdot k_0} + 1}{\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} + \frac{-\frac{i \cdot x}{a^2 \cdot k_0} + 1}{-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1} =$ $\frac{\left(\frac{i \cdot x}{a^2 \cdot k_0} + 1\right)\left(-\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1\right) + \left(-\frac{i \cdot x}{a^2 \cdot k_0} + 1\right)\left(\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1\right)}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} =$ $\frac{-\frac{i \cdot x}{a^2 \cdot k_0}\frac{i \cdot t \cdot \hbar}{m \cdot a^2} + \frac{i \cdot x}{a^2 \cdot k_0} - \frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1 - \frac{i \cdot x}{a^2 \cdot k_0}\frac{i \cdot t \cdot \hbar}{m \cdot a^2} - \frac{i \cdot x}{a^2 \cdot k_0} + \frac{i \cdot t \cdot \hbar}{m \cdot a^2} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} =$ $2 \cdot \frac{\frac{x \cdot t \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1}$
We combine and get the result:	$\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^* = i \cdot k_0 \cdot \rho \cdot 2 \cdot \frac{\frac{x \cdot t \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1}$
We now have j:	$j = \frac{\hbar}{2mi} i \cdot k_0 \cdot \rho \cdot 2 \cdot \frac{\frac{x \cdot t \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} =$ $\frac{\hbar}{m} k_0 \cdot \rho \cdot \frac{\frac{x \cdot t \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1}$
We note that:	$\frac{\hbar}{m} k_0 = v_0$
We rewrite:	$j = v_0 \cdot \rho \cdot \frac{\frac{x \cdot t \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1}$

We interpret

On the one hand, the gradual change of the current density with time is due to the fact that ρ changes. Therefore, there is more or less matter at the location in question.

On the other hand, the average velocity changes at each position. We know the maximum of the wave packet being at position $x = v_0 \cdot t$.

We insert this into j and use $v_0 = \frac{k_0 \cdot \hbar}{m}$:

$$j_{\left(x=v_0 \cdot t, v_0=\frac{k_0 \cdot \hbar}{m}\right)} = v_0 \ \cdot \rho \cdot \frac{\frac{v_0 \cdot t^2 \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} = v_0 \ \cdot \rho \cdot \frac{\frac{\frac{k_0 \cdot \hbar}{m} \cdot t^2 \cdot \hbar}{k_0 \cdot m \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} =$$

$$v_0 \ \cdot \rho \cdot \frac{\frac{\hbar^2 \cdot t^2}{m^2 \cdot a^4} + 1}{\left(\frac{t \cdot \hbar}{m \cdot a^2}\right)^2 + 1} = v_0 \ \cdot \rho$$

At position $x = v_0 \cdot t$ the probability density is $v_0 \ \cdot \rho$.

The deviations therefore are located at the front and rear slopes.

Remark:

Left of the maximum we have a flow back of the probability current reducing the velocity, right of the maximum we have an additional flow forward that adds to the velocity.