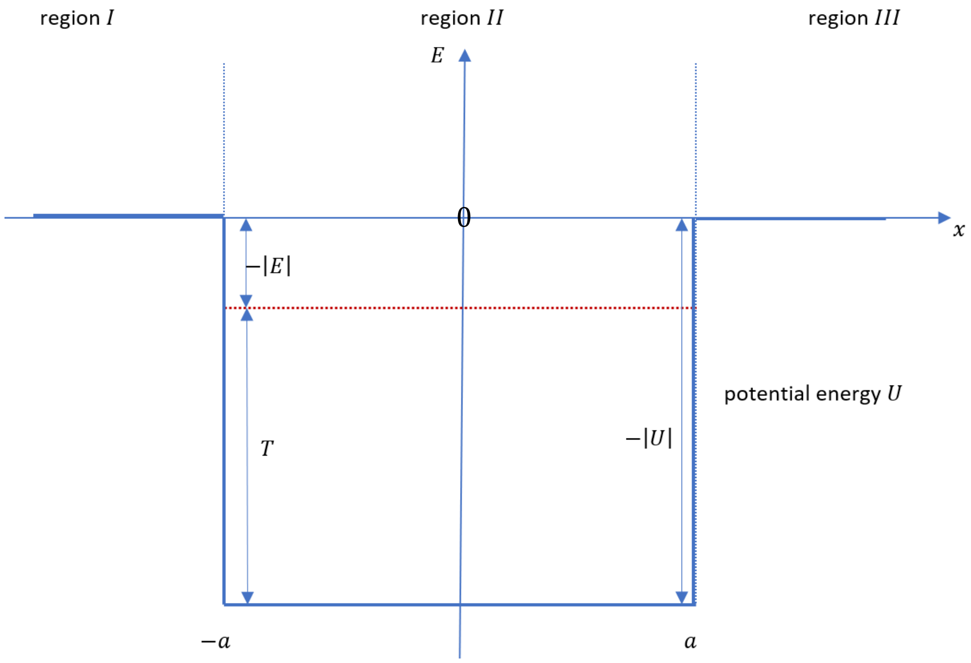
These worked examples follow “Rechenmethoden der Quantentheorie, Siegfried Flügge, Springer, Berlin, 1965”. Library of Congress Catalog Card Number 65-24546, title-nr. 7288

Hope I can help you with learning quantum mechanics.

Calculate energy values and eigenfunctions for the one-dimensional potential well.



Maybe it is helpful to have the following picture in mind:

- We have a moving particle. This particle has positive kinetic energy $+|T|$. The particle is located in region II (it can't be located in region I or III).
- Inside region II we have a constant potential V. It is negative, we name it $-|U|$. We use a potential with $|U| > |T|$.
- From classic physics we know that the force F from potential V on the particle is given by its derivation: $F = \frac{dV(x)}{dx}$. In our case the potential is constant, $\frac{dV(x)}{dx} = 0$. The particle will move inside the well undisturbed.
- At the borders $a, -a$ we have the derivation of the potential with infinite value (the slope is infinite). This gives an infinite force acting on the particle (reversing its direction in our 1D example).
- If we look from outside (region I or III) at this particle inside the well (region II), the particle there has total energy $E = E_{kin} + E_{pot}$. As the potential is negative $(-U)$ the total energy E, viewed from the regions outside is negative, namely $E = |T| - |U|$. With $|U| > |T|$ we get that E is negative. We write this as $-|E|$.
- But within region II the energy is positive with respect to the (negative) potential $-|U|$. It is the kinetic energy $|T|$.

With this picture in mind we take a look at the Schrödinger equation adapted to our 1D example:

$$u'' + \frac{2mE}{\hbar^2} u = 0$$

usually written as:

$$u'' + k^2 u = 0, \qquad k^2 = \frac{2mE}{\hbar^2}$$

Viewed from outside, regions I or III, the total energy $-|E| < 0$ is negative, so we write using κ instead of k:

$$\kappa^2 = \frac{2m|E|}{\hbar^2}$$

Consistently get the differential equation:

$$u'' - \kappa^2 u = 0$$

$$u'' - \frac{2m|E|}{\hbar^2} u = 0$$

This Schrödinger equation has the solutions in region I resp. region III:

$$u_I = A_I e^{\kappa x} + B_I e^{-\kappa x}$$

$$u_{III} = A_{III} e^{\kappa x} + B_{III} e^{-\kappa x}$$

In region II we have the (regular) Schrödinger equation:

$$u'' + k^2 u = 0$$

Solution, written as exponentials:

$$u_{II} = A_{II} e^{ikx} + B_{II} e^{-ikx}$$

Written as trigonometric functions:

$$u_{II} = A_{II} cos(kx) + B_{II} sin(kx)$$

Note: The equivalence of exponential and trigonometric solution you may find at:

https://quantum-abc.de/twice_potential_well.pdf

We know that eigenfunctions need to be normalizable. Necessary condition for this is that they decay to zero for $x \to \pm\infty$.

Assuming this we get:

$B_I = 0$ because we have negative values of x there.

$A_{III} = 0$ because we have positive values of x there.

So there remain three equations:

$$u_I = A_I e^{\kappa x}$$

$$u_{II} = A_{II} cos(kx) + B_{II} sin(kx)$$

$$u_{III} = B_{III} e^{-\kappa x}$$

We need the derivatives:

$$u_I' = \kappa A_I e^{\kappa x}$$

$$u_{II}' = -k A_{II} sin(kx) + k B_{II} cos(kx)$$

$$u_{III}' = -\kappa B_{III} e^{-\kappa x}$$

Any valid solution requests that at the boundaries left $(-a)$ and right (a) the functions must match in terms of the functions themselves (continuity) and their first derivatives (slopes):

$$u_I(-a) = u_{II}(-a)$$

$$u_I{}'(-a) = u_{II}{}'(-a)$$

$$u_{II}(a) = u_{III}(a)$$

$$u_{II}{}'(a) = u_{III}{}'(a)$$

We insert the definitions:

$$A_I e^{-\kappa a} = A_{II} cos(-ka) + B_{II} sin(-ka)$$

$$\kappa A_I e^{-\kappa a} = -kA_{II} sin(-ka) + kB_{II} cos(-ka)$$

$$A_{II} cos(ka) + B_{II} sin(ka) = B_{III} e^{-\kappa a}$$

$$-kA_{II} sin(ka) + kB_{II} cos(ka) = \kappa B_{III} e^{-\kappa a}$$

We want the arguments of the trigonometric functions to be positive. We use symmetries of sin and cos:

$$\sin(x) = -\sin(-x)$$

$$\cos(x) = \cos(-x)$$

We rewrite:

$$A_I e^{-\kappa a} = A_{II} cos(ka) - B_{II} sin(ka)$$

$$\kappa A_I e^{-\kappa a} = kA_{II} sin(ka) + kB_{II} cos(ka)$$

$$B_{III} e^{-\kappa a} = A_{II} cos(ka) + B_{II} sin(ka)$$

$$\kappa B_{III} e^{-\kappa a} = -kA_{II} sin(ka) + kB_{II} cos(ka)$$

These are four equations for four variables.

We rearrange:

$$A_I \cdot e^{-\kappa a} - A_{II} \cdot cos(ka) + B_{II} \cdot sin(ka) + B_{III} \cdot 0 = 0$$

$$A_I \cdot \kappa \cdot e^{-\kappa a} - A_{II} \cdot k \cdot sin(ka) - B_{II} \cdot k \cdot cos(ka) + B_{III} \cdot 0 = 0$$

$$A_I \cdot 0 - A_{II} \cdot cos(ka) - B_{II} \cdot sin(ka) + B_{III} \cdot e^{-\kappa a} = 0$$

$$A_I \cdot 0 + A_{II} \cdot k \cdot sin(ka) - B_{II} \cdot k \cdot cos(ka) - B_{III} \cdot \kappa \cdot e^{-\kappa a} = 0$$

This homogeneous system of linear equations has a solution if the determinant is zero.

We calculate the determinant by help of wxmaxima:

$$\left| \begin{pmatrix} e^{-\kappa a} & -cos(ka) & sin(ka) & 0 \\ \kappa \cdot e^{-\kappa a} & -k \cdot sin(ka) & -k \cdot cos(ka) & 0 \\ 0 & -cos(ka) & -sin(ka) & e^{-\kappa a} \\ 0 & k \cdot sin(ka) & -k \cdot cos(ka) & -\kappa \cdot e^{-\kappa a} \end{pmatrix} \right|$$

We get the determinant:

$$e^{-a\kappa}(k\cos(ak)\,(\cos(ak)\,\kappa e^{-a\kappa} - k\sin(ak)\,e^{-a\kappa}) - k\sin(ak)\,(\sin(ak)\,\kappa e^{-a\kappa} + k\cos(ak)\,e^{-a\kappa})) + \cos(ak)\,\kappa\%e^{-a\kappa}(\sin(ak)\,\kappa e^{-a\kappa} + k\cos(ak)\,e^{-a\kappa}) + \sin(ak)\,\kappa e^{-a\kappa}(\cos(ak)\,\kappa e^{-a\kappa} - k\sin(ak)\,e^{-a\kappa})$$

We simplify this by help of wxmaxima:

$$(\cos(ak)\sin(ak)\,\kappa^2 + (\cos(ak)^2 - \sin(ak)^2)k\kappa - k^2\cos(ak)\sin(ak))2e^{-2a\kappa}$$

We solve with respect to κ and get two solutions:

$$\kappa = -k\frac{\cos(ak)}{\sin(ak)}, \kappa = k\frac{\sin(ak)}{\cos(ak)}$$

We rewrite:

$$\kappa = -k\,cot(ak), \kappa = k\,tan(ak)$$

These are the eigenvalue conditions, the conditions necessary that eigenvalues exist.

Note: The following transformations are guided by the goal of determining the conditions at the boundaries of $-a$ and a. There, exponential functions are to be connected to trigonometric functions.

We check $\kappa = -k\,cot(ak)$:

$$A_I \cdot e^{ak\,cot(ak)} - A_{II} \cdot cos(ka) + B_{II} \cdot sin(ka) = 0$$

$$-A_I \cdot k\,cot(ak) \cdot e^{ak\,cot(ak)} - A_{II} \cdot k \cdot sin(ka) - B_{II} \cdot k \cdot cos(ka) = 0$$

$$-A_{II} \cdot cos(ka) - B_{II} \cdot sin(ka) + B_{III} \cdot e^{ak\,cot(ak)} = 0 \qquad (3)$$

$$A_{II} \cdot k \cdot sin(ka) - B_{II} \cdot k \cdot cos(ka) + B_{III} \cdot k\,cot(ak) \cdot e^{ak\,cot(ak)} = 0$$

We subtract equation one and three:

$$A_I \cdot e^{ak\,cot(ak)} + 2 \cdot B_{II} \cdot sin(ka) - B_{III} \cdot e^{ak\,cot(ak)} = 0 \Rightarrow \qquad (1)$$

We add equation two and four we get:

$$-A_I \cdot k\,cot(ak) \cdot e^{ak\,cot(ak)} - 2 \cdot B_{II} \cdot k \cdot cos(ka) + B_{III} \cdot k\,cot(ak) \cdot e^{ak\,cot(ak)} = 0 \Rightarrow$$

$$A_I \cdot e^{ak\,cot(ak)} + 2 \cdot B_{II} \cdot \frac{k \cdot cos(ka)}{k\,cot(ak)} - B_{III} \cdot e^{ak\,cot(ak)} = 0 \qquad (2)$$

From (1) and (2) we see that B_{II} must be zero and $B_{III} = A_I$. With this we go into (3):

$$A_{II} \cdot cos(ka) = A_I \cdot e^{ak\,cot(ak)}$$

Similar we get for $\kappa = k\,tan(ak)$:

$$A_{II} = 0, B_{III} = -A_I, B_{II} \cdot sin(ka) = A_I \cdot e^{-ak\,cot(ak)}$$

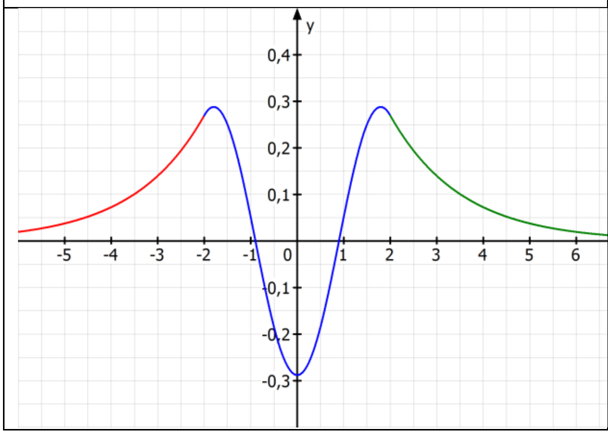
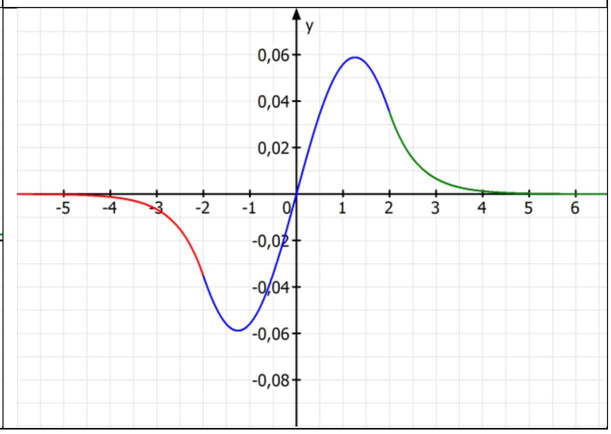
The eigenvalue conditions give us two kinds eigenfunctions, the first type being symmetric, the second antisymmetric with respect to a switch of x to $-x$.

Let us compare this behavior:

$\kappa = k\,tan(ak)$	$\kappa = -k\,cot(ak)$
$u_I = A_I \cdot e^{k\,tan(ak)\cdot x}$	$u_I = -A_I \cdot e^{-k\,cot(ak)\cdot x}$
$u_{II} = A_I \cdot e^{-ak\,tan(ak)} \cdot \frac{cos(kx)}{cos(ka)}$	$u_{II} = A_I \cdot e^{ak\,cot(ak)} \cdot \frac{sin(kx)}{sin(ka)}$
$u_{III} = A_I \cdot e^{-k\,tan(ak)\cdot x}$	$u_{III} = A_I \cdot e^{k\,cot(ak)\cdot x}$
Eigenfunctions are symmetric $u(x) = u(-x)$	Eigenfunctions are antisymmetric $u(x) = -u(-x)$

For better readability we write this shorter:

$\kappa = k\,tan(ak)$	$\kappa = -k\,cot(ak)$
$u_I = A_I \cdot e^{\kappa\cdot x}$	$u_I = -A_I \cdot e^{\kappa\cdot x}$
$u_{II} = A_I \cdot e^{-a\kappa} \cdot \frac{cos(kx)}{cos(ka)}$	$u_{II} = A_I \cdot e^{-a\kappa} \cdot \frac{sin(kx)}{sin(ka)}$
$u_{III} = A_I \cdot e^{-\kappa\cdot x}$	$u_{III} = A_I \cdot e^{-\kappa\cdot x}$
Eigenfunctions are symmetric $u(x) = u(-x)$	Eigenfunctions are antisymmetric $u(x) = -u(-x)$

We plot a symmetric eigenfunction with $A_I = 1, a = 2, k = 1.75$:	We plot an antisymmetric eigenfunction with $A_I = 1, a = 2, k = 1.25$:
	

We calculate the amplitude A_I from the normalization restriction for the symmetric function:	We calculate the amplitude A_I from the normalization restriction for the antisymmetric function:
$\int_{-\infty}^{\infty} u^2 dx = \int_{-\infty}^{-a} {u_I}^2 dx + \int_{-a}^{a} {u_{II}}^2 dx + \int_{a}^{\infty} {u_{III}}^2 dx = 1$	$\int_{-\infty}^{\infty} u^2 dx = \int_{-\infty}^{-a} {u_I}^2 dx + \int_{-a}^{a} {u_{II}}^2 dx + \int_{a}^{\infty} {u_{III}}^2 dx = 1$

$\int_{-\infty}^{-a} {u_I}^2 dx = {A_I}^2 \cdot \int_{-\infty}^{-a} e^{2\kappa \cdot x} dx =$	$\int_{-\infty}^{-a} {u_I}^2 dx = {A_I}^2 \cdot \int_{-\infty}^{-a} e^{2\kappa \cdot x} dx =$
${A_I}^2 \cdot \frac{1}{2\kappa} \cdot [e^{2\kappa \cdot x}]_{-\infty}^{-a} =$	${A_I}^2 \cdot \frac{1}{2\kappa} \cdot [e^{2\kappa \cdot x}]_{-\infty}^{-a} =$
${A_I}^2 \cdot \frac{1}{2\kappa} \cdot e^{-2\kappa a} =$	${A_I}^2 \cdot \frac{1}{2\kappa} \cdot e^{-2\kappa a} =$
${A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa}$	${A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa}$

$\int_{-a}^{a} {u_{II}}^2 dx = \frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \int_{-a}^{a} cos^2(kx) dx =$	$\int_{-a}^{a} {u_{II}}^2 dx = \frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \int_{-a}^{a} cos^2(kx) dx =$
$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left[\frac{x}{2} + \frac{1}{2k} \cdot sin(kx)cos(kx)\right]_{-a}^{a} =$	$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left[\frac{x}{2} + \frac{1}{2k} \cdot sin(kx)cos(kx)\right]_{-a}^{a} =$
$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left(\frac{a}{2} + \frac{1}{4k} \cdot sin(2ka) + \frac{a}{2} - \frac{1}{4k} \cdot sin(-2ka)\right) =$	$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left(\frac{a}{2} + \frac{1}{4k} \cdot sin(2ka) + \frac{a}{2} - \frac{1}{4k} \cdot sin(-2ka)\right) =$
$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left(a + \frac{1}{2k} \cdot sin(2ka)\right) =$	$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{cos^2(ka)} \cdot \left(a + \frac{1}{2k} \cdot sin(2ka)\right) =$
$\frac{{A_I}^2 \cdot e^{-2a\kappa}}{2 \cdot k \cdot cos^2(ka)} \cdot (2ka + sin(2ka))$	$-\frac{{A_I}^2 \cdot e^{-2a\kappa}}{2 \cdot k \cdot sin^2(ka)} \cdot (-2ka + sin(2ka))$

$\int_{a}^{\infty} {u_{III}}^2 dx = {A_I}^2 \cdot \int_{a}^{\infty} e^{-2\kappa \cdot x} dx =$	$\int_{a}^{\infty} {u_{III}}^2 dx = {A_I}^2 \cdot \int_{a}^{\infty} e^{-2\kappa \cdot x} dx =$
$-{A_I}^2 \cdot \frac{1}{2\kappa} \cdot [e^{-2\kappa \cdot x}]_{a}^{\infty} =$	$-{A_I}^2 \cdot \frac{1}{2\kappa} \cdot [e^{-2\kappa \cdot x}]_{a}^{\infty} =$
${A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa}$	${A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa}$

Result:	Result:
$\int_{-\infty}^{\infty} u^2 dx = {A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa} + \frac{{A_I}^2 \cdot e^{-2a\kappa}}{2 \cdot k \cdot cos^2(ka)} \cdot (2ka + sin(2ka)) + {A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa} =$	$\int_{-\infty}^{\infty} u^2 dx = {A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa} - \frac{{A_I}^2 \cdot e^{-2a\kappa}}{2 \cdot k \cdot sin^2(ka)} \cdot (-2ka + sin(2ka)) + {A_I}^2 \cdot \frac{e^{-2\kappa a}}{2\kappa} =$
$\frac{{A_I}^2 e^{-2a\kappa}}{\kappa} + \frac{{A_I}^2(\sin(2ak) + 2ak)e^{-2a\kappa}}{2k\cos(ak)^2} =$	$\frac{{A_I}^2 e^{-2a\kappa}}{\kappa} - \frac{{A_I}^2(\sin(2ak) - 2ak)e^{-2a\kappa}}{2k sin^2(ka)} =$
${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{2ka + sin(2ka)}{2 \cdot k \cdot cos^2(ka)}\right) =$	${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} - \frac{sin(2ka) - 2ka}{2 \cdot k \cdot sin^2(ka)}\right) =$
${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{2ka}{2 \cdot k \cdot cos^2(ka)} + \frac{sin(2ka)}{2 \cdot k \cdot cos^2(ka)}\right) =$	${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{2ka}{2 \cdot k \cdot sin^2(ka)} - \frac{sin(2ka)}{2 \cdot k \cdot sin^2(ka)}\right) =$
${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{a}{cos^2(ka)} + \frac{1}{k}\frac{sin(2ka)}{2 \cdot cos^2(ka)}\right) =$	${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{a}{sin^2(ka)} - \frac{1}{k}\frac{sin(2ka)}{2 \cdot sin^2(ka)}\right) =$
${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{a}{cos^2(ka)} + \frac{1}{k}\tan(ka)\right) = 1 \rightarrow$	${A_I}^2 \cdot e^{-2\kappa a} \cdot \left(\frac{1}{\kappa} + \frac{a}{sin^2(ka)} - \frac{1}{k}\cot(ka)\right) = 1 \rightarrow$
$A_I = \sqrt{\frac{e^{2\kappa a}}{\left(\frac{1}{\kappa} + \frac{a}{cos^2(ka)} + \frac{1}{k}\tan(ka)\right)}}$	$A_I = \sqrt{\frac{e^{2\kappa a}}{\left(\frac{1}{\kappa} + \frac{a}{sin^2(ka)} - \frac{1}{k}\cot(ka)\right)}}$

This obviously is not the same.[1]

[1] Note: In the original text is stated that the constant A_1 is the same for the symmetric and the antisymmetric case: $1/{A_I}^2 = a \cdot exp(-2\kappa a) \cdot \left(1 + 1/\kappa a + \kappa/k^2 a + \kappa^2/k^2\right)$. I couldn't reproduce this.

We take a look at the symmetric case, $u_I = A_I \cdot e^{k\,tan(ak)\cdot x}$.	We take a look at the asymmetric case, $u_I = -A_I \cdot e^{-k\,cot(ak)\cdot x}$.
u_1 must go to zero for $x \to -\infty$, so $tan(ak)$ must be a positive value. a and k are positive numbers. Similar for u_3.	u_1 must go to zero for $x \to -\infty$, so $tan(ak)$ must be a positive value. a and k are positive numbers. Similar for u_3.
This restricts the product ak to value between 0 and $\frac{\pi}{2}$: $$0 < ak < \frac{\pi}{2}$$	This restricts the product ak to value between 0 and $\frac{\pi}{2}$: $$0 < ak < \frac{\pi}{2}$$

In both cases we have:

$$0 < ak < \frac{\pi}{2}$$

a is an arbitrary value, twice the width of the well.

k is the energy in region II:

$$k = \sqrt{\frac{2mT}{\hbar^2}}$$

We check the units of k:

$$[k] = \sqrt{\frac{kg \cdot kg \cdot m^2 \cdot s^2}{s^2 \cdot kg^2 \cdot m^4}} = \sqrt{\frac{1}{m^2}} = \frac{1}{m}$$

This is ok, an inverted wavelength as expected, ak then is unit-free.

We calculate the eigenvalues.

We use the necessary eigenvalue conditions:

$\kappa = k\,tan(ak)$	$\kappa = -k\,cot(ak)$

We remember:

$$\kappa^2 = \frac{2m|E|}{\hbar^2}$$

$$k^2 = \frac{2mT}{\hbar^2}$$

$\kappa = -k\,cot(ak), \kappa = k\,tan(ak)$ are transcendental equations.

We see that:

$$\kappa^2 + k^2 = \frac{2m|E|}{\hbar^2} + \frac{2mT}{\hbar^2} = \frac{2m|U|}{\hbar^2}$$

This is a constant depending on the potential.

We multiply by the square of the width of the well, a^2 to get a dimensionless constant:

$$a^2(\kappa^2 + k^2) = \frac{2a^2m|U|}{\hbar^2} = C^2$$

We check the units:

$$\left[\frac{2am|U|}{\hbar^2}\right] = \frac{m^2 \cdot kg \cdot \frac{kg \cdot m^2}{s^2}}{\frac{kg^2 \cdot m^4}{s^4} \cdot s^2} = 1$$

$C^2$is a dimensionless constant.

We now get rid of κ in the necessary eigenvalue conditions $\kappa = -k\, cot(ak), \kappa = k\, tan(ak)$:

$$a^2(\kappa^2 + k^2) = C^2$$

$$\kappa^2 + k^2 = \frac{C^2}{a^2}$$

$$\kappa^2 = \frac{C^2}{a^2} - k^2$$

$$\kappa = \sqrt{\frac{C^2}{a^2} - k^2}$$

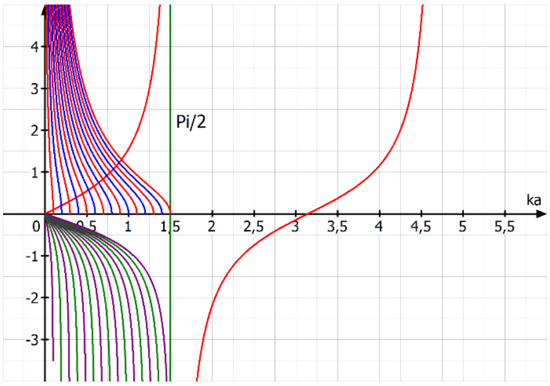
We insert this in the necessary eigenvalue conditions and get an expression for $tan(ak)$:

$\kappa = k\, tan(ak)$	$\kappa = -k\, cot(ak)$
$\sqrt{\frac{C^2}{a^2} - k^2} = k\, tan(ak)$	$\sqrt{\frac{C^2}{a^2} - k^2} = -k\, cot(ak)$
$\frac{\sqrt{\frac{C^2}{a^2} - k^2}}{k} = tan(ak)$	$-\sqrt{\frac{C^2}{k^2a^2} - 1} = cot(ak)$
$tan(ak) = \sqrt{\frac{C^2}{k^2a^2} - 1}$	$tan(ak) = -\frac{1}{\sqrt{\frac{C^2}{k^2a^2} - 1}}$

We solve this graphically.

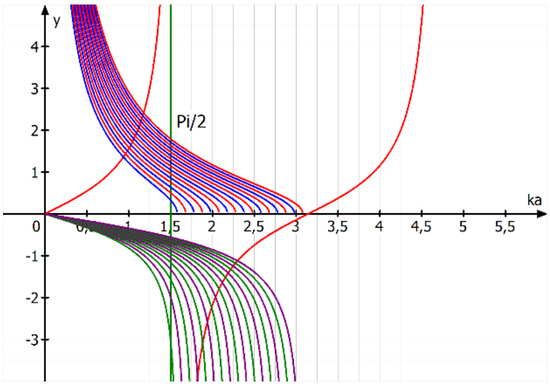
We plot: $0 < C < \frac{\pi}{2}$.
In the upper half we have the symmetric eigenvalue condition $\kappa = k\, tan(ak)$, in the lower half the antisymmetric eigenvalue condition $\kappa = -k\, cot(ak)$.

We see that in this range we have only intersections (solutions) of the symmetric eigenvalue condition.



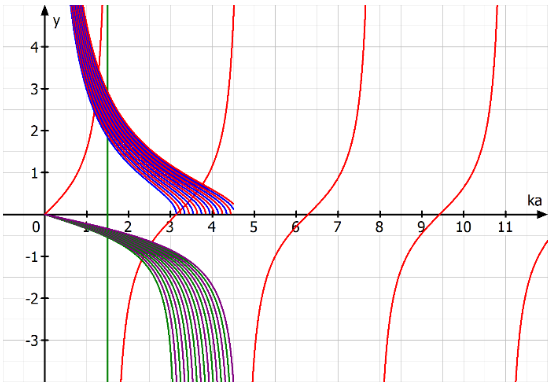
We plot: $\frac{\pi}{2} < C < \pi$.
In the upper half we have the symmetric eigenvalue condition $\kappa = k\, tan(ak)$, in the lower half the antisymmetric eigenvalue condition $\kappa = -k\, cot(ak)$.

We see that in this range we have one intersection (solution) of the symmetric eigenvalue condition as well as one intersection (solution) of the antisymmetric eigenvalue condition.



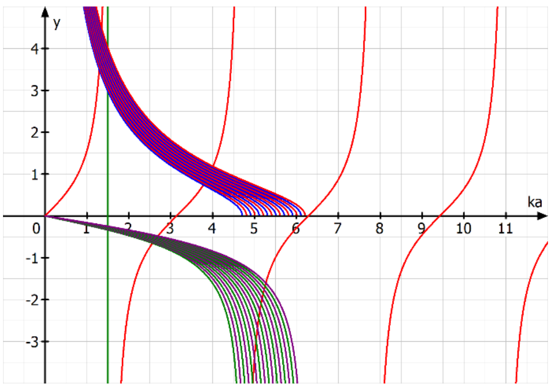
We plot: $\pi < C < \frac{3}{2}\pi$.
In the upper half we have the symmetric eigenvalue condition $\kappa = k\, tan(ak)$, in the lower half the antisymmetric eigenvalue condition $\kappa = -k\, cot(ak)$.

We see that in this range we have two intersections (solutions) of the symmetric eigenvalue condition as well as one intersection (solution) of the antisymmetric eigenvalue condition.



We plot: $\frac{3}{2}\pi < C < 2\pi$.
In the upper half we have the symmetric eigenvalue condition $\kappa = k\, tan(ak)$, in the lower half the antisymmetric eigenvalue condition $\kappa = -k\, cot(ak)$.

We see that in this range we have two intersections (solutions) of the symmetric eigenvalue condition as well as two intersections (solutions) of the antisymmetric eigenvalue condition.



Interpretation

The dimensionless constant C^2 is built by:

$$\frac{2a^2m|U|}{\hbar^2} = C^2$$

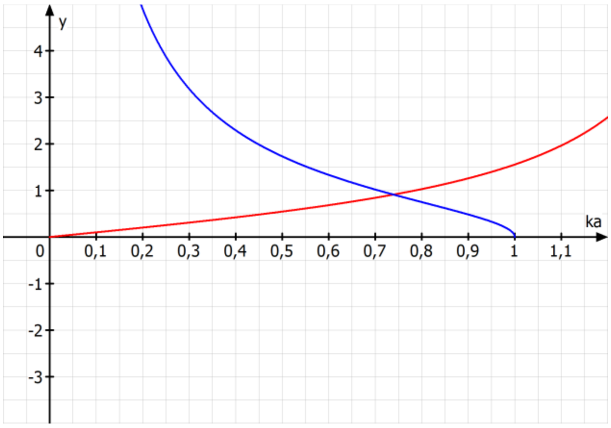
$\frac{2a^2m|U|}{\hbar^2}$ is the amount of potential energy within the well in units of h (Planck's constant), so the lowest possible value of C^2 is one. In this case we have only a symmetric solution.

We calculate the energy for $C^2 = 1$.

We use:

$$-|E| = -|U| + |T| \rightarrow |E| = |U| - |T|$$

In a well with $C^2 = 1$ exists only one symmetric eigenfunction.



We calculate the kinetic energy E_α:

$$|E_\alpha| = |U| - |T_\alpha| =;$$

We assume the kinetic energy being smaller then the potential energy and omit the absolute value signs:

$$E_\alpha = U - T_\alpha =;$$

We have the kinetic energy:

$${k_\alpha}^2 = \frac{2mT_\alpha}{\hbar^2} \rightarrow T_\alpha = \frac{{k_\alpha}^2\hbar^2}{2m}$$

In our graphic scheme we use ka instead of k:

$$T_\alpha = \frac{\hbar^2}{2ma^2} \cdot (ak_\alpha)^2$$

The energy level of the first symmetric eigenfunction:

$$E_\alpha = U - T_\alpha = U - \frac{\hbar^2}{2ma^2} \cdot (ak_\alpha)^2$$

Using $\frac{2a^2m|U|}{\hbar^2} = C^2$ we get:

$$\frac{2a^2m|U|}{\hbar^2} = C^2 \rightarrow \frac{\hbar^2}{2a^2m} = \frac{|U|}{C^2}$$

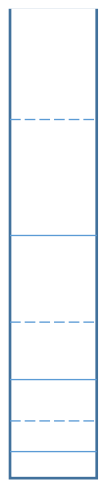
We omit the absolute value signs for U and write:

$$\frac{\hbar^2}{2ma^2} = \frac{U}{C^2}$$

With this we rewrite:

$$E_\alpha = U - \frac{\hbar^2}{2ma^2} \cdot (ak_\alpha)^2 = U\left(1 - \frac{(ak_\alpha)^2}{C^2}\right)$$

The deeper the well the more eigenfunctions are possible. We get the picture:



Note: Dotted lines belong to asymmetric eigenfunctions.

Additional remark

The probability of the particle to be found in the interval dx is u^2dx. For $E < 0$, it decays exponentially like e^{-2xa} outside $|x| = a$. According to quantum mechanics, the particle can move into these areas forbidden by classical mechanics. With noticeable probability, a distance outside of $|x| = a$ can still be reached:

$$\Delta x \sim \frac{\hbar}{\sqrt{2m|E|}}$$

The deeper the well, the larger $|E|$ and the smaller Δx.