These worked examples follow “Rechenmethoden der Quantentheorie, Siegfried Flügge, Springer, Berlin, 1965”. Library of Congress Catalog Card Number 65-24546, title-nr. 7288

Remark

The treatment of the quantum bouncing ball, a high-school problem of classic physics, exceeds the scope of what can be expected of beginners. Nevertheless, beginners are also invited to read this text. They can learn from it how physicists proceed to adapt mathematics to the conditions of nature.

We here are dealing with this topic based on a book from 1965. In this book, the basics are worked out in a more complicated way. Modern books use the Airy function, which combines many properties of Bessel and Hankel functions.

You may find a modern approach to this topic at: https://ia600208.us.archive.org/3/items/arxiv-0909.2209/0909.2209.pdf

Hope I can help you with learning quantum mechanics.

Note: This is an experimental text that may contain errors.

A particle (a super hard ball) bounces over a super hard surface. No damping, no loss of energy. Calculate the movement with quantum mechanics methods.

Classic physics

A simulation using classic physics and made with geogebra you find here.

The particle with mass m and heigth x over the surface lives in a gravitational potential $m \cdot g \cdot x$.

We assume the potential g to be constant. The potential is the total energy as we assume no initial velocity.

When the particle starts falling we have potential energy only.

When the particle reaches the ground we have kinetic energy only.

We have elastic reflection, so the energy is conserved: $E_{pot} + E_{kin} = E = const.$

We get the laws of motion:

$$x(t) = x_0 - \frac{1}{2} \cdot g \cdot t^2$$

$$v(t) = -g \cdot t$$

$$a(t) = const.$$

We get the time the particle needs to reach zero:

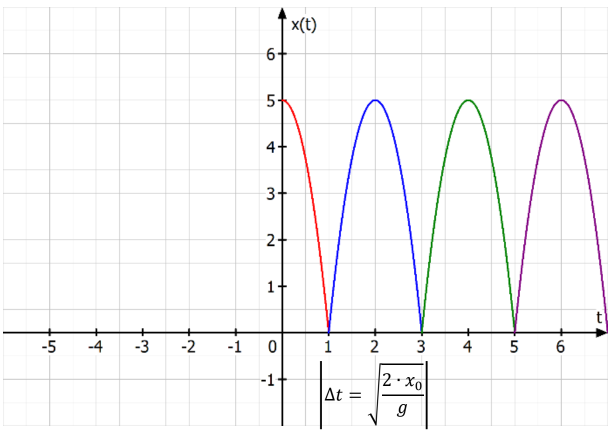
$$t = \sqrt{\frac{2 \cdot x_0}{g}}$$

Due to the reflection on the ground we get in the next period:

$$x(t) = x_0 - \frac{1}{2} \cdot g \cdot \left(t - 2 \cdot \sqrt{\frac{2 \cdot x_0}{g}}\right)^2$$

$$v(t) = -g \cdot \left(t - 2 \cdot \sqrt{\frac{2 \cdot x_0}{g}}\right)$$

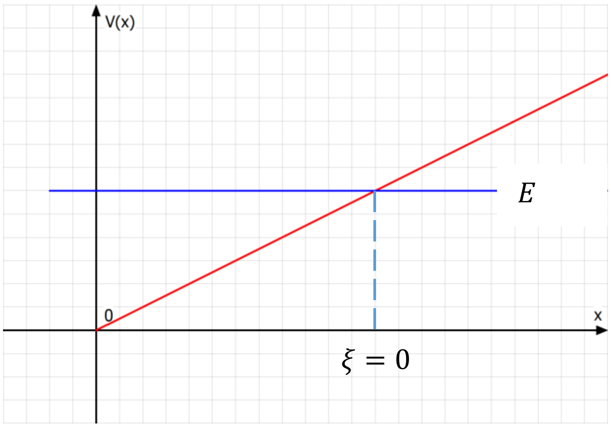
We plot this:



Note: What might be problematic for classic physics is the point of reflection.

Quantum mechanics

We use the following picture:



Note: Any given energy E determines the height the ball is bouncing and separates the classically forbidden range right of $x > \xi$ from the classically allowed range $x < \xi$.

We use abbreviations:

$$\frac{2 \cdot m^2 \cdot g}{\hbar^2} = \frac{1}{l^3}$$

$$\frac{2 \cdot m \cdot E}{\hbar^2} = \frac{\lambda}{l^2}$$

We check units. (mass) (meter)

$$\left[\frac{m^2 \cdot g}{\hbar^2}\right] = \frac{kg^2 \cdot m \cdot s^2}{s^2 \cdot kg^2 \cdot m^4} = \frac{1}{m^3}$$

$$\left[\frac{m \cdot E}{\hbar^2}\right] = \frac{kg \cdot kg \cdot m^2 \cdot s^2}{s^2 \cdot kg^2 \cdot m^4} = \frac{1}{m^2}$$

Note: l is of unit lentgh, λ a real number.

Note: We can look at λ as the ratio of the total energy E of the particle divided by the potential energy in length unit l:

$$\lambda = \frac{E}{m \cdot g \cdot l}$$

Note: We can look at the characteristic length unit l as the ratio of total energy E divided by the force acting multiplied by the ratio λ:

$$l = \frac{E}{m \cdot g \cdot \lambda}$$

We get ξ:

$$\xi = \frac{x}{l} - \lambda = \frac{x - l \cdot \lambda}{l}$$

Note: ξ is unit free.

We shift the x-axis by $l \cdot \lambda$ to the right and rescale it by $\frac{1}{l}$.

We note:

$$\xi < 0 \rightarrow x < \frac{\lambda}{l}$$

The turning point ξ becomes the new zero. This helps because we get the energy E negative within the allowed area left and positive in the forbidden area right.

The Schrödinger equation for the original problem:

$$-\frac{\hbar^2}{2 \cdot m} \cdot \frac{d^2 u}{dx^2} + (m \cdot g \cdot x - E) \cdot u = 0$$

We use boundary conditions:

$$u(0) = 0, u(x \rightarrow \infty) \rightarrow 0$$

Note: the Schrödinger equation does not calculate a position but a probability amplitude for the particle at position x. The square of the probability amplitude gives the probability density. The integral over the probability density within an area then gives the probability the particle being in this area.

We apply the coordinate change and get a new differential equation:

$$-\frac{d^2 u}{d\xi^2} + \xi \cdot u = 0$$

We check this in detail.

We use:

$$\frac{2 \cdot m^2 \cdot g}{\hbar^2} = \frac{1}{l^3}$$

$$\frac{2 \cdot m \cdot E}{\hbar^2} = \frac{\lambda}{l^2}$$

$$\xi = \frac{x}{l} - \lambda$$

We rearrange:

$$\xi = \frac{x}{l} - \lambda \rightarrow x = l \cdot (\xi + \lambda)$$

$$\frac{2 \cdot m \cdot E}{\hbar^2} = \frac{\lambda}{l^2} \rightarrow E = \frac{\lambda \cdot \hbar^2}{2 \cdot m \cdot l^2}$$

$$\frac{2 \cdot m^2 \cdot g}{\hbar^2} = \frac{1}{l^3} \rightarrow m \cdot g = \frac{\hbar^2}{2 \cdot m \cdot l^3}$$

We combine:

$$m \cdot g \cdot x = \frac{\hbar^2 \cdot l \cdot (\xi + \lambda)}{2 \cdot m \cdot l^3} = \frac{\hbar^2 \cdot (\xi + \lambda)}{2 \cdot m \cdot l^2}$$

$$E = \frac{\lambda \cdot \hbar^2}{2 \cdot m \cdot l^2}$$

$$m \cdot g \cdot x - E = \frac{\hbar^2 \cdot (\xi + \lambda)}{2 \cdot m \cdot l^2} - \frac{\lambda \cdot \hbar^2}{2 \cdot m \cdot l^2} =$$

$$\frac{\hbar^2 \cdot (\xi + \lambda) - \lambda \cdot \hbar^2}{2 \cdot m \cdot l^2} = \frac{\hbar^2 \cdot (\xi + \lambda - \lambda)}{2 \cdot m \cdot l^2} = \frac{\hbar^2 \cdot \xi}{2 \cdot m \cdot l^2}$$

We examine $\frac{du}{d\xi}$:

$$\frac{du}{dx} = \frac{du}{d\xi} \cdot \frac{d\xi}{dx} = \frac{du}{d\xi} \cdot \frac{1}{l}$$

We examine $\frac{d^2u}{d\xi^2}$:

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dx}\left(\frac{du}{d\xi} \cdot \frac{1}{l}\right) =$$

$$\frac{1}{l} \cdot \frac{d}{dx}\left(\frac{du}{d\xi}\right) = \frac{1}{l} \cdot \frac{d}{d\xi}\left(\frac{du}{d\xi}\right) \cdot \frac{d\xi}{dx} = \frac{1}{l^2} \cdot \frac{d^2u}{d\xi^2}$$

We insert into the original Schrödinger equation:

$$-\frac{\hbar^2}{2 \cdot m} \cdot \frac{d^2u}{dx^2} + (m \cdot g \cdot x - E) \cdot u = 0 \rightarrow$$

$$-\frac{\hbar^2}{2 \cdot m} \cdot \frac{1}{l^2} \cdot \frac{d^2u}{d\xi^2} + \left(\frac{\hbar^2 \cdot \xi}{2 \cdot m \cdot l^2}\right) \cdot u = 0 \rightarrow$$

$$-\frac{d^2u}{d\xi^2} + \xi \cdot u = 0$$

This is a second order linear ordinary differential equation.

We set the boundary conditions:

$$u(\xi = -\lambda) = 0, u(\xi \to \infty) = 0$$

Note: $\xi = -\lambda$ implies $x = 0$ and we are back to the boundary condition of the original differential equation.

The solution for $-\frac{d^2u}{d\xi^2} + \xi \cdot u = 0$:

$$u(\xi) = C \cdot \sqrt{\xi} \cdot Z_{\frac{1}{3}}\left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}\right)$$

Note: $Z_{\frac{1}{3}}$ are Bessel resp. Hankel functions.

> To get the solutions for $-\frac{d^2u}{d\xi^2} + \xi \cdot u = 0$ we need knowledge about Bessel functions and Hankel functions and are definitely out of the scope of this paper. You may find more information at:
> https://www.johndcook.com/blog/2013/09/01/relating-airy-and-bessel-functions/.
> Another source (in german) might be: https://itp.tugraz.at/~schnizer/AnalyticalMethods/AnMe26.pdf.
> Another source (in geman) maight be:
> https://support.ptc.com/help/mathcad/r10.0/de/index.html#page/PTC_Mathcad_Help/example_hankel_functions.html#wwID0ETL46

We will discuss separately the areas $\xi > 0$, the classical inaccessible area and $\xi < 0$, the classical allowed area.

First part: $\xi > 0$

For $\xi \gg 1$ we can use the asymptotic formulas of the Hankel function of first kind:

$$H_{\frac{1}{3}}^{(1)}(z) = \sqrt{\frac{2}{\pi \cdot z}} \cdot exp\left(i \cdot \left(z - \frac{5}{12} \cdot \pi\right)\right)$$

The Hankel function becomes:

$$H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}\right) = \sqrt{\frac{2}{\pi \cdot \frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}}} \cdot exp\left(i \cdot \left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}} - \frac{5}{12} \cdot \pi\right)\right)$$

We will replace this expression in our solution for $u(\xi)$:

$$u(\xi) = C \cdot \sqrt{\xi} \cdot H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}\right) \rightarrow$$

$$u(\xi) = C \cdot \sqrt{\xi} \cdot \sqrt{\frac{2}{\pi \cdot \frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}}} \cdot exp\left(i \cdot \left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}} - \frac{5}{12} \cdot \pi\right)\right) =$$

$$C \cdot \xi^{\frac{1}{2}} \cdot \xi^{\frac{1}{2}\left(-\frac{3}{2}\right)} \cdot \sqrt{\frac{3}{\pi \cdot i}} \cdot exp\left(i \cdot \left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}} - \frac{5}{12} \cdot \pi\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi \cdot i}} \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}} - i \cdot \frac{5}{12} \cdot \pi\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot i^{-\frac{1}{2}} \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right) \cdot exp\left(\left(-i \cdot \frac{5}{12} \cdot \pi\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot exp\left(\left(-i \cdot \frac{1}{4} \cdot \pi\right)\right) \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right) \cdot exp\left(\left(-i \cdot \frac{5}{12} \cdot \pi\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot exp\left(\left(-i \cdot \frac{1}{4} \cdot \pi - i \cdot \frac{5}{12} \cdot \pi\right)\right) \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot exp\left(\left(-i \cdot \pi\left(\frac{1}{4} + \frac{5}{12}\right)\right)\right) \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right) =$$

$$C \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot exp\left(\left(-i \cdot \frac{2}{3} \cdot \pi\right)\right) \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right)$$

This solution for positive ξ becomes real if we set C to eliminate the complex term $\left(-i \cdot \frac{2}{3} \cdot \pi\right)$:

$$C = A \cdot exp\left(\left(i \cdot \frac{2}{3} \cdot \pi\right)\right)$$

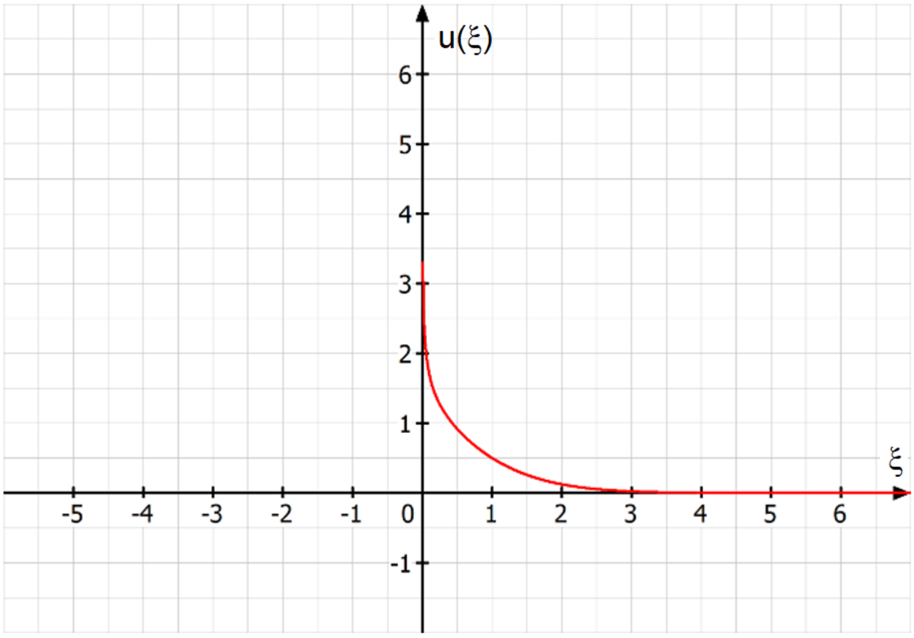
We get the real solution:

$$u(\xi) = A \cdot \xi^{-\frac{1}{4}} \cdot \sqrt{\frac{3}{\pi}} \cdot exp\left(\left(-\frac{2}{3} \cdot \xi^{\frac{3}{2}}\right)\right)$$

Note: A is a real value.

This solution has the correct behavior, it goes to zero for growing ξ.

Let us take a look at the graph:



The graph shows an exponential decay for positive values of ξ and has a singularity at $\xi = 0$.

Second part: $\xi < 0$

The solution for $\xi > 0$ was:

$$u(\xi) = C \cdot \sqrt{\xi} \cdot Z_{\frac{1}{3}}\left(\frac{2}{3} \cdot i \cdot \xi^{\frac{3}{2}}\right)$$

In the complex plane we write $\xi < 0$ as $|\xi| \cdot (-1)$ or $|\xi| \cdot e^{-i\pi}$.

The solution for $\xi < 0$ then is:

$$u(\xi < 0) = C \cdot \sqrt{-|\xi|} \cdot H_{\frac{1}{3}}^{(1)}\left(-\frac{2}{3} \cdot |\xi|^{\frac{3}{2}}\right)$$

Note: $H_{\frac{1}{3}}^{(1)}$ is the Hankel function or Bessel funciton of the third kind.

Note: You may find more information at: Hankel functions - Encyclopedia of Mathematics.

We use the general Hankel function for noninteger ν:

$$H_\nu^{(1)}(z \cdot e^{-i\pi}) = \frac{i}{sin(\pi \cdot \nu)}\left(exp(-2 \cdot \pi \cdot i \cdot \nu) \cdot J_\nu(z) - exp(\pi \cdot i \cdot \nu) \cdot J_{-\nu}(z)\right)$$

Note: $J_\nu(z), J_{-\nu}(z)$ are Bessel functions of first kind.

For $\nu = \frac{1}{3}$ we get:

$$H_{\frac{1}{3}}^{(1)}\left(z \cdot e^{-i\pi}\right) = \frac{i}{sin\left(\frac{\pi}{3}\right)}\left(exp\left(-\frac{2}{3} \cdot \pi \cdot i\right) \cdot J_{\frac{1}{3}}(z) - exp\left(\pi \cdot i \cdot \frac{1}{3}\right) \cdot J_{-\frac{1}{3}}(z)\right) =$$

$$\frac{i \cdot exp\left(-\frac{2}{3} \cdot \pi \cdot i\right)}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}(z) - exp\left(\pi \cdot i \cdot \frac{3}{3}\right) \cdot J_{-\frac{1}{3}}(z)\right) =$$

$$\frac{i \cdot exp\left(-\frac{2}{3} \cdot \pi \cdot i\right)}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)\right) =$$

$$\frac{exp\left(-\frac{1}{2} \cdot \pi \cdot i\right) \cdot exp\left(-\frac{2}{3} \cdot \pi \cdot i\right)}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)\right) =$$

$$\frac{exp\left(-\frac{7}{6} \cdot \pi \cdot i\right)}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z)\right)$$

We get:

$$u\left(z \cdot e^{-i\pi}\right) = C \cdot \sqrt{z \cdot e^{-i\pi}} \cdot H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3} \cdot \left(z \cdot e^{-i\pi}\right)^{\frac{3}{2}}\right) =$$

$$C \cdot \sqrt{z \cdot e^{-i\pi}} \cdot \frac{exp\left(-\frac{7}{6} \cdot \pi \cdot i\right)}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}\left(\frac{2}{3} \cdot \left(z \cdot e^{-i\pi}\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \cdot \left(z \cdot e^{-i\pi}\right)^{\frac{3}{2}}\right)\right)$$

Again, this solution is real if we choose C:

$$C = A \cdot exp\left(\frac{7}{6} \cdot \pi \cdot i\right)$$

We get:

$$u\left(z \cdot e^{-i\pi}\right) = \sqrt{z \cdot e^{-i\pi}} \cdot \frac{A}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}\left(\frac{2}{3} \cdot \left(z \cdot e^{-i\pi}\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \cdot \left(z \cdot e^{-i\pi}\right)^{\frac{3}{2}}\right)\right)$$

We use that $z \cdot e^{-i\pi} = -z$, replace z by ξ and resubstitute:

$$\xi = \frac{x}{l} - \lambda \rightarrow -\xi = \lambda - \frac{x}{l}$$

We get:

$$u\left(\lambda - \frac{x}{l}\right) = \sqrt{\lambda - \frac{x}{l}} \cdot \frac{A}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right)\right)$$

This is a real eigenfunction with eigenvalue zero. This forces λ to fulfill the condition:

$$u(x=0)=0 \rightarrow u(\lambda)=0 \rightarrow$$

$$J_{\frac{1}{3}}\left(\frac{2}{3}\cdot\lambda^{\frac{3}{2}}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\cdot\lambda^{\frac{3}{2}}\right)=0$$

Let us harvest the results.

Eigenfunction

We have the length unit:

$$l=\left(\frac{\hbar^2}{2\cdot g\cdot m^2}\right)^{\frac{1}{3}}$$

For an electron we get:

$$\frac{\hbar^2}{2\cdot g\cdot m^2}=\frac{(1.055\cdot 10^{-34})^2}{2\cdot 9.81\cdot(9.1094\cdot 10^{-31})^2}\sim 0.00088\text{ (meter)}$$

We check units:

$$\left[\left(\frac{\hbar^2}{g\cdot m^2}\right)^{\frac{1}{3}}\right]=\frac{kg^2\cdot m^4\cdot s^2\cdot s^2}{s^4\cdot m\cdot kg^2}=(m^3)^{\frac{1}{3}}=m$$

According to classic physics the position $x_0=\lambda\cdot l=\frac{E}{m\cdot g}$ is the highest point of the trajectory.

Going beyond this position only for few millimeter results in $|\xi|\gg 1$ and thus a very small probability amplitude. Therefore we are interested in the asymptotic behavior of the solution near the classical limit x_0.

Case $\xi>0$

$$u(\xi)=A\cdot\xi^{-\frac{1}{4}}\cdot\sqrt{\frac{3}{\pi}}\cdot exp\left(\left(-\frac{2}{3}\cdot\xi^{\frac{3}{2}}\right)\right)$$

We resubstitute to get a function of x:

$$u(x)=A\cdot\left(\frac{x-x_0}{l}\right)^{-\frac{1}{4}}\cdot\sqrt{\frac{3}{\pi}}\cdot exp\left(\left(-\frac{2}{3}\cdot\left(\frac{x-x_0}{l}\right)^{\frac{3}{2}}\right)\right)$$

This means roughly an exponential decay for $x>x_0$.

Case $\xi<0$

We have the solution for $\xi<0$:

$$u\left(\lambda-\frac{x}{l}\right)=\sqrt{\lambda-\frac{x}{l}}\cdot\frac{A}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\cdot\left(\lambda-\frac{x}{l}\right)^{\frac{3}{2}}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\cdot\left(\lambda-\frac{x}{l}\right)^{\frac{3}{2}}\right)\right)$$

We have asymptotic behavior:

$$J_{\frac{1}{3}}(z) \rightarrow \sqrt{\frac{2}{\pi \cdot z}} \cdot cos\left(z - \frac{5 \cdot \pi}{12}\right)$$

$$J_{-\frac{1}{3}}(z) \rightarrow \sqrt{\frac{2}{\pi \cdot z}} \cdot cos\left(z - \frac{\pi}{12}\right)$$

We use:

$$z = \frac{2}{3} \cdot \left(\frac{x - x_0}{l}\right)$$

If the electron is some millimeter below the classical point of return, we get:

$$u(x) = \left(\frac{x - x_0}{l}\right)^{-\frac{1}{4}} \cdot \frac{A}{sin\left(\frac{\pi}{3}\right)} \cdot \sqrt{\frac{3}{\pi}} \cdot \left(cos\left(\frac{2}{3} \cdot \left(\frac{x_0 - x}{l}\right)^{\frac{3}{2}} - \frac{5 \cdot \pi}{12}\right) + cos\left(\frac{2}{3} \cdot \left(\frac{x_0 - x}{l}\right)^{\frac{3}{2}} - \frac{\pi}{12}\right)\right)$$

We build the average of the expectation value of u^2 over a range $\Delta x \gg l$:

$$\overline{u^2} = \frac{4}{\pi} \cdot A^2 \cdot \sqrt{\frac{l}{x - x_0}}$$

The classical residence time dt in an interval dx is proportional $\overline{u^2}$:

$$dt \sim \frac{dx}{\sqrt{x_0 - x}}$$

We get the speed of the particle:

$$\frac{dx}{dt} \sim \sqrt{x_0 - x}$$

We use that $x_0 - x \coloneqq s$ is the distance of the moving particle and get finally the classical formula:

$$\frac{dx}{dt} \sim \sqrt{2 \cdot g \cdot s}$$

We compare with the classic solution:

We get the laws of motion (with the origin x_0 set to zero)

$x(t) = \frac{1}{2} \cdot g \cdot t^2$	$v(t) = -g \cdot t$	$a(t) = const.$	$t = \sqrt{\frac{2 \cdot x}{g}}$

$$v(t) = -g \cdot t = -g \cdot \sqrt{\frac{2 \cdot x}{g}} = -\sqrt{2 \cdot x \cdot g}$$

Eigenvalue

We have discrete eigenvalues for $\xi < 0$.

The zeros are given by the condition:

$$u\left(\lambda - \frac{x}{l}\right) = 0 \rightarrow$$

$$\sqrt{\lambda - \frac{x}{l}} \cdot \frac{A}{sin\left(\frac{\pi}{3}\right)}\left(J_{\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right)\right) = 0 \rightarrow$$

$$J_{\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \cdot \left(\lambda - \frac{x}{l}\right)^{\frac{3}{2}}\right) = 0$$

We write:

$$f(z) := J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z) = 0$$

We assume the zeros $z_n(n = 1,2,3, \dots)$.

We have the according energy in terms of λ_n:

$$E_n = \frac{\hbar^2}{2 \cdot m \cdot l^2} \cdot \lambda_n$$

We use:

$$z_n = \left(\frac{2}{3} \cdot \lambda_n\right)^{\frac{3}{2}} \rightarrow \lambda_n = \left(\frac{3}{2} \cdot z_n\right)^{\frac{2}{3}}$$

We get the energy:

$$E_n = \frac{\hbar^2}{2 \cdot m \cdot l^2} \cdot \left(\frac{3}{2} \cdot z_n\right)^{\frac{2}{3}} = \frac{\hbar^2}{2 \cdot m \cdot l^2} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}} \cdot z_n^{\frac{2}{3}} = \varepsilon \cdot z_n^{\frac{2}{3}}$$

We use the length unit:

$$l = \left(\frac{\hbar^2}{2 \cdot g \cdot m^2}\right)^{\frac{1}{3}}$$

We get the energy units ε:

$$\frac{\hbar^2}{2 \cdot m \cdot l^2} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}} \rightarrow \frac{\hbar^2}{2 \cdot m \cdot \left(\frac{\hbar^2}{2 \cdot g \cdot m^2}\right)^{\frac{2}{3}}} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}} =$$

$$\frac{\left(\hbar^{\frac{6}{2}} \cdot 2 \cdot g \cdot m^2 \cdot \frac{3}{2}\right)^{\frac{2}{3}}}{\left(2^{\frac{3}{2}} \cdot m^{\frac{3}{2}} \cdot \hbar^2\right)^{\frac{2}{3}}} = \frac{\left(\hbar^6 \cdot 2^2 \cdot g^2 \cdot m^4 \cdot \frac{9}{4}\right)^{\frac{1}{3}}}{(2^3 \cdot m^3 \cdot \hbar^4)^{\frac{1}{3}}} =$$

$$\left(\frac{\hbar^6 \cdot 2^2 \cdot g^2 \cdot m^4 \cdot \frac{9}{4}}{2^3 \cdot m^3 \cdot \hbar^4}\right)^{\frac{1}{3}} = \left(\frac{\hbar^2 \cdot g^2 \cdot m \cdot \frac{9}{4}}{2}\right)^{\frac{1}{3}} =$$

$$\left(\hbar^2 \cdot g^2 \cdot m \cdot \frac{9}{8}\right)^{\frac{1}{3}}$$

Result:

$$\varepsilon = \left(\hbar^2 \cdot g^2 \cdot m \cdot \frac{9}{8}\right)^{\frac{1}{3}}$$

We check units:

$$\left[(\hbar^2 \cdot g^2 \cdot m)^{\frac{1}{3}}\right] = \left(\frac{kg^2 \cdot m^4 \cdot m^2 \cdot kg}{s^2 \cdot s^4}\right)^{\frac{1}{3}} = \left(\frac{kg^3 \cdot m^6}{s^6}\right)^{\frac{1}{3}} = \frac{kg \cdot m^2}{s^2} = Joule$$

For electrons we get:

$$\varepsilon_{el} \approx \left((1.055 \cdot 10^{-34})^2 \cdot 9.81^2 \cdot 9.109 \cdot 10^{-31} \cdot \frac{9}{8}\right)^{\frac{1}{3}} = 1.032 \cdot 10^{-32}\, Joule$$

If n becomes big we have $z \gg 1$. In this case the particle reaches the point of return, the highest point in the allowed area in our solution, meaning $x \to 0$.

We take the solution for $u(x)$:

$$u(x) = \left(\frac{x - x_0}{l}\right)^{-\frac{1}{4}} \cdot \frac{A}{sin\left(\frac{\pi}{3}\right)} \cdot \sqrt{\frac{3}{\pi}} \cdot \left(cos\left(\frac{2}{3} \cdot \left(\frac{x_0 - x}{l}\right)^{\frac{3}{2}} - \frac{5 \cdot \pi}{12}\right) + cos\left(\frac{2}{3} \cdot \left(\frac{x_0 - x}{l}\right)^{\frac{3}{2}} - \frac{\pi}{12}\right)\right)$$

We apply the zero condition and $x = 0$ and get the condition for $f(z) = 0$:

$$cos\left(\frac{2}{3} \cdot \left(\frac{x_0}{l}\right)^{\frac{3}{2}} - \frac{5 \cdot \pi}{12}\right) + cos\left(\frac{2}{3} \cdot \left(\frac{x_0}{l}\right)^{\frac{3}{2}} - \frac{\pi}{12}\right) = 0$$

Trigonometric identities give:

$$cos\left(\frac{2}{3} \cdot \left(\frac{x_0}{l}\right)^{\frac{3}{2}} - \frac{\pi}{4}\right) = 0$$

We get:

$$z_n = \left(\frac{2}{3} \cdot \lambda_n\right)^{\frac{3}{2}} = \left(\frac{2}{3} \cdot \frac{x_0}{l}\right)^{\frac{3}{2}} = \frac{\pi}{4} + (2 \cdot n + 1) \cdot \frac{\pi}{2}$$

We get energy levels E_n:

$$E_n = \varepsilon \cdot \left(\left(2 \cdot n + \frac{3}{2}\right) \cdot \frac{\pi}{2}\right)^{\frac{2}{3}}$$