This paper deals with the space quantum mechanics takes place.

Related information you may find at:

https://math.mit.edu/~rbm/18-102-S14/Chapter3.pdf

https://mathweb.ucsd.edu/~bdriver/231-02-03/Lecture_Notes/Hilbert-Spaces.pdf

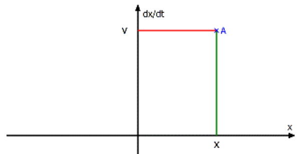
Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German…)

Hope I can help you with learning quantum mechanics.

Contents

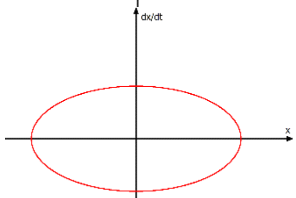
Hilbert space

Vector space

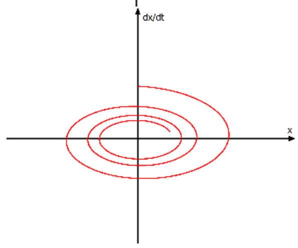


Imagine a particle that can move in one dimension. The particle has position and speed. We describe this particle and its state best by taking position and speed as independent coordinates and build a 2D coordinate system.

Every point (vector) in the plane represents a particle in a unique state with position and velocity. A free particle can have any position in this space.



If we fix the particle with a spring to the origin, give it constant mass and energy, the particle is restricted to a path in the plane. Every position on the path is possible, all other positions are impossible, given mass, energy and spring parameters does not change.



If we have friction the energy will decay, and the particle come to rest.

"Measuring" in this picture mathematically takes place by projecting the vector to one of its basis axes, a task performed by matrix operations. In our example: If we want to extract the position out of the vector, we must project it to the x-axis.

If we work with more particles, more parameters, the vector space will get more dimensions, but properties like subspace, orthogonality, projection, norm, basis will remain and help imagine the behavior of the system.

Complex numbers

Every complex number has a complex conjugated companion. We chose $a, b \in \mathbb{R}$, i the imaginary unit and write:

In cartesian style	In polar style
$z = a + ib \to z^* = a - ib$	$z = ae^{i\varphi} \to z^* = ae^{-i\varphi}$

Note: $*$ denotes complex conjugation.

Vector space

Real space vector in column or row presentation:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \; or \; (1\ 2\ 3)$$

This vector can be visualized as an arrow, starting at the origin $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and ending at $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

The representation of a vector always is made upon a basis. Throughout this text we use the standard orthonormal basis:

$$\begin{pmatrix}1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\1\end{pmatrix}$$

We take the variable $a \in \mathbb{R}$. All vectors $a \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ build a line in space, a 1D subspace.

Dot product

In real space we use the scalar product or dot product.

$$a := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, b := \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, a \cdot b = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 6$$

In general:

$$a \cdot b := \sum_{i=1}^{n} a_i b_i$$

Note: the presentation depends on the basis we are working with. We use the orthonormal basis.

Complex space vector:

$$|u\rangle := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 + i \cdot 3 \\ 2 - i \cdot 4 \\ 3 + i \cdot 2 \end{pmatrix} \; or \; \langle u| = \big((1 - i \cdot 3)(2 + i \cdot 4)(3 - i \cdot 2)\big)$$

Note: $|u\rangle$ is called a ket, $\langle u|$ its corresponding bra.

Note: switching from column vector to row vector resp. from ket $|u\rangle$ to bra $\langle u|$ is requires complex conjugation.

A second vector $|v\rangle$:

$$|v\rangle := \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 4 + i \\ 5 - i \cdot 5 \\ 6 + i \cdot 6 \end{pmatrix} or \; \langle v| = \big((4 - i)(5 + i \cdot 5)(6 - i \cdot 6)\big)$$

Note: switching from column vector to row vector needs complex conjugation.

The scalar product or dot product:

$$\langle u|v\rangle := \sum_{i=1}^{n} {\alpha_i}^* \beta_i$$

Note: α_i and β_i are complex numbers.

We get:

$$\langle u|v\rangle = (1 - i \cdot 3)(4 + i) + (2 + i \cdot 4)(5 - i \cdot 5) + (3 - i \cdot 2)(6 + i \cdot 6) =$$

$$7 - i \cdot 11 + 30 + i \cdot 10 + 30 + i \cdot 6 =$$

$$67 + 5i$$

$$\langle v|u\rangle = (4 - i)(1 + i \cdot 3) + (5 + i \cdot 5)(2 - i \cdot 4) + (6 - i \cdot 6)(3 + i \cdot 2) =$$

$$67 - 5i$$

Note: $\langle u|v\rangle = \langle v|u\rangle^*$

Note: the dot product gives a (complex) number.

Note: this requires a basis of the vector space. Different bases will give different results.

The dot product of a vector with itself will give a real number, the square of the absolute value:

$$\langle u|u\rangle \coloneqq \sum_{i=1}^{n} \alpha_i{}^* \alpha_i = \|u\|^2$$

We check with our example:

$$\langle u|u\rangle = (1 - i \cdot 3)(1 + i \cdot 3) + (2 + i \cdot 4)(2 - i \cdot 4) + (3 - i \cdot 2)(3 + i \cdot 2) =$$

$$10 + 20 + 13 = 43$$

Hilbert space

Hilbert space $\mathcal{H}$ is a complete vector space over complex numbers $\mathbb{C}$. “Complete” means that every Cauchy sequence converges. This is a mathematical property that guarantees mathematics in this space is working properly.

The dot product denotes a function $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ with the following properties:

1. The dot product is antilinear in the first argument:

$$\langle \alpha u|v\rangle = \alpha^* \langle u|v\rangle$$

2. The dot product is linear in the second argument:

$$\langle u|\alpha v\rangle = \alpha \langle u|v\rangle$$

3. The dot product is Hermitian:

$$\langle u|v\rangle = \langle v|u\rangle^*$$

Note:

$$\langle v|v\rangle \in \mathbb{R}$$
$$\langle v|v\rangle \geq 0$$
$$\langle v|v\rangle = 0 \leftrightarrow |v\rangle = 0$$

A Hilbert space of finite dimension has a finite number of orthonormal basis vectors $|e_i\rangle$.

We rewrite $|v\rangle$:

$$|v\rangle = \sum_{i=1}^{n} \beta_i |e_i\rangle$$

Note: different bases will give different representations of the same vector $|v\rangle$.

Note: β_i are complex numbers.

Note: We use the standard orthonormal basis.

We can disassemble the vector $|v\rangle$:

$$|v\rangle = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

We can identify the components:

$$|v\rangle_i = \beta_i$$

If we chose an orthonormal basis the dot product of the basis vectors becomes simple:

$$\langle e_i | e_j \rangle = \delta_{ij}$$

Note: δ_{ij} is the Kronecker δ:

$$\delta_{ij} = \begin{cases} 1 \; for \; i = j \\ 0 \; for \; i \neq j \end{cases}$$

In case we have an orthonormal basis, the dot product simplifies:

$$\langle u|v\rangle = \left\langle \sum_{i=1}^{n} \alpha_i e_i \,\middle|\, \sum_{j=1}^{n} \beta_j e_j \right\rangle =$$

$$\sum_{i,j=1}^{n} {\alpha_i}^* \beta_j \langle e_i | e_j \rangle = \sum_{i,j=1}^{n} {\alpha_i}^* \beta_j \delta_{ij} =$$

$$\sum_{i=1}^{n} {\alpha_i}^* \beta_j$$

Note: ${\alpha_i}^*$ because of antilinearity of the dot product in the first argument.

The norm or absolute value of $|v\rangle$:

$$\|v\| = \sqrt{\langle v|v\rangle}$$

For the norm we have two inequalities.

The Schwarz inequality	The triangle inequality
$\langle u\|v\rangle\langle v\|u\rangle \leq \langle u\|u\rangle\langle v\|v\rangle$ or $\|\langle u\|v\rangle\| \leq \|u\|\|v\|$	$\|u+v\| \leq \|u\| + \|v\|$

Operators

Operators extract observables.

Matrices

Quantum mechanics uses operators. Operators constitute (extract) observables. The position of a particle e. g. is an observable.

In a Hilbert space with finite dimension operators are matrices.

The real valued 3D matrix A:

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We can transpose the matrix A and get A^T:

$$A := \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

The Hilbert space is a vector space over complex numbers $\mathbb{C}$. The complex valued matrix B:

$$B \coloneqq \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

Note: $a, b, c, d, e, f, g, h, k$ can be complex numbers.

We can complex conjugate the matrix B:

$$B^* \coloneqq \begin{pmatrix} a^* & b^* & c^* \\ d^* & e^* & f^* \\ g^* & h^* & k^* \end{pmatrix}$$

We call a matrix M hermitian if it has the following shape:

$$M \coloneqq \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

Note: $a, b, c \in \mathbb{R}$, $d, e, f \in \mathbb{C}$.

Note: for a Hermitian matrix we have: $a_{ij} = {a_{ji}}^*$.

Note: the diagonal entries are real numbers.

If we transpose M and complex conjugate it, we call the result the adjoint matrix $M^\dagger$.

Hermitian matrices survive this process unchanged:

$$M \coloneqq \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} \rightarrow M^\dagger = \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

Note: a real number does not change by complex conjugation.

Any explicit representation of a matrix uses a specific basis. If the basis changes, the representation of the matrix changes. This holds for vectors too.

Hermitian matrix

A (Hermitian) matrix M applied to a vector $\vec{v}$ produces a new vector $\vec{w}$:

$$M \cdot \vec{v} = \vec{w}$$

We calculate the example above:

$$M \cdot \vec{v} = \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} \begin{pmatrix} 4 + i \\ 5 - i \cdot 5 \\ 6 + i \cdot 6 \end{pmatrix} =$$

$$\begin{pmatrix} a \cdot (4 + i) + d \cdot (5 - i \cdot 5) + e \cdot (6 + i \cdot 6) \\ d^* \cdot (4 + i) + b \cdot (5 - i \cdot 5) + f \cdot (6 + i \cdot 6) \\ e^* \cdot (4 + i) + f^* \cdot (5 - i \cdot 5) + c \cdot (6 + i \cdot 6) \end{pmatrix} =$$

$$\begin{pmatrix} (4a + 5d + 6e) + i(a - 5d + 6e) \\ (4d^* + 5b + 6f) + i(d^* - 5b + 6f) \\ (4e^* + 5f^* + 6c) + i(e^* - 5f^* + 6c) \end{pmatrix}$$

Result:

$$\vec{w} = \begin{pmatrix} (4a + 5d + 6e) + i(a - 5d + 6e) \\ (4d^* + 5b + 6f) + i(d^* - 5b + 6f) \\ (4e^* + 5f^* + 6c) + i(e^* - 5f^* + 6c) \end{pmatrix}$$

We calculate:

$$\vec{v}^* \cdot M = \big((4-i)(5+i\cdot 5)(6-i\cdot 6)\big)\begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} =$$

$$\begin{pmatrix} (4-i)\cdot a + (5+i\cdot 5)\cdot d^* + (6-i\cdot 6)\cdot e^* \\ (4-i)\cdot d + (5+i\cdot 5)\cdot b + (6-i\cdot 6)\cdot f^* \\ (4-i)\cdot e + (5+i\cdot 5)\cdot f + (6-i\cdot 6)\cdot c \end{pmatrix} =$$

$$\begin{pmatrix} (4a + 5d^* + 6e^*) - i(a - 5d^* + 6e^*) \\ (4d + 5b + 6f^*) - i(d - 5b + 6f^*) \\ (4e + 5f + 6c) - i(e - 5f + 6c) \end{pmatrix}$$

Result:

$$\vec{v}^* \cdot M = \vec{w}^*$$

In the bra-ket notation we write for any Hermitian matrix M:

$$M|v\rangle = |w\rangle \Leftrightarrow \langle v|M = \langle w|$$

Adjoint operators

For any operator (matrix) A, B and any complex number a we have:

$$\left(A^\dagger\right)^\dagger = A$$

$$(a \cdot A)^\dagger = a^* \cdot A^\dagger$$

$$(A + B)^\dagger = A^\dagger + B^\dagger$$

$$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$$

Note: the factor a goes out with its complex conjugated.

Note: for Hermitian matrices A, B this gives commutativity: $A \cdot B = B \cdot A$

Eigenvalue

Results of measurements are eigenvalues of a Hermitian operator:

$$M|v\rangle = \lambda|v\rangle$$

Note: λ traditionally used for eigenvalues.

Note: for a Hermitian operator, eigenvalues are real numbers.

We calculate eigenvalues λ:

$$det(\lambda \cdot id - A) = 0$$

Note: det is the determinant of a matrix, id is the identity matrix.

We choose $A = \begin{pmatrix} 1 & 2+3i \\ 2-3i & 4 \end{pmatrix}$:

$$det(\lambda \cdot id - A) =$$

$$det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2+3i \\ 2-3i & 4 \end{pmatrix}\right) =$$

$$det\begin{pmatrix} \lambda - 1 & -2-3i \\ -2+3i & \lambda - 4 \end{pmatrix} =$$

$$(\lambda - 1)(\lambda - 4) - (-2-3i)(-2+3i) =$$

$$\lambda^2 - 5\lambda - 9 \rightarrow$$

$\lambda_1 = \frac{5+\sqrt{61}}{2}$	$\lambda_2 = \frac{5-\sqrt{61}}{2}$

Eigenvector

We choose the first eigenvalue λ_1 and calculate its eigenvector:

$$\begin{pmatrix} \frac{5+\sqrt{61}}{2} - 1 & -2-3i \\ -2+3i & \frac{5+\sqrt{61}}{2} - 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{3+\sqrt{61}}{2} & -2-3i \\ -2+3i & \frac{-3+\sqrt{61}}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{pmatrix} 3+\sqrt{61} & -4-6i \\ -4+6i & -3+\sqrt{61} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

We get two equations:

$I. \left(3+\sqrt{61}\right) \cdot a - (4+6i)b = 0$	$II. (-4+6i) \cdot a + \left(-3+\sqrt{61}\right)b = 0$
$a = b\frac{(4+6i)}{\left(3+\sqrt{61}\right)}$	$a = -b\frac{\left(-3+\sqrt{61}\right)}{(-4+6i)} = b\frac{\left(3-\sqrt{61}\right)}{(-4+6i)}$

We check whether the factor with b is the same (and it is):

$$\frac{\frac{(4+6i)}{\left(3+\sqrt{61}\right)}}{\frac{\left(3-\sqrt{61}\right)}{(-4+6i)}} = \frac{(4+6i)}{\left(3+\sqrt{61}\right)} \cdot \frac{(-4+6i)}{\left(3-\sqrt{61}\right)} =$$

$$\frac{-16+24i-24i-36}{9-3\sqrt{61}+3\sqrt{61}-61} = \frac{-52}{-52} = 1$$

We get the eigenvectors to eigenvalue $\lambda_1 = \frac{5+\sqrt{61}}{2}$:

$$\begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix}$$

We check whether the matrix A applied to this vector reproduces it by the factor λ_1:

$$A \cdot \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix} = \frac{5+\sqrt{61}}{2} \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix} ?$$

Left side:

$$\begin{pmatrix} 1 & 2+3i \\ 2-3i & 4 \end{pmatrix} \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{(4+6i)}{(3+\sqrt{61})} + (2+3i) \\ (2-3i)\frac{(4+6i)}{(3+\sqrt{61})} + 4 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{(4+6i)}{(3+\sqrt{61})} + \frac{(2+3i)(3+\sqrt{61})}{(3+\sqrt{61})} \\ \frac{(2-3i)(4+6i)}{(3+\sqrt{61})} + \frac{4(3+\sqrt{61})}{(3+\sqrt{61})} \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} \begin{pmatrix} (4+6i) + (2+3i)(3+\sqrt{61}) \\ (2-3i)(4+6i) + 4(3+\sqrt{61}) \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} \begin{pmatrix} 4+6i+6+2\sqrt{61}+9i+i3\sqrt{61} \\ 8+18+12+4\sqrt{61} \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} \begin{pmatrix} 10+2\sqrt{61}+15i+i3\sqrt{61} \\ 38+4\sqrt{61} \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} \begin{pmatrix} 10+2\sqrt{61}+i(15+3\sqrt{61}) \\ 38+4\sqrt{61} \end{pmatrix}$$

Right side:

$$\frac{5+\sqrt{61}}{2} \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} \frac{5+\sqrt{61}}{2} \begin{pmatrix} 4+6i \\ 3+\sqrt{61} \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})} (5+\sqrt{61}) \begin{pmatrix} 2+3i \\ \frac{3+\sqrt{61}}{2} \end{pmatrix} =$$

$$\frac{1}{(3+\sqrt{61})}\begin{pmatrix}(5+\sqrt{61})(2+3i)\\(5+\sqrt{61})\left(\frac{3+\sqrt{61}}{2}\right)\end{pmatrix}=$$

$$\frac{1}{(3+\sqrt{61})}\begin{pmatrix}10+2\sqrt{61}+i(15+3\sqrt{61})\\\frac{15+8\sqrt{61}+61}{2}\end{pmatrix}=$$

$$\frac{1}{(3+\sqrt{61})}\begin{pmatrix}10+2\sqrt{61}+i(15+3\sqrt{61})\\\frac{76+8\sqrt{61}}{2}\end{pmatrix}=$$

$$\frac{1}{(3+\sqrt{61})}\begin{pmatrix}10+2\sqrt{61}+i(15+3\sqrt{61})\\38+4\sqrt{61}\end{pmatrix}$$

Right side and left side are equal. Result:

The vector	is eigenvector to the matrix	with (real) eigenvalue.
$\begin{pmatrix}\left(\frac{(4+6i)}{(3+\sqrt{61})}\right)\\1\end{pmatrix}$	$\begin{pmatrix}1 & 2+3i\\2-3i & 4\end{pmatrix}$	$\frac{5+\sqrt{61}}{2}$

We choose the second eigenvalue λ_1:

$$\begin{pmatrix}\frac{5-\sqrt{61}}{2}-1 & -2-3i\\-2+3i & \frac{5-\sqrt{61}}{2}-4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$$

$$\begin{pmatrix}\frac{3-\sqrt{61}}{2} & -2-3i\\-2+3i & \frac{-3-\sqrt{61}}{2}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$$

$$\begin{pmatrix}3-\sqrt{61} & -4-6i\\-4+6i & -3-\sqrt{61}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$$

$I.(3-\sqrt{61})\cdot a-(4+6i)b=0$	$II.(-4+6i)\cdot a-(3+\sqrt{61})b=0$
$a=b\frac{(4+6i)}{(3-\sqrt{61})}$	$a=b\frac{(3+\sqrt{61})}{(-4+6i)}$

We check whether the factor with b is the same (and it is):

$$\frac{\frac{(4+6i)}{(3-\sqrt{61})}}{\frac{(3+\sqrt{61})}{(-4+6i)}}=\frac{(4+6i)}{(3-\sqrt{61})}\cdot\frac{(-4+6i)}{(3+\sqrt{61})}=$$

$$\frac{-16+24i-24i-36}{9-3\sqrt{61}+3\sqrt{61}-61}=\frac{-52}{-52}=1$$

We get the eigenvectors to eigenvalue $\lambda_2=\frac{5-\sqrt{61}}{2}$:

$$\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\end{pmatrix}\atop 1\right)$$

We check whether the matrix A applied to this vector reproduces it by the factor λ_2:

$$A\cdot\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\end{pmatrix}\atop 1\right)=\frac{5-\sqrt{61}}{2}\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\end{pmatrix}\atop 1\right)\ ?$$

Left side:

$$\begin{pmatrix}1 & 2+3i\\ 2-3i & 4\end{pmatrix}\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\end{pmatrix}\atop 1\right)=$$

$$\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}+(2+3i)\\ (2-3i)\frac{(4+6i)}{(3-\sqrt{61})}+4\end{pmatrix}=$$

$$\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}+\frac{(2+3i)(3-\sqrt{61})}{(3-\sqrt{61})}\\ \frac{(2-3i)(4+6i)}{(3-\sqrt{61})}+\frac{4(3+\sqrt{61})}{(3-\sqrt{61})}\end{pmatrix}=$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix}(4+6i)+(2+3i)(3-\sqrt{61})\\ (2-3i)(4+6i)+4(3-\sqrt{61})\end{pmatrix}=$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix}4+6i+6-2\sqrt{61}+9i-i3\sqrt{61}\\ 8+18+12-4\sqrt{61}\end{pmatrix}=$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix}10-2\sqrt{61}+15i-i3\sqrt{61}\\ 38-4\sqrt{61}\end{pmatrix}=$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix}10-2\sqrt{61}+i(15-3\sqrt{61})\\ 38-4\sqrt{61}\end{pmatrix}$$

Right side:

$$\frac{5-\sqrt{61}}{2}\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\end{pmatrix}\atop 1\right)=$$

$$\frac{1}{(3-\sqrt{61})}\frac{5-\sqrt{61}}{2}\begin{pmatrix}4+6i\\ 3-\sqrt{61}\end{pmatrix}=$$

$$\frac{1}{(3-\sqrt{61})}(5-\sqrt{61})\begin{pmatrix} 2+3i \\ \frac{3-\sqrt{61}}{2} \end{pmatrix} =$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix} (5-\sqrt{61})(2+3i) \\ (5-\sqrt{61})\left(\frac{3-\sqrt{61}}{2}\right) \end{pmatrix} =$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix} 10-2\sqrt{61}+i(15-3\sqrt{61}) \\ \frac{15-8\sqrt{61}+61}{2} \end{pmatrix} =$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix} 10-2\sqrt{61}+i(15-3\sqrt{61}) \\ \frac{76-8\sqrt{61}}{2} \end{pmatrix} =$$

$$\frac{1}{(3-\sqrt{61})}\begin{pmatrix} 10-2\sqrt{61}+i(15-3\sqrt{61}) \\ 38-4\sqrt{61} \end{pmatrix}$$

Right side and left side are equal. Result:

The vector	is eigenvector to the matrix	with (real) eigenvalue.
$\begin{pmatrix} \left(\frac{(4+6i)}{(3-\sqrt{61})}\right) \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2+3i \\ 2-3i & 4 \end{pmatrix}$	$\frac{5-\sqrt{61}}{2}$

We have two eigenvectors with two different eigenvalues:

$\lambda_1 = \frac{5+\sqrt{61}}{2}; \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix}$	$\lambda_2 = \frac{5-\sqrt{61}}{2}; \begin{pmatrix} \left(\frac{(4+6i)}{(3-\sqrt{61})}\right) \\ 1 \end{pmatrix}$

Subspaces

The optimal case concerning eigenvector and eigenvalue: if a $n \times n$-matrix has n eigenvectors with different eigenvalues. The eigenvectors are spanning the whole space.

In our example the vector space is $\mathbb{C} \times \mathbb{C}$ and has two dimensions.

We got two eigenvectors with two different eigenvalues. We have a Hermitian matrix, the eigenvectors should span the whole space.

Every eigenvector spans a 1D subspace. Together they span the whole space if they are linear independent:

$$a \cdot \begin{pmatrix} \left(\frac{(4+6i)}{(3+\sqrt{61})}\right) \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} \left(\frac{(4+6i)}{(3-\sqrt{61})}\right) \\ 1 \end{pmatrix} = 0 \Leftrightarrow a = b = 0$$

We check linear independency by calculating the determinant of both vectors.

$$det\begin{pmatrix}\begin{pmatrix}\frac{(4+6i)}{(3+\sqrt{61})}\\1\end{pmatrix} & \begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\\1\end{pmatrix}\end{pmatrix}=$$

$$\left(\frac{(4+6i)}{(3+\sqrt{61})}\right)\cdot 1-\left(\frac{(4+6i)}{(3-\sqrt{61})}\right)\cdot 1=$$

$$\frac{(4+6i)}{(3+\sqrt{61})}-\frac{(4+6i)}{(3-\sqrt{61})}=$$

$$\frac{(4+6i)(3-\sqrt{61})-(4+6i)(3+\sqrt{61})}{(3+\sqrt{61})(3-\sqrt{61})}=$$

$$\frac{12-4\sqrt{61}+18i-6i\sqrt{61}-(12+4\sqrt{61}+18i+6i\sqrt{61})}{9-61}=$$

$$\frac{12-4\sqrt{61}+18i-6i\sqrt{61}-12-4\sqrt{61}-18i-6i\sqrt{61}}{-52}=$$

$$\frac{-8\sqrt{61}-12i\sqrt{61}}{-52}=\frac{2\sqrt{61}+3i\sqrt{61}}{13}$$

The determinant is not equal zero, both vectors are linear independent. They span the whole 2D space.

We check orthogonality with the dot product:

$$\left(\left(\frac{(4-6i)}{(3+\sqrt{61})}\right)(1)\right)\left(\begin{pmatrix}\frac{(4+6i)}{(3-\sqrt{61})}\\1\end{pmatrix}\right)=$$

$$\frac{(4-6i)}{(3+\sqrt{61})}\frac{(4+6i)}{(3-\sqrt{61})}+1=$$

$$\frac{16+36}{9-61}+1=-\frac{52}{52}+1=0$$

Note: switching from column to row vectors requires complex conjugation.

Result: the vectors are orthogonal.

We check the size of the first vector by remembering $\|v\|=\sqrt{\langle v|v\rangle}$:

$$\sqrt{\left(\left(\frac{(4-6i)}{(3+\sqrt{61})}\right)(1)\right)\left(\begin{pmatrix}\frac{(4+6i)}{(3+\sqrt{61})}\\1\end{pmatrix}\right)}=$$

$$\sqrt{\frac{(4-6i)}{(3+\sqrt{61})}\frac{(4+6i)}{(3+\sqrt{61})}+1}=$$

$$\sqrt{\frac{52}{70+6\sqrt{61}}+1}\neq 1$$

Note: the same holds for the second eigenvector.

Result: the eigenvectors are orthogonal but not orthonormal. By dividing them by their size we can build an orthonormal basis.

Note: it is an advantage of Hermitian matrices that their eigenvectors can easily be used to build an orthonormal basis of the whole space.

Measuring

The spin of an electron is an observable fitting well with a 2-dimensional Hilbert space. The spin itself is a spatial 3-dimensional vector with components in x, y and z direction. This information is condensed in a 2-dimensional vector in Hilbert space.

Pauli matrices

We have three Pauli-matrices:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli-matrices have eigenvectors and eigenvalues:

matrix	eigenvalues	eigenvectors	normalized
$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$+1, -1$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$+1, -1$	$\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$+1, -1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Note: Pauli-matrices are Hermitian.

The spatial components of spin, the observables:

$$s_x, s_y, s_z$$

The operators resp. matrices:

$$S_x = \frac{\hbar}{2}\sigma_x;\ S_y = \frac{\hbar}{2}\sigma_y;\ S_z = \frac{\hbar}{2}\sigma_z$$

Note: $\frac{\hbar}{2}$ is a factor needed to be dimensionally correct.

Note: all operators have eigenvalues $+\frac{\hbar}{2}, -\frac{\hbar}{2}$.

Measuring the z-component of the spin

Quantum mechanics postulates that the outcome of a measurement (the observation) are the eigenvalues of σ_z.

We will obtain the value $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$. No mixture.

After the measurement, the spin orients in the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if the system previously was undisturbed.

Traditionally the eigenvectors are written as:

$$\begin{pmatrix}1\\0\end{pmatrix} \coloneqq |z+\rangle, \qquad \begin{pmatrix}0\\1\end{pmatrix} \coloneqq |z-\rangle$$

The spin itself, before we measure it, has any orientation:

$$\begin{pmatrix}a\\b\end{pmatrix}$$

Note: a, b are complex numbers.

Projection

If we repeatedly measure the undisturbed spin, we will get a series of results $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in the relation to a and b.

Mathematically this is done by projection.

We assume the state of the spin:

$$|v\rangle \coloneqq \begin{pmatrix}a\\b\end{pmatrix}$$

Note: a, b are complex numbers.

The projection operator P_{z+} onto $\begin{pmatrix}1\\0\end{pmatrix}$ resp. $\|z+\rangle$	The projection operator P_{z-} onto $\begin{pmatrix}0\\1\end{pmatrix}$ resp. $\|z-\rangle$
$P_{z+} \coloneqq \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$P_{z-} \coloneqq \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$
The effect of the projection operator P_{z+} onto $\|v\rangle$	The effect of the projection operator P_{z-} onto $\|v\rangle$
$P_{z+}\|v\rangle = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\0\end{pmatrix}$	$P_{z-}\|v\rangle = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\b\end{pmatrix}$

Note: the origin of P_{z+} and P_{z-} follows later …

We get the probabilities for the outcome p of a measurement:

$p(z+) = \langle v\|P_{z+}\|v\rangle =$ $(a^* \; b^*)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} =$ $(a^* \; b^*)\begin{pmatrix}a\\0\end{pmatrix} = a^*a$	$p(z-) = \langle v\|P_{z-}\|v\rangle =$ $(a^* \; b^*)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} =$ $(a^* \; b^*)\begin{pmatrix}0\\b\end{pmatrix} = b^*b$

Note: $p(z+)$ is $\|a\|^2$, $p(z-)$ is$\|b\|^2$.

The sum of the probabilities $p(z+)$ and $p(z+)$:

$$p(z+) + p(z-) = a^*a + b^*b = 1$$

Note: normalization of $|v\rangle = \begin{pmatrix}a\\b\end{pmatrix}$ guarantees the statistically correct result.

Constant phase

If we multiply $|v\rangle$ by a constant (time independent) phase $e^{i\varphi}$ we get the same probabilities:

$$\langle e^{i\varphi}v|e^{i\varphi}v\rangle = e^{-i\varphi}e^{i\varphi}\langle v|v\rangle = \langle v|v\rangle$$

Note: the dot product is antilinear in the first component.

Note: $e^{i\varphi}v \neq v$, only the probability (dot product) remains unchanged.

Projection operators in detail

We use orthonormal basis vectors written as $|i\rangle$. The vector space is finite with n dimensions.

We remember: a vector $|v\rangle$ can be written as (using the orthonormal basis …):

$$|v\rangle = \sum_i a_i|i\rangle = \sum_i |i\rangle a_i$$

Note: multiplication by a (complex) number is commutative.

The components $a_i|i\rangle$ are result of a projection of $|v\rangle$ onto a basis vector $|i\rangle$.

The coefficients a_i we get by the dot-product with the basis vector $\langle i|$:

$$a_i = \langle i|v\rangle$$

We rewrite the sum:

$$|v\rangle = \sum_i |i\rangle\langle i|v\rangle \Rightarrow \cdots$$

Note: id is the identity matrix.

$\langle i|v\rangle$ is the dot product of a basis vector $\langle i|$ with $|v\rangle$. $\langle i|v\rangle$ gives the coefficient a_i.

$$\sum_i |i\rangle\langle i|v\rangle = \left(\sum_i |i\rangle\langle i|\right)|v\rangle$$

Note: we can draw out $|v\rangle$ of all elements of the sum.

What we get is:

$$\sum_i |i\rangle\langle i| = id$$

We have a sum of projection operators P_i that project $|v\rangle$ onto the basis vector $|i\rangle$:

$$P_i = |i\rangle\langle i|$$

We can insert the identity matrix into every dot-product:

$$\langle u|v\rangle = \langle u|id|v\rangle = \sum_i u|i\rangle\langle i|v\rangle$$

Note: this is called "resolving the identity" and frequently used in quantum mechanics.

By resolving the identity, we can disassemble operators A:

$$A = id \cdot A \cdot id = \sum_{i,j} |i\rangle\langle i|A|j\rangle\langle j| =$$

$$\sum_{i,j} |i\rangle A_{ij} \langle j|$$

We calculate $\langle u|A|v\rangle$. We chose $|u\rangle := \sum_i a_i |i\rangle$, $|v\rangle := \sum_i b_i |i\rangle$:

$$\langle u|A|v\rangle = \sum_{i,j} \langle u|i\rangle A_{ij} \langle j|v\rangle =$$

$$\sum_{i,j} \langle u|i\rangle A_{ij} \langle j|v\rangle =$$

$$\sum_{i,j} {a_i}^* A_{ij} b_j$$

Constructing projection operators

The projection operator that projects $|v\rangle$ onto the basis vector $|i\rangle$:

$$P_i = |i\rangle\langle i|$$

We apply this to the spin example:

$$P_{z+} = |z+\rangle\langle z+| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_{z-} = |z-\rangle\langle z-| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: $P_{z+} + P_{z-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the identity matrix.

Measuring the x-component of the spin

We use eigenvectors:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} := |x+\rangle, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} := |x-\rangle$$

The spin before we measure it has any orientation: $\begin{pmatrix} a \\ b \end{pmatrix}$

Note: a, b are complex numbers.

Note: the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is normalized.

The projection operator P_{x+} onto $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ resp. $\lvert x+\rangle$:	The projection operator P_{x-} onto $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ resp. $\lvert x-\rangle$:
$P_{x+} = \lvert x+\rangle\langle x+\rvert = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(1\ 1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$P_{x-} = \lvert x-\rangle\langle x-\rvert = \frac{1}{\sqrt{2}}(1\ (-1))\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

The effect of the projection operator P_{x+} onto $\|v\rangle$:	The effect of the projection operator P_{x-} onto $\|v\rangle$:
$P_{x+}\|v\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a+b\\ a+b\end{pmatrix} =$ $\frac{a+b}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$	$P_{x-}\|v\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a-b\\ -(a-b)\end{pmatrix} =$ $\frac{a-b}{2}\begin{pmatrix}1\\ -1\end{pmatrix}$

Note: $P_{x+} + P_{x-} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. This is the identity matrix.

We build the dot product of $P_{x+}|v\rangle$ and $P_{x-}|v\rangle$:

$$\langle v|P_{x+}P_{x-}|v\rangle =$$

$$\left(\frac{a^*+b^*}{2}(1\ \ 1)\right)\left(\frac{a-b}{2}\begin{pmatrix}1\\ -1\end{pmatrix}\right) =$$

$$\frac{(a^*+b^*)(a-b)}{4}(1\ \ 1)\begin{pmatrix}1\\ -1\end{pmatrix} =$$

$$\frac{(a^*+b^*)(a-b)}{4}(1-\ 1) = 0$$

The projection operator produces a pair of orthogonal vectors.

We get the probabilities for the outcome p of a measurement:

$p(x+) = \langle v\|P_{x+}\|v\rangle =$ $(a^*\ b^*)\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} =$ $\frac{1}{2}(a^*\ b^*)\begin{pmatrix}a+b\\ a+b\end{pmatrix} =$ $\frac{1}{2}\big(a^*(a+b) + b^*(a+b)\big) =$ $\frac{1}{2}\big((a^*+\ b^*)(a+b)\big)$	$p(x-) = \langle v\|P_{x-}\|v\rangle =$ $(a^*\ b^*)\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} =$ $\frac{1}{2}(a^*\ b^*)\begin{pmatrix}a-b\\ -(a-b)\end{pmatrix} =$ $\frac{1}{2}\big(a^*(a-b) - b^*(a-b)\big) =$ $\frac{1}{2}(a^*-\ b^*)(a-b)$

The probability of $p(x+)$ or $p(x-)$:

$$p(x+) + p(x-) =$$

$$\frac{1}{2}(a^*+\ b^*)(a+b) + \frac{1}{2}(a^*-\ b^*)(a-b) =$$

$$\frac{1}{2}\big((a^*+\ b^*)(a+b) + (a^*-\ b^*)(a-b)\big) =$$

$$\frac{1}{2}(a^*a + a^*b + b^*a + b^*b + a^*a - a^*b - b^*a + b^*b) =$$

$$\frac{1}{2}(a^*a + b^*b + a^*a + b^*b) =$$

$$a^*a + b^*b = 1$$

Note: Normalization of $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ guarantees the statistical correct result.

Measuring the y-component of the spin

We use eigenvectors:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} := |y+\rangle, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} := |y-\rangle$$

The spin itself, before we measure it, has any orientation: $\begin{pmatrix} a \\ b \end{pmatrix}$

Note: a, b are complex numbers.

Note: the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is normalized.

The projection operator P_{y+} onto $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ resp. $\|y+\rangle$:	The projection operator P_{y-} onto $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ resp. $\|y-\rangle$:
$P_{y+} = \|y+\rangle\langle y+\| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}}(1\ -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ Note: switching from ket to bra needs complex conjugation.	$P_{y-} = \|y-\rangle\langle y-\| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\frac{1}{\sqrt{2}}(1\ i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ Note: switching from ket to bra needs complex conjugation.
The effect of the projection operator P_{y+} onto$\|v\rangle$:	The effect of the projection operator P_{y-} onto $\|v\rangle$:
$P_{y+}\|v\rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a-ib \\ ia+b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a-ib \\ i(a-ib) \end{pmatrix} = \frac{a-ib}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$	$P_{y-}\|v\rangle = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+ib \\ -ia+b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+ib \\ -i(a+ib) \end{pmatrix} = \frac{a+ib}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Note: $P_{y+} + P_{y-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the identity matrix.

We build the dot product of $P_{y+}|v\rangle$ and $P_{y-}|v\rangle$:

$$\langle v|P_{y+}P_{y-}|v\rangle =$$

$$\frac{a^* + ib^*}{2}(1\ (-i))\left(\frac{a+ib}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right) =$$

$$\frac{(a^* + ib^*)(a+ib)}{4}(1\ (-i))\begin{pmatrix} 1 \\ -i \end{pmatrix} =$$

$$\frac{(a^* + ib^*)(a+ib)}{4}(1 - \ 1) = 0$$

The projection operator produces a pair of orthogonal vectors. It projects the vector $|v\rangle$ onto an eigenvector with eigenvalue λ.

We get the probabilities for the outcome p of a measurement:

$p(y+) = \langle v\|P_{y+}\|v\rangle =$ $(a^*\ b^*)\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} =$ $\frac{1}{2}(a^*\ b^*)\begin{pmatrix}a-ib\\ ia+b\end{pmatrix} =$ $\frac{1}{2}\left(a^*(a-ib)+b^*(ia+b)\right) =$ $\frac{1}{2}(a^*a - ia^*b + iab^* + bb^*) =$ $\frac{1}{2}\left(a^*a + bb^* + i(ab^* - a^*b)\right) =$ $\frac{1}{2} + \frac{i(ab^* - a^*b)}{2}$ Note: $(ab^* - a^*b)$ is a pure imaginary number $\rightarrow\ i(ab^* - a^*b)$ is a real number.	$p(y-) = \langle v\|P_{y-}\|v\rangle =$ $(a^*\ b^*)\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} =$ $\frac{1}{2}(a^*\ b^*)\begin{pmatrix}a+ib\\ -ia+b\end{pmatrix} =$ $\frac{1}{2}\left(a^*(a+ib)+b^*(-ia+b)\right) =$ $\frac{1}{2}(a^*a + ia^*b - iab^* + b^*b) =$ $\frac{1}{2}\left(a^*a + bb^* + i(a^*b - ab^*)\right)$ $\frac{1}{2} + \frac{i(a^*b - ab^*)}{2}$ Note: $(a^*b - ab^*)$ is a pure imaginary number $\rightarrow\ i(a^*b - ab^*)$ is a real number.

Remark: please note that the probability $p(y+)$ has an interesting shape:

$$\frac{1}{2} + \frac{i(ab^* - a^*b)}{2}$$

The range of the component $i(ab^* - a^*b)$ extends from -1 to 1. Analog for $p(y-)$.

We calculate the sum of probabilities:

$$p(y+) + p(y-) =$$

$$\frac{1}{2}\left(a^*a + bb^* + i(ab^* - a^*b)\right) + \frac{1}{2}\left(a^*a + bb^* + i(a^*b - ab^*)\right) =$$

$$\frac{1}{2}(a^*a + bb^* + a^*a + bb^*) =$$

$$a^*a + b^*b = 1$$

Note: Normalization of $|v\rangle = \begin{pmatrix}a\\ b\end{pmatrix}$ guarantees the statistical correct result.

Expectation value

If we perform a series of measurements A on an undisturbed state $|v\rangle$ (or parallel on a multiple of identical states), we get an expectation value for the result:

$$\langle A\rangle_v = \sum_i p(\lambda_i)\lambda_i$$

Note: this is the arithmetic average.

The probabilities $p(\lambda_i)$ are the effect of the projection operator P_{λ_i}. The projection operator P_{λ_i} onto $|v\rangle$ results in an eigenvector with eigenvalue λ_i:

$$\sum_i p(\lambda_i)\lambda_i = \sum_i \langle v|P_{\lambda_i}|v\rangle\lambda_i =$$

$$\sum_i \langle v|\lambda_i P_{\lambda_i}|v\rangle$$

The projection operator P_{λ_i} consist of eigenvectors of the operator A. The operator A applied to the projection operator P_{λ_i} reproduces it by multiples of eigenvalues λ_i:

$$AP_{\lambda_i} = \lambda_i P_{\lambda_i}$$

We can rewrite:

$$\sum_i \langle v|\lambda_i P_{\lambda_i}|v\rangle = \sum_i \langle v|AP_{\lambda_i}|v\rangle$$

The sum over the projections operators results in the identity matrix:

$$\sum_i P_{\lambda_i} = id$$

We get the expectation value:

$$\langle A\rangle_v = \langle v|A|v\rangle$$

Note: The expectation value depends on the state the operator is applicated to.

Uncertainty

The standard deviation or uncertainty:

$$(\Delta A)_v = \sqrt{\langle (A - \langle A\rangle_v)^2\rangle_v} =$$

$$\sqrt{\langle A^2 - 2A\langle A\rangle_v + {\langle A\rangle_v}^2\rangle_v} =$$

$$\sqrt{\langle A^2\rangle_v - \langle 2A\langle A\rangle_v\rangle_v + \langle {\langle A\rangle_v}^2\rangle_v} =;$$

Note: $A\langle A\rangle_v = {\langle A\rangle_v}^2$, $\langle {\langle A\rangle_v}^2\rangle_v = {\langle A\rangle_v}^2$

We get:

$$(\Delta A)_v = \sqrt{\langle A^2\rangle_v - {\langle A\rangle_v}^2}$$

If the vector $|v\rangle$ is eigenvector to the operator A with eigenvalue :

$$\langle A^2\rangle_v = \lambda^2 = {\langle A\rangle_v}^2$$

In this case the uncertainty is zero.