Time-independent vs. time-dependent operators.

Related information you may find at:

https://phys.libretexts.org/Bookshelves/Quantum_Mechanics/Advanced_Quantum_Mechanics_(Kok)/03%3A_Schrodinger_and_Heisenberg_Pictures

https://ocw.mit.edu/courses/8-05-quantum-physics-ii-fall-2013/resources/mit8_05f13_chap_05/

Griffiths, 6.8.1 The Heisenberg Picture

Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German…)

Hope I can help you with learning quantum mechanics.

Contents

Operators in the Schroedinger picture are time independent. An exception from this is the time dependent Hamiltonian that comes into account if we change the entire system in time, e. g. change the potential, apply magnetic fields. In these cases, the Hamiltonian becomes explicit time dependent. We must distinguish this kind of time dependency from the time dependency in the Heisenberg picture.

Throughout this text we use the subscript S for states and operators in the Schroedinger picture, the subscript H for those in the Heisenberg picture.

Heisenberg picture

We have the Schroedinger time development operator $U(t, t_0)$:

$$|v_s(t)\rangle = U(t, t_0)|v_s(t_0)\rangle$$

We neutralize the effect of the Schroedinger time development operator by applying the transposed:

$$|v_s(t_0)\rangle = U^\dagger(t, t_0)U(t, t_0)|v_s(t_0)\rangle$$

We name this:

$$|v_H\rangle = U^\dagger(t, t_0)U(t, t_0)|v_s(t_0)\rangle$$

Note: $|v_H\rangle$ is time independent.

The corresponding bra:

$$\langle v_H| = \langle v_s(t_0)|U^\dagger(t, t_0)U(t, t_0)$$

If the states are time independent, we need time dependent operators A:

$$A_H(t) \coloneqq U^\dagger(t, t_0)A_S U(t, t_0)$$

Note: The operator A_S is time independent.

With these definitions we get that the expectation values do not change:

$$\langle u_H|A_H(t)|v_H\rangle = \langle u_S(t)|A_S|v_S(t)\rangle$$

Note: u_H, v_H, A_S are time independent. u_S, v_S, A_H are time dependent.

We check this:

$$\langle u_H|A_H(t)|v_H\rangle = \left\langle u_H\middle|U^\dagger(t, t_0)A_S U(t, t_0)\middle|v_H\right\rangle =$$

$$\langle u_s(t_0)|U^\dagger(t, t_0)U(t, t_0)\left|U^\dagger(t, t_0)A_S U(t, t_0)\right|U^\dagger(t, t_0)U(t, t_0)|v_s(t_0)\rangle =$$

$$\left\langle u_s(t_0)U^\dagger(t, t_0)\middle|A_S\middle|U(t, t_0)v_s(t_0)\right\rangle =$$

$$\langle u_S(t)|A_S|v_S(t)\rangle$$

An example might be a magnetic field and a rotating spin.

The Schroedinger picture works with the state of the spin rotating within the x, y-plane. The spin changes from $|x+\rangle$ to $|x+\rangle$ to $|x-\rangle$ to $|y-\rangle$ to …

Heisenberg operator

In the Heisenberg picture the state remains constant e. g. $|x+\rangle$. The operators are changing:

$$(S_x)_H \neq \frac{\hbar}{2}\sigma_x$$

Instead σ_x becomes a matrix constantly changing from σ_x to $-\sigma_y$ to $-\sigma_x$ to σ_y to …

We check this with the spin in a constant magnetic field, oriented in z-direction.

We know the unitary time development operator:

$$U(t,0) = e^{-i\omega t\sigma_z}$$

Note: $\omega = \frac{aB_z}{2}$, a is a positive real number representing the strength of the magnetic field.

We expand the Pauli matrix:

$$U(t,0) = e^{-i\omega t\sigma_z} = e^{-i\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} = \begin{pmatrix}e^{-i\omega t} & 0\\ 0 & e^{i\omega t}\end{pmatrix}$$

We need the adjoint:

$$U^\dagger(t,0) = e^{i\omega t\sigma_z} = e^{i\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} = \begin{pmatrix}e^{i\omega t} & 0\\ 0 & e^{-i\omega t}\end{pmatrix}$$

Note: for diagonal matrices we can exchange the exponential of a matrix with the matrix of the exponentials.

We know $(S_x)_S$:

$$(S_x)_S = \frac{\hbar}{2}\sigma_x$$

Our rule to make operators time dependent:

$$A_H(t) := U^\dagger(t,0)A_S U(t,0)$$

We transform $(S_x)_S$ to the time dependent $(S_x)_H$:

$$(S_x)_H = U^\dagger(t,0)(S_x)_S U(t,0) =$$

$$U^\dagger(t,0)\frac{\hbar}{2}\sigma_x U(t,0) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix}e^{i\omega t} & 0\\ 0 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega t} & 0\\ 0 & e^{i\omega t}\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix}0 & e^{i\omega t}\\ e^{-i\omega t} & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega t} & 0\\ 0 & e^{i\omega t}\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix}0 & e^{2i\omega t}\\ e^{-2i\omega t} & 0\end{pmatrix}\right)$$

We transform $(S_y)_S$ to the time dependent $(S_y)_H$:

$$(S_y)_H = U^\dagger(t,0)(S_y)_S U(t,0) =$$

$$U^\dagger(t,0)\frac{\hbar}{2}\sigma_y U(t,0) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t}\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{i\omega t+\frac{3}{2}\pi} \\ e^{-i\omega t+\frac{1}{2}\pi} & 0\end{pmatrix}\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t}\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\omega t+\frac{3}{2}\pi} \\ e^{-2i\omega t+\frac{1}{2}\pi} & 0\end{pmatrix}\right)$$

Note: $-i = e^{i\frac{3}{2}\pi}, i = e^{i\frac{1}{2}\pi}$

Result: the operators in the Heisenberg picture are “rotating.”

We take $\omega := \frac{2\cdot\pi}{T}$ and calculate for various times t:

For $t = 0$:

$$(S_x)_H(t=0) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{2\cdot\pi}{T}t} \\ e^{-2i\frac{2\cdot\pi}{T}t} & 0\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right) = \frac{\hbar}{2}\sigma_x = (S_x)_S$$

For $t = \frac{T}{8}$:

$$(S_x)_H\left(t=\frac{T}{4}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{2\cdot\pi\cdot T}{T\cdot 8}} \\ e^{-2i\frac{2\cdot\pi\cdot T}{T\cdot 8}} & 0\end{pmatrix}\right)$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{i\frac{\pi}{2}} \\ e^{-i\frac{\pi}{2}} & 0\end{pmatrix}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}\right) =$$

$$-\frac{\hbar}{2}\sigma_y = -\left(S_y\right)_S$$

For $t = \frac{T}{4}$:

$$(S_x)_H\left(t=\frac{T}{4}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{2\cdot\pi\cdot T}{T\cdot 4}} \\ e^{-2i\frac{2\cdot\pi\cdot T}{T\cdot 4}} & 0\end{pmatrix}\right)$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{\pi}{2}} \\ e^{-2i\frac{\pi}{2}} & 0\end{pmatrix}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{i\pi} \\ e^{-i\pi} & 0\end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}\right) = -\frac{\hbar}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right) =$$

$$-\frac{\hbar}{2}\sigma_x = -(S_x)_S$$

For $t = \frac{3\cdot T}{8}$:

$$(S_x)_H\left(t = \frac{T}{4}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{2\cdot\pi\cdot 3\cdot T}{T\cdot 8}} \\ e^{-2i\frac{2\cdot\pi\cdot 3\cdot T}{T\cdot 8}} & 0 \end{pmatrix}\right)$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{i\frac{3\cdot\pi}{2}} \\ e^{-i\frac{3\cdot\pi}{2}} & 0 \end{pmatrix}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\sigma_y = -(S_y)_S$$

For $t = \frac{T}{2}$:

$$(S_x)_H\left(t = \frac{T}{2}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\frac{2\cdot\pi\cdot T}{T\cdot 2}} \\ e^{-2i\frac{2\cdot\pi\cdot T}{T\cdot 2}} & 0 \end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\left(\begin{pmatrix} 0 & e^{2i\pi} \\ e^{-2i\pi} & 0 \end{pmatrix}\right) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) =$$

$$\frac{\hbar}{2}\sigma_x = (S_x)_S$$

Please note the double angle effect. We get a full rotation in time $t = \frac{T}{2}$.

Heisenberg equation

We take the Schroedinger equation:

$$i\hbar\frac{d}{dt}|v_S(t)\rangle = H(t)|v_S(t)\rangle$$

We use:

$$|v_S(t)\rangle = U(t, t_0)|v_S(t_0)\rangle$$

We get:

$$i\hbar\frac{d}{dt}|v_S(t)\rangle = i\hbar\frac{d}{dt}(U(t, t_0)|v_S(t_0)\rangle)$$

Note: $|v_S(t_0)\rangle = const$:

$$i\hbar\frac{d}{dt}U(t, t_0)|v_S(t_0)\rangle = H(t)U(t, t_0)|v_S(t_0)\rangle$$

Note again: $|v_S(t_0)\rangle = const$.

We get:

$$i\hbar\frac{d}{dt}U(t, t_0) = H(t)U(t, t_0)$$

This is a differential equation with initial condition $U(t_0, t_0) = 1$.

We apply it to an arbitrary Heisenberg operator A_H.

Note: We omit the factor $i\hbar$ and add it in the end:

$$\frac{d}{dt}A_H(t) = \frac{d}{dt}\left(U^\dagger(t)A_S U(t)\right) =$$

$$\left(\frac{d}{dt}U^\dagger(t)\right)A_S U(t) + U^\dagger(t)\left(\frac{d}{dt}A_S\right)U(t) + U^\dagger(t)A_S\left(\frac{d}{dt}U(t)\right) =$$

$$H_S(t)U^\dagger(t)A_S U(t) + U^\dagger(t)\left(\frac{d}{dt}A_S\right)U(t) + U^\dagger(t)A_S H_S(t)U(t) =$$

$$-U^\dagger(t)H_S(t)A_S U(t) + U^\dagger(t)\left(\frac{d}{dt}A_S\right)U(t) + U^\dagger(t)A_S H_S(t)U(t) =$$

$$U^\dagger(t)A_S H_S(t)U(t) - U^\dagger(t)H_S(t)A_S U(t) + U^\dagger(t)\left(\frac{d}{dt}A_S\right)U(t) =;$$

Note: $\left(\frac{d}{dt}A_S\right)$ might be an explicit time dependency of the Schrödinger operator due to e. g. time varying potentials. If there is no such dependency, this term becomes zero.

Note:

$$U^\dagger(t)\left(\frac{d}{dt}A_S\right)U(t) \coloneqq \frac{\partial}{\partial t}A_H$$

We resolve the difference:

$$U^\dagger(t)A_S H_S(t)U(t) - U^\dagger(t)H_S(t)A_S U(t) =$$

$$U^\dagger(t)A_S U(t)U^\dagger(t)H_S(t)U(t) - U^\dagger H_S(t)A_S U(t)U^\dagger(t)U(t) =$$

$$A_H H_H - H_H A_H =$$

$$[A_H, H_H]$$

Result: We get the Heisenberg equation (now with the added $i\hbar$):

$$i\hbar\frac{d}{dt}A_H(t) = [A_H, H_H] + i\hbar\frac{\partial}{\partial t}A_H$$

We can rewrite this:

$$\frac{d}{dt}A_H(t) = -\frac{i}{\hbar}[A_H, H_H] + \frac{\partial}{\partial t}A_H$$

Ehrenfest

Dot product and expectation values are the same whether we work with the Schroedinger or the Heisenberg picture:

$$\langle A\rangle_v = \langle v_S|A_S|v_S\rangle = \langle v_H|A_H|v_H\rangle$$

We check this. The operator A has no explicit time dependency: $i\hbar\frac{\partial}{\partial t}A_H = 0$.

We begin with the Heisenberg equation:

$$\frac{d}{dt}\langle A\rangle_v = \frac{d}{dt}\langle v_H|A_H|v_H\rangle =;$$

Note: in the Heisenberg picture the states are constant.

$$\frac{d}{dt}\langle v_H|A_H|v_H\rangle = \left\langle v_H\middle|\frac{d}{dt}A_H\middle|v_H\right\rangle =$$

$$-\frac{i}{\hbar}\langle v_H|[A_H, H_H]|v_H\rangle$$

Note: the expectation value is independent of the picture we choose:

$$\langle u_H|A_H(t)|v_H\rangle = \langle u_S(t)|A_S|v_S(t)\rangle$$

We can omit the indices S, H and write:

$$\frac{d}{dt}\langle A\rangle_v = -\frac{i}{\hbar}\langle[A, H]\rangle_v$$

Note: The index v is a reminder to the basis the representation takes place.

Result: We got the Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_v = -\frac{i}{\hbar}\langle[A, H]\rangle_v$$

The expectation value does not change if the operator A commutes with the Hamiltonian, its expectation value is conserved.

We can make this claim stronger:

$$[A, H] = 0 \rightarrow [A^n, H] = 0 \rightarrow [f(A), H] = 0$$

Note: $f(A)$ can be any reasonable function of A.

Note: this claim not proved here.

Note: $[H, H] = 0$, the energy of the system is conserved.

We check the Ehrenfest theorem with the rotating spin in a magnetic field.

We chose the Hamiltonian for a constant magnetic field in z-direction:

$$H = B_z\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

We know the spin operator:

$$S_z := \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

We build the commutator:

$$[S_z, H] = S_zH - HS_z =$$

$$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}B_z\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} - B_z\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} =$$

$$\frac{B_z\hbar}{2}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right) = 0$$

Result: The spin in direction z is conserved.

Energy time uncertainty

We use an operator A and the Hamiltonian H, both without explicit time dependencies.

The Heisenberg uncertainty relation:

$$(\Delta A)_v(\Delta B)_v \geq \frac{1}{2}|\langle v|[A,B]|v\rangle|$$

We replace the operator B by the Hamiltonian and remember that the uncertainty ΔH usually is written as ΔE:

$$(\Delta A)_v(\Delta E)_v \geq \frac{1}{2}|\langle v|[A,H]|v\rangle| = \frac{1}{2}|\langle [A,H]\rangle_v|$$

Note: The index v is a reminder to the dependency of a chosen basis …

We use the Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_v = -\frac{i}{\hbar}\langle [A,H]\rangle_v \rightarrow -\frac{\hbar}{i}\frac{d}{dt}\langle A\rangle_v = \langle [A,H]\rangle_v$$

We get:

$$(\Delta A)_v(\Delta E)_v \geq \frac{1}{2}|\langle [A,H]\rangle_v| = \frac{\hbar}{2}\left|\frac{d}{dt}\langle A\rangle_v\right|$$

Note: We lost the i by help of the absolute value.

We examine the expression

$$(\Delta A)_v(\Delta E)_v \geq \frac{\hbar}{2}\left|\frac{d}{dt}\langle A\rangle_v\right|$$

On the right side we have up to a factor $\left|\frac{d}{dt}\langle A\rangle_v\right|$. This is the time needed by the expectation value of A_v to change by the quantity $(\Delta A)_v$ because $(\Delta E)_v$ should be constant in time.

Note: if $(\Delta E)_v = 0$ we have $0 \geq \frac{\hbar}{2}\left|\frac{d}{dt}\langle A\rangle_v\right|$ and therefore $\langle A\rangle_v = const$. The system is in an energy eigenstate.

We divide both sides of the inequality by $\left|\frac{d}{dt}\langle A\rangle_v\right|$:

$$\frac{(\Delta A)_v}{\left|\frac{d}{dt}\langle A\rangle_v\right|}(\Delta E)_v \geq \frac{\hbar}{2}$$

We define:

$$(\Delta\tau)_v = \frac{(\Delta A)_v}{\left|\frac{d}{dt}\langle A\rangle_v\right|}$$

Note: $(\Delta\tau)_v$ has the dimension of time.

We get the energy-time-uncertainty:

$$(\Delta\tau)_v(\Delta E)_v \geq \frac{\hbar}{2}$$

Interpretation: The smaller $(\Delta E)_v$ the bigger $(\Delta\tau)_v$ and thus the smaller $\left|\frac{d}{dt}\langle A\rangle_v\right|$.

The change of the average $\langle A \rangle_v$, $\left|\frac{d}{dt}\langle A \rangle_v\right|$ is reciprocal to the uncertainty of the energy $(\Delta E)_v$.

For small $(\Delta E)_v$ we have only slow variation of $\langle A \rangle_v$.

> The magnetic interaction between the spins actually breaks this degeneracy and produces the so-called "hyperfine" splitting. This is a very tiny split: $5{,}88 \cdot 10^{-6}\ ev$ (compare with about $13.6\ ev$ for the ground state energy). For a hyperfine atomic transition, the emitted photon carries the energy difference: $E_y = 5{,}88 \cdot 10^{-6}\ ev$ resulting in a wavelength of $21.1\ cm$ and a frequency $\nu = 1420.405751786(30) MHz$. The eleven significant digits of this frequency attest to the sharpness of the emission line.
>
> The issue of uncertainty arises because the excited state of the hyperfine splitting has a lifetime τ_H for decay to the ground state and emission of a photon. This lifetime is extremely long, in fact $\tau_H \sim$ 11 million years ($= 3.4\ 10^{14}$ sec, recalling that a year is about $\pi \times 10^7$ sec, accurate to better than 1%). This lifetime can be viewed as the time that takes some observable of the electron-proton system to change significantly (its total spin angular momentum, perhaps) so by the uncertainty principle it must be related to some energy uncertainty $\Delta E \sim \frac{\hbar}{\tau_H} \cong 2 \times 10^{-30}\ ev$ of the original excited state of the hydrogen atom.

Citation from B. Zwiebach,

https://ocw.mit.edu/courses/8-05-quantum-physics-ii-fall-2013/resources/mit8_05f13_chap_05/