Tensor products in quantum mechanics.

Related information you may find at:

Susskind & Friedman, Quantum Mechanics, Penguin Science, ISBN 978-0-141-97781-2

http://www.math.ucdenver.edu/~rrosterm/q-clans/node5.html

Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German …)

A comprehensive discussion concerning tensors that goes far beyond our purpose you may find at

https://kconrad.math.uconn.edu/blurbs/linmultialg/tensorprod.pdf

Hope I can help you with learning quantum mechanics.

Contents

“Disclaimer”

There often is a confusion between “entanglement” and “correlation”. Maybe the following citation helps.

“Correlations are a property of a set of measurements. Two measurement outcomes are correlated if the associated probability distribution cannot be factorized, that is, when the outcome of one measurement gives information about the outcome of the other measurement. The concept of "correlations", in this sense, is not inherently quantum, although quantum mechanics can make for correlations stronger than those allowed by classical probability theory.

On the other hand, entanglement is a property of a state, with respect to some partition on the underlying space. A bipartite state ρ is said to be entangled if it cannot be written as a convex combination of product states, that is, if it cannot be written in the form $\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$ for some $p_k \geq 0, \sum_k p_k = 1$ and states ρ_k^A, ρ_k^B. The bipartite structure is usually, although not necessarily, taken to refer to degrees of freedom of spatially separated particles. It can however refer to any pair of degrees of freedom of a quantum system.”

glS (https://physics.stackexchange.com/users/58382/gls), Correlation vs. entanglement for composite quantum system, URL (version: 2020-01-09): https://physics.stackexchange.com/q/524010

Tensor product

We have a state of phase, a Hilbert space $\mathcal{H}_1$ of dimension d. If we combine two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of dimension D each, we get a Hilbert space $\mathcal{H}$ of dimension d^2:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Example

Let A and B be two 3×3 matrices: $A := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B := \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

The matrix version of the tensor product, sometimes called the Kronecker product:

$$A \otimes B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{13}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{23}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\ a_{31}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{32}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} & a_{33}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} & a_{13}b_{11} & a_{13}b_{12} & a_{13}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} & a_{13}b_{21} & a_{13}b_{22} & a_{13}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} & a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} & a_{23}b_{11} & a_{23}b_{12} & a_{23}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & a_{23}b_{21} & a_{23}b_{22} & a_{23}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} & a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} & a_{32}b_{11} & a_{32}b_{12} & a_{32}b_{13} & a_{33}b_{11} & a_{33}b_{12} & a_{33}b_{13} \\ a_{31}b_{21} & a_{31}b_{22} & a_{31}b_{23} & a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} & a_{33}b_{21} & a_{33}b_{22} & a_{33}b_{23} \\ a_{31}b_{31} & a_{31}b_{32} & a_{31}b_{33} & a_{32}b_{31} & a_{32}b_{32} & a_{32}b_{33} & a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{pmatrix}$$

Direct sum

We compare this with classical physics. The direct sum of two matrices A and B:

$$A \oplus B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{23} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Note the difference. We use V_A as the vector space related to the matrix A, V_B as the vector space related to matrix B and $V = V_A \oplus V_B$.

In classical physics any vector $w \in V$ can be written as direct sum of two vectors $\vec{u} \in V_A$ and $\vec{v} \in V_B$:

$$\vec{w} = \vec{u} \oplus \vec{v}$$

This establishes a principal difference between classical physics and quantum mechanics because not all vectors of the combined space by tensor product can be expressed by vectors of the underlying spaces. The quantum space is richer than the classical space.

Properties of the tensor product

Let A, B, C, D be matrices and c, d, e, f (complex) constants, then:

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$c(A \otimes B) = (cA \otimes B) = (A \otimes cB)$$

$$(cA + dB) \otimes (eC + fD) = ceA \otimes C + cfA \otimes D + deB \otimes C + dfB \otimes D$$

Working with vectors $\vec{u}, \vec{v}, \vec{w}$ and a, b being (complex) constants we get the distributive law:

$$\vec{u} \otimes (a\vec{v} + b\vec{w}) = a\vec{u} \otimes \vec{v} + b\vec{u} \otimes \vec{w}$$

Example:

$\vec{w}$ is a vector of the combined space and assembled by two basis vectors:

$$\vec{w} = \vec{e}_{11} \otimes \vec{e}_{21} + \vec{e}_{12} \otimes \vec{e}_{22} - \vec{e}_{12} \otimes \vec{e}_{21} - \vec{e}_{11} \otimes \vec{e}_{22}$$

note: $\vec{e}_{11}$ is the basis vector $\vec{e}_1$ of space V_A, $\vec{e}_{21}$ is the basis vector $\vec{e}_1$ of space V_B etc.

We can represent $\vec{w}$ as $\vec{u}\otimes\vec{v}$:

$$\vec{u} = \vec{e}_{11} - \vec{e}_{12}$$

$$\vec{v} = \vec{e}_{21} - \vec{e}_{22}$$

We check:

$$\vec{u}\otimes\vec{v} = (\vec{e}_{11} - \vec{e}_{12})\otimes(\vec{e}_{21} - \vec{e}_{22}) =$$

$$\vec{e}_{11}\otimes\vec{e}_{21} - \vec{e}_{11}\otimes\vec{e}_{22} - \vec{e}_{12}\otimes\vec{e}_{21} + \vec{e}_{12}\otimes\vec{e}_{22}$$

Result:

The vector $\vec{w}$ as part of the space $\mathcal{H}_1\otimes\mathcal{H}_2$ can be assembled by two vectors of space $\mathcal{H}_1$ and $\mathcal{H}_2$.

Counterexample:

$$\vec{x} = \vec{e}_{11}\otimes\vec{e}_{21} - \vec{e}_{12}\otimes\vec{e}_{22}$$

$\vec{x}$ is part of the space $\mathcal{H}_1\otimes\mathcal{H}_2$. It cannot be represented by two vectors of space $\mathcal{H}_1$ and $\mathcal{H}_2$.

We check this. Note: $a, b, c\ d$ are (complex) values.

We try to assemble the vector $\vec{x}$ by vectors of the underlying spaces:

$$\vec{x} = (a\vec{e}_{11} + b\vec{e}_{12})\otimes(c\vec{e}_{21} + d\vec{e}_{22})$$

We resolve the tensor product:

$$\vec{x} = ac\vec{e}_{11}\otimes\vec{e}_{21} + ad\vec{e}_{11}\otimes\vec{e}_{22} + bc\vec{e}_{12}\otimes\vec{e}_{21} + bd\vec{e}_{12}\otimes\vec{e}_{22}$$

We check the coefficients:

$$ac = 1, bd = -1$$

$$ad = bc = 0$$

From $ac = 1$ and $bd = -1$ we get that none of the factors can be zero.

From $ad = bc = 0$ we get that either a or d and b or c must be zero.

This is a contradiction. The vector $\vec{x} = \vec{e}_{11}\otimes\vec{e}_{21} - \vec{e}_{12}\otimes\vec{e}_{22}$ cannot be assembled by vectors of the underlying spaces via the tensor product.

We recall: any spin state can be represented by linear combination of basis vectors $|z+\rangle := \begin{pmatrix}1\\0\end{pmatrix}$ and $|z-\rangle := \begin{pmatrix}0\\1\end{pmatrix}$:

$|x+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$|y+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	$|z+\rangle := \begin{pmatrix}1\\0\end{pmatrix}$
$|x-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$|y-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	$|z-\rangle := \begin{pmatrix}0\\1\end{pmatrix}$

We use the Pauli matrices:

$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$	$\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

Measuring a spin state requires an operator:

$S_x = \frac{\hbar}{2}\sigma_x$	$S_y = \frac{\hbar}{2}\sigma_y$	$S_z = \frac{\hbar}{2}\sigma_z$

Tensor product of spin basis vectors

We build all possible tensor products of the basis vectors:

Tensor-Table:

$a \otimes b$	$\vert x+\rangle$	$\vert x-\rangle$	$\vert y+\rangle$	$\vert y-\rangle$	$\vert z+\rangle$	$\vert z-\rangle$
$\vert x+\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\i\\1\\i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-i\\1\\-i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$
$\vert x-\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$
$\vert y+\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\i\\i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\i\\-i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\i\\i\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-i\\i\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\i\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\i\end{pmatrix}$
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$\vert z+\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$
$\vert z-\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\i\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-i\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Note: The tensor product is not commutative. All combinations are unique.

Note: We will need this table in the end, so we name it Tensor-Table.

For the tensor product holds:

$$(A\otimes B)(\vec{u}\otimes\vec{v}) = A\vec{u}\otimes B\vec{v}$$

Note: The operator A is acting on the Hilbert space $\mathcal{H}_1$, the operator B is acting on the Hilbert space $\mathcal{H}_2$. Vector $\vec{u}$ is part of Hilbert space $\mathcal{H}_1$, vector $\vec{v}$ is part of Hilbert space $\mathcal{H}_2$.

Note: This works for vectors $\vec{w} = \vec{u}\otimes\vec{v}$ only. Remember that there are vectors in the combined space that cannot be written as $(\vec{u}\otimes\vec{v})$.

If we measure a single spin, the apparatus acts on this spin only. Accordingly, the Hermitian operator acts on the Hilbert space $\mathcal{H}_1$ resp. $\mathcal{H}_2$ of that spin only. We achieve this effect by help of the tensor product:

$$A_1 = A \otimes id_2$$

$$B_2 = id_1 \otimes B$$

Note: A and B are the operators acting on the Hilbert spaces $\mathcal{H}_1$ resp. $\mathcal{H}_2$.

Note: id_1 and id_2 are the identity matrices in the Hilbert spaces $\mathcal{H}_1$ resp. $\mathcal{H}_2$.

Note: Operators A_1 and B_2 are acting on the whole system but modify only their subsystem because they are built with the tensor product.

With this we get:

$$[A, B] = 0$$

We check this by help of the definition of the tensor product:

$$(A \otimes B)(\vec{u} \otimes \vec{v}) = A\vec{u} \otimes B\vec{v}$$

$$[A_1, B_1] = (A \otimes id_2)(id_1 \otimes B)(\vec{u} \otimes \vec{v}) - (id_1 \otimes B)(A \otimes id_2)(\vec{u} \otimes \vec{v}) =$$

$$(A \otimes id_2)(\vec{u} \otimes B\vec{v}) - (id_1 \otimes B)(A\vec{u} \otimes \vec{v}) =$$

$$(A\vec{u} \otimes id_2 B\vec{v}) - (id_1 A\vec{u} \otimes B\vec{v}) =$$

$$(A\vec{u} \otimes B\vec{v}) - (A\vec{u} \otimes B\vec{v}) = 0$$

A combination often used is:

$$C = A_1 + B_2 = (A \otimes id_2) + (id_1 \otimes B)$$

Examples for this:

In case of two spins the operator needed to measure the first spin only:

$$S_{z1} = S_z \otimes id_2 = \frac{\hbar}{2}\sigma_z \otimes id_2$$

The operator for measuring the second spin only:

$$S_{z2} = id_1 \otimes S_z = id_1 \otimes \frac{\hbar}{2}\sigma_z$$

If we want to measure the total spin in z-direction, given by the sum of the two spins:

$$S_{z(1+2)} = S_{z1} + S_{z2} = S_z \otimes id_2 + id_1 \otimes S_z =$$

$$\frac{\hbar}{2}\sigma_z \otimes id_2 + id_1 \otimes \frac{\hbar}{2}\sigma_z$$

If both systems do not interact then the Hamiltonian of the combined system is given by the sum of the Hamiltonians of the individual systems:

$$H = H_1 + H_2 = H \otimes id_2 + id_1 \otimes H$$

To simplify reading we use the bra-ket notation and mark by position:

Vectors in the dual space:

$$|\overrightarrow{uv}\rangle := |\vec{u}\rangle \otimes |\vec{v}\rangle$$

$$\langle\overrightarrow{uv}| := \langle\vec{u}| \otimes \langle\vec{v}|$$

Basis vectors are written as:

$$|ij\rangle := |i\rangle \otimes |j\rangle$$

$$\langle ij| := \langle i| \otimes \langle j|$$

The scalar product in the combined space is defined as the product of two scalars of the individual spaces:

$$\langle\overrightarrow{uv}|\overrightarrow{xy}\rangle = \langle\vec{u}|\vec{x}\rangle\langle\vec{v}|\vec{y}\rangle$$

We assume the basis vectors orthonormal:

$$\langle ij|kl\rangle = \delta_{ij}\delta_{kl}$$

Eigenvalues and eigenvectors of tensor products

<table>
<tr>
<td>We calculate eigenvalues and eigenvectors of operator C built by the tensor product of two operators, A acting on $\mathcal{H}_{+1}$, B acting on $\mathcal{H}_{+2}$:
$$C := A \otimes B$$</td>
<td>We calculate eigenvalues and eigenvectors of operator C built by the sum of two operators, A acting on $\mathcal{H}_{+1}$, B acting on $\mathcal{H}_{+2}$:
$$C = A_1 + B_2 = (A \otimes id_2) + (id_1 \otimes B)$$</td>
</tr>
<tr>
<td colspan="2">We remember:
$$(A \otimes B)(\vec{u} \otimes \vec{v}) = A\vec{u} \otimes B\vec{v}$$</td>
</tr>
<tr>
<td>Eigenvalues of operator C are product of eigenvalues of operator A and eigenvalues of operator B.</td>
<td>Eigenvalues of operator C are sum of eigenvalues of operator A and eigenvalues of operator B.</td>
</tr>
<tr>
<td colspan="2">We use:
$\mathcal{H}_{1\lambda_1}$ is eigenspace of $\mathcal{H}_1$ to eigenvalue $\lambda_1 \to A|\vec{u}\rangle = \lambda_1|\vec{u}\rangle$
$\mathcal{H}_{2\lambda_2}$ is eigenspace of $\mathcal{H}_2$ to eigenvalue $\lambda_2 \to B|\vec{v}\rangle = \lambda_2|\vec{v}\rangle$</td>
</tr>
<tr>
<td>We get:
$$C|\overrightarrow{uv}\rangle = (A|\vec{u}\rangle \otimes B|\vec{v}\rangle) =$$
$$\lambda_1\vec{u} \otimes \lambda_2\vec{v} = \lambda_1\lambda_2\vec{u} \otimes \vec{v} =$$
$$\lambda_1\lambda_2|\overrightarrow{uv}\rangle$$</td>
<td>We get:
$$C|\overrightarrow{uv}\rangle = A|\vec{u}\rangle \otimes id_2|\vec{v}\rangle + id_1|\vec{u}\rangle \otimes B|\vec{v}\rangle =$$
$$\lambda_1|\vec{u}\rangle \otimes id_2|\vec{v}\rangle + id_1|\vec{u}\rangle \otimes \lambda_2|\vec{v}\rangle =$$
$$\lambda_1|\overrightarrow{uv}\rangle + \lambda_2|\overrightarrow{uv}\rangle =$$
$$(\lambda_1 + \lambda_2)|\overrightarrow{uv}\rangle$$</td>
</tr>
<tr>
<td>In case the product $\lambda_1 \cdot \lambda_2$ is unique , Eigen space to the product of eigenvalues $\lambda = \lambda_1 \cdot \lambda_2$ is the tensor product of the eigenspace to eigenvalue λ_1 and eigenspace to eigenvalue λ_2:
$$\mathcal{H}_\lambda = \mathcal{H}_{1\lambda_1} \otimes \mathcal{H}_{2\lambda_2}$$</td>
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$$\mathcal{H}_\lambda = \mathcal{H}_{1\lambda_1} \otimes \mathcal{H}_{2\lambda_2}$$</td>
</tr>
<tr>
<td>In case the product $\lambda_1 \cdot \lambda_2$ is not unique, we build the eigenspace by the direct sum of all possible combinations:
$$\mathcal{H}_\lambda = \bigoplus_{\lambda_1, \lambda_2 : \lambda_1 \cdot \lambda_2 = \lambda} \left(\mathcal{H}_{1\lambda_1} \otimes \mathcal{H}_{2\lambda_2}\right)$$</td>
<td>In case the sum $\lambda_1 + \lambda_2$ is not unique, we build the eigenspace by the direct sum of all possible combinations:
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</tr>
</table>

Example two spins: The operator $C := \sigma_z \otimes \sigma_z$ has eigenvalues: $\{1, -1\}$ Any vector of the combined system $|\vec{w}_+\rangle := a|\overrightarrow{z_+z_+}\rangle + b|\overrightarrow{z_-z_-}\rangle$ is eigenvector to eigenvalue $+1$. Any vector of the combined system $|\vec{w}_-\rangle := a|\overrightarrow{z_+z_-}\rangle + b|\overrightarrow{z_-z_+}\rangle$ is eigenvector to eigenvalue -1. Note: $\vec{w}_+, \vec{w}_-$ are vectors of the combined space. Note: a, b are (complex) numbers. We can combine eigenvalue 1: $(\lambda_1 = 1) \cdot (\lambda_2 = 1)$ $(\lambda_1 = -1) \cdot (\lambda_2 = -1)$ Analog, we combine eigenvalue -1: $(\lambda_1 = 1) \cdot (\lambda_2 = -1)$ $(\lambda_1 = -1) \cdot (\lambda_2 = 1)$	Example two spins: The operator $C := \sigma_z \otimes id_2 + id_1 \otimes \sigma_z$ has eigenvalues: $\{+2, 0, -2\}$ Eigenvalues $+2$ and -2 are unique. Eigenvalue 0 can be built by $(\lambda_1 = 1) + (\lambda_2 = -1)$ $(\lambda_1 = -1) + (\lambda_2 = 1)$ Any vector of the combined system $|\vec{w}_0\rangle := a|\overrightarrow{z_+z_-}\rangle + b|\overrightarrow{z_-z_+}\rangle$ is eigenvector to eigenvalue 0.
We get eigenspaces: $\mathcal{H}_{+1} = (\mathcal{H}_{1,+1} \otimes \mathcal{H}_{2,+1}) \oplus (\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,-1})$ $\mathcal{H}_{-1} = (\mathcal{H}_{1,+1} \otimes \mathcal{H}_{2,-1}) \oplus (\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,+1})$ $\mathcal{H}_{1,+1} \otimes \mathcal{H}_{2,+1}$ is a 1-D vector space built by $|\overrightarrow{z_+z_+}\rangle$. $\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,-1}$ is a 1-D vector space built by $|\overrightarrow{z_-z_-}\rangle$. Eigenspace $\mathcal{H}_{+1}$ is a 2-D vector space, the direct sum: $\mathcal{H}_{+1} = (\mathcal{H}_{1,+1} \otimes \mathcal{H}_{2,+1}) \oplus (\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,-1})$ Any vector in this space can be written as the sum of two independent vectors: $a|\overrightarrow{z_+z_+}\rangle + b|\overrightarrow{z_-z_-}\rangle$ The same holds for $\mathcal{H}_{-1}$.	We get eigenspaces: $\mathcal{H}_{+2} = (\mathcal{H}_{1,1} \otimes \mathcal{H}_{2,1})$ $\mathcal{H}_{-2} = (\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,-1})$ $\mathcal{H}_{+2}$ and $\mathcal{H}_{-2}$ are 1-D vector spaces built by $|\overrightarrow{z_+z_+}\rangle\ resp.\ |\overrightarrow{z_-z_-}\rangle$ Eigenspace $\mathcal{H}_0$ is a 2-D vector space, the direct sum: $\mathcal{H}_0 = (\mathcal{H}_{1,+1} \otimes \mathcal{H}_{2,-1}) \oplus (\mathcal{H}_{1,-1} \otimes \mathcal{H}_{2,+1})$ Any vector in this space can be written as the sum of two independent vectors: $a|\overrightarrow{z_+z_-}\rangle + b|\overrightarrow{z_-z_+}\rangle$

Correlation in tensor product states

Example

We work with a two-spin-system and the state $|\vec{w}\rangle$:

$$|\vec{w}\rangle := \frac{1}{\sqrt{2}}(|\overrightarrow{z_+z_+}\rangle + |\overrightarrow{z_-z_-}\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

We can choose the basis and express the state $|\vec{w}\rangle$ with the x-basis and the y-basis too:

$|\vec{w}\rangle := \frac{1}{\sqrt{2}}(|\overrightarrow{x_+x_+}\rangle + |\overrightarrow{x_-x_-}\rangle) =$	$|\vec{w}\rangle := \frac{1}{\sqrt{2}}(|\overrightarrow{y_+y_-}\rangle + |\overrightarrow{y_-y_+}\rangle) =$

$\frac{1}{\sqrt{2}}\left(\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right)=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\left(\frac{1}{2}\begin{pmatrix}1\\-i\\i\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\i\\-i\\1\end{pmatrix}\right)=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

Note: the state $|\vec{w}\rangle$ cannot be achieved by $\frac{1}{\sqrt{2}}(|\overrightarrow{y_+y_+}\rangle+|\overrightarrow{y_-y_-}\rangle)$:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{2}\begin{pmatrix}1\\i\\i\\-1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

For independent systems we calculate the probabilities for spin-up resp. spin-down by help of the projection operators in the $\overrightarrow{z_+}$ and $\overrightarrow{z_-}$ system:

$$P_{1z_+}=(|z_+\rangle\langle z_+|)\otimes id_2=\left(\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\right)\otimes id_2=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$P_{2z_+}=id_1\otimes(|z_+\rangle\langle z_+|)=id_1\otimes\left(\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$

$$P_{1z_-}=(|z_-\rangle\langle z_-|)\otimes id_2=\left(\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\right)\otimes id_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$P_{2z_-}=id_1\otimes(|z_-\rangle\langle z_-|)=id_1\otimes\left(\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

With this we get the expectation values with the vector $|\vec{w}\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$:

$$\langle w|P_{1z_+}|w\rangle=$$

$$\frac{1}{\sqrt{2}}(1\ 0\ 0\ 1)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=$$

$$\frac{1}{2}(1\ 0\ 0\ 1)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=$$

$$\frac{1}{2}(1\ 0\ 0\ 1)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{1}{2}$$

The other expectation values are the same:

$$\langle w|P_{1z_+}|w\rangle=\langle w|P_{2z_+}|w\rangle=\langle w|P_{1z_-}|w\rangle=\langle w|P_{2z_-}|w\rangle=\frac{1}{2}$$

The expectation values, the probabilities for spin-up and spin-down are 50%.

Note: The probabilities for the result of a measurement in system 1 are not independent of the result of a measurement in system 2:

$$\langle w|P_{1z_-}P_{2z_+}|w\rangle=$$

$$\frac{1}{\sqrt{2}}(1\ 0\ 0\ 1)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=$$

$$\frac{1}{2}(1\ 0\ 0\ 1)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=$$

$$\frac{1}{2}(1\ 0\ 0\ 1)\begin{pmatrix}0\\0\\0\\0\end{pmatrix}=0$$

Note: the order of the projection operators is irrelevant. Both are acting on different systems and thus commute.

Conclusion:

If we measure “spin-up” in system 2 then we know that the spin in system 1 is up too.

If we measure “spin-down” in system 2 then we know that the spin in system 1 is down too.

We can express the state $|\vec{w}\rangle$ in the x-basis and the y-basis. From this we can conclude that the measurement of the spin in x-direction and y-direction is not independent too.

We check the x-basis:

$$\langle w|P_{1x_+}P_{2x_-}|w\rangle=;$$

We need the operators:

$$P_{1x_+}=(|x_+\rangle\langle x_+|)\otimes id_2=\frac{1}{2}\left(\begin{pmatrix}1\\1\end{pmatrix}(1\ 1)\right)\otimes id_2=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}$$

$$P_{2x_+}=id_1\otimes(|x_+\rangle\langle x_+|)=id_1\otimes\frac{1}{2}\left(\begin{pmatrix}1\\1\end{pmatrix}(1\ 1)\right)=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&1\\1&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$$

$$P_{1x_-} = (|x_-\rangle\langle x_-|)\otimes id_2 = \frac{1}{2}\left(\begin{pmatrix}1\\-1\end{pmatrix}(1-1)\right)\otimes id_2 = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -1\\-1 & 0 & 1 & 0\\0 & -1 & 0 & 1\end{pmatrix}$$

$$P_{2x_-} = id_1\otimes(|x_-\rangle\langle x_-|) = id_1\otimes\frac{1}{2}\left(\begin{pmatrix}1\\-1\end{pmatrix}(1-1)\right) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 & -1 & 0 & 0\\-1 & 1 & 0 & 0\\0 & 0 & 1 & -1\\0 & 0 & -1 & 1\end{pmatrix}$$

We calculate:

$$\langle w|P_{1x_+}P_{2x_-}|w\rangle =$$

$$\frac{1}{\sqrt{2}}(1\,0\,0\,1)\frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1 & 0 & 0\\-1 & 1 & 0 & 0\\0 & 0 & 1 & -1\\0 & 0 & -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} =$$

$$\frac{1}{8}(1\,0\,0\,1)\begin{pmatrix}1 & -1 & 1 & -1\\-1 & 1 & -1 & 1\\1 & -1 & 1 & -1\\-1 & 1 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} =$$

$$\frac{1}{8}(1\,0\,0\,1)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 0$$

Conclusion

The spin in x-direction has the same behavior as the spin in z-direction. Measurement of the spin in x-direction in system 1 will fix the spin in system 2 parallel.

The measurement of the spin in y-direction shows antiparallel behavior.

We check the y-basis:

$$\langle w|P_{1y_+}P_{2y_-}|w\rangle =;$$

We need the operators. We remember that switching from bra to ket needs complex conjugation:

$$P_{1y_+} = (|y_+\rangle\langle y_+|)\otimes id_2 = \left(\begin{pmatrix}1\\-i\end{pmatrix}(1\,i)\right)\otimes id_2 = \frac{1}{2}\begin{pmatrix}1 & i\\-i & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & i & 0\\0 & 1 & 0 & i\\-i & 0 & 1 & 0\\0 & -i & 0 & 1\end{pmatrix}$$

$$P_{2y_+} = id_1\otimes(|y_+\rangle\langle y_+|) = id_1\otimes\left(\begin{pmatrix}1\\-i\end{pmatrix}(1\,i)\right) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix}1 & i\\-i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & 0 & 0\\-i & 1 & 0 & 0\\0 & 0 & 1 & i\\0 & 0 & -i & 1\end{pmatrix}$$

$$P_{1y_-} = (|y_-\rangle\langle y_-|)\otimes id_2 = \left(\begin{pmatrix}1\\ i\end{pmatrix}(1 \; -i)\right)\otimes id_2 = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & -i & 0\\ 0 & 1 & 0 & -i\\ i & 0 & 1 & 0\\ 0 & i & 0 & 1\end{pmatrix}$$

$$P_{2y_-} = id_1\otimes(|y_-\rangle\langle y_-|) = id_1\otimes\frac{1}{2}\left(\begin{pmatrix}1\\ i\end{pmatrix}(1 \; -i)\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i & 0 & 0\\ i & 1 & 0 & 0\\ 0 & 0 & 1 & -i\\ 0 & 0 & i & 1\end{pmatrix}$$

We do this in depth and calculate the expectation value of all possible combinations:

$\langle w\vert P_{1y_+}P_{2y_+}\vert w\rangle$	$\langle w\vert P_{1y_+}P_{2y_-}\vert w\rangle$
$\langle w\vert P_{1y_-}P_{2y_+}\vert w\rangle$	$\langle w\vert P_{1y_-}P_{2y_-}\vert w\rangle$

$\langle w\vert P_{1y_+}P_{2y_+}\vert w\rangle =$ $\frac{1}{\sqrt{2}}(1\;0\;0\;1)\frac{1}{2}\begin{pmatrix}1 & 0 & i & 0\\ 0 & 1 & 0 & i\\ -i & 0 & 1 & 0\\ 0 & -i & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & i & 0 & 0\\ -i & 1 & 0 & 0\\ 0 & 0 & 1 & i\\ 0 & 0 & -i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}1 & i & i & -1\\ -i & 1 & 1 & i\\ -i & 1 & 1 & i\\ -1 & -i & -i & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix} = 0$	$\langle w\vert P_{1y_+}P_{2y_-}\vert w\rangle =$ $\frac{1}{\sqrt{2}}(1\;0\;0\;1)\frac{1}{2}\begin{pmatrix}1 & 0 & i & 0\\ 0 & 1 & 0 & i\\ -i & 0 & 1 & 0\\ 0 & -i & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -i & 0 & 0\\ i & 1 & 0 & 0\\ 0 & 0 & 1 & -i\\ 0 & 0 & i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}1 & -i & i & 1\\ i & 1 & -1 & i\\ -i & -1 & 1 & -i\\ 1 & -i & i & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}2\\ 2i\\ -2i\\ 2\end{pmatrix} = \frac{4}{8} = \frac{1}{2}$
$\langle w\vert P_{1y_-}P_{2y_+}\vert w\rangle =$ $\frac{1}{\sqrt{2}}(1\;0\;0\;1)\frac{1}{2}\begin{pmatrix}1 & 0 & -i & 0\\ 0 & 1 & 0 & -i\\ i & 0 & 1 & 0\\ 0 & i & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & i & 0 & 0\\ -i & 1 & 0 & 0\\ 0 & 0 & 1 & i\\ 0 & 0 & -i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}1 & i & -i & 1\\ -i & 1 & -1 & -i\\ i & -1 & 1 & i\\ 1 & i & -i & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}2\\ -2i\\ 2i\\ 2\end{pmatrix} = \frac{4}{8} = \frac{1}{2}$	$\langle w\vert P_{1y_-}P_{2y_-}\vert w\rangle =$ $\frac{1}{\sqrt{2}}(1\;0\;0\;1)\frac{1}{2}\begin{pmatrix}1 & 0 & -i & 0\\ 0 & 1 & 0 & -i\\ i & 0 & 1 & 0\\ 0 & i & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -i & 0 & 0\\ i & 1 & 0 & 0\\ 0 & 0 & 1 & -i\\ 0 & 0 & i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}1 & -i & -i & -1\\ i & 1 & 1 & -i\\ i & 1 & 1 & -i\\ -1 & i & i & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} =$ $\frac{1}{8}(1\;0\;0\;1)\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix} = 0$

Conclusion

Measurement of the spin in y-direction in system 1 will fix the spin in system 2 antiparallel. The expectation value of $\langle w|P_{1y_+}P_{2y_-}|w\rangle$ and $\langle w|P_{1y_-}P_{2y_+}|w\rangle$ is 50% each but the expectation values of $\langle w|P_{1y_+}P_{2y_+}|w\rangle$ and $\langle w|P_{1y_-}P_{2y_-}|w\rangle$ are zero.

Note: The systems are correlated.

Statistical correlation

The conditional probability $P(A|B)$ describes the probability for an event A occurring, given that another event B has already occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Note: $P(A \cap B)$ is the probability that both A and B are occurring. $P(B)$ is the probability that B occurs.

We can express the conclusion above by help of conditional probability.

We write $P_{2z_+}(z_1 +)$ as the probability for measuring spin-up in System 1. With the conditional probability we get that the probability to measure spin-up in system 2, given that we measured spin-up in system 1:

$$P_{2z_+}(z_1 +) = \frac{P(z_+ z_+)}{P(z_2 +)} =$$

$$\frac{\langle w | P_{1z_+} P_{2z_+} | w \rangle}{\langle w | P_{2z_+} | w \rangle} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

The conditional probability that we measure spin-up in system 2, given that we measured spin-up in system 1, is one or 100%.

Time development of a correlated system

Given a system is at time $t = 0$ in a tensor product state $|\vec{w}\rangle = |\vec{u}\rangle \otimes |\vec{v}\rangle$. Given further we have a time independent Hamiltonian H.

We examine two cases the system remaining in a non-superposition state.

Case 1: $|\vec{w}\rangle$ is an eigenstate of the Hamiltonian H:

$$|\vec{w}(t)\rangle = e^{-i\frac{E}{\hbar}t}(|\vec{u}\rangle \otimes |\vec{v}\rangle)$$

Case 2: The Hamiltonian is the sum of the Hamiltonians of the subsystems:

$$H \coloneqq H_{11} + H_{22} = H_1 \otimes id_2 + id_1 \otimes H_2$$

In this case H_{11} and H_{22} are commuting and we get:

$$|\vec{w}(t)\rangle = e^{-i\frac{H}{\hbar}t}|\vec{w}(0)\rangle =$$

$$e^{-i\frac{H_{11}+H_{22}}{\hbar}t}|\vec{w}(0)\rangle =$$

$$\left(e^{-i\frac{H_{11}}{\hbar}t} e^{-i\frac{H_{22}}{\hbar}t}\right)|\vec{w}(0)\rangle =$$

$$\left(e^{-i\frac{H_1}{\hbar}t} \otimes id_2\right)\left(id_1 \otimes e^{-i\frac{H_2}{\hbar}t}\right)(|\vec{u}\rangle \otimes |\vec{v}\rangle) =$$

$$\left(e^{-i\frac{H_1}{\hbar}t}|\vec{u}\rangle\right) \otimes \left(e^{-i\frac{H_2}{\hbar}t}|\vec{v}\rangle\right)$$

Interaction

We choose a system of two spins.

The Energy of a two-spin system:

$$E = a \cdot \overrightarrow{s_1} \cdot \overrightarrow{s_2} =$$

$$a\left(s_{1x}s_{2x} + s_{1y}s_{2y} + s_{1z}s_{2z}\right)$$

Note: a is a (complex) number.

The Hamiltonian:

$$H = \frac{a\hbar^2}{4}\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) =;$$

We need the tensor products of the Pauli matrices:

$a \otimes b$	σ_x	σ_y	σ_z
σ_x	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
σ_y	$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$
σ_z	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note: all products are unique.

The Hamiltonian becomes:

$$H = \frac{a\hbar^2}{4}\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) =$$

$$\frac{a\hbar^2}{4}\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) =$$

$$\frac{a\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We calculate the eigenvalues:

$$det\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -1-\lambda & 2 & 0 \\ 0 & 2 & -1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} =$$

$$(1-\lambda)\big((-1-\lambda)(-1-\lambda)(1-\lambda)-2\cdot 2(1-\lambda)\big)=$$

$$(1-\lambda)^2\big((-1-\lambda)(-1-\lambda)-4\big)=$$

$$(1-\lambda)^2((-1-\lambda)^2-4)=$$

$$(1-\lambda)^2(\lambda^2+2\lambda-3)=;$$

Note: $\lambda^2+2\lambda-3=(\lambda-1)(\lambda+3)$

Note: $(1-\lambda)^2=(\lambda-1)^2$

We get:

$$(\lambda-1)(\lambda-1)(\lambda-1)(\lambda+3)$$

We get eigenvalues $\lambda_1=1$ and $\lambda_2=-3$.

Eigenvalue λ_1 is threefold degenerate, eigenvalue λ_2 is straight.

The eigenspace to eigenvalue 1 has 3 dimensions, the eigenspace to eigenvalue 2 has 1 dimension.

The eigenvalues of the Hamiltonian:

$$E_1=\frac{a\hbar^2}{4}$$

$$E_2=-\frac{3a\hbar^2}{4}$$

Eigen space to eigenvalue 1:

$$\begin{pmatrix}0&0&0&0\\0&-2&2&0\\0&2&-2&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$-2b+2c=0$$

$$2b-2c=0$$

We get:

$$b=c$$

This is a 3-D Eigen space. A possible orthonormal basis for this:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

We refer to the tensor product table above and identify the basis vectors:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=|\overrightarrow{z_+z_+}\rangle:=|\overrightarrow{e_1}\rangle$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |\overrightarrow{z_-z_-}\rangle := |\overrightarrow{e_2}\rangle$$

$$\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|\overrightarrow{z_+z_-}\rangle + |\overrightarrow{z_-z_+}\rangle) :=:= |\overrightarrow{e_3}\rangle$$

Eigen space to eigenvalue -3:

$$\begin{pmatrix}4 & 0 & 0 & 0\\0 & 2 & 2 & 0\\0 & 2 & 2 & 0\\0 & 0 & 0 & 4\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$4a = 0 \to a = 0$$

$$2b + 2c = 0 \to c = -b$$

$$2b + 2c = 0 \to c = -b$$

$$4d = 0 \to d = 0$$

We get:

$$c\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

This is a 1-D Eigen space. We refer to the tensor product table above and identify the basis vector as the combination:

$$\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|\overrightarrow{z_+z_-}\rangle - |\overrightarrow{z_-z_+}\rangle) := |\overrightarrow{e_4}\rangle$$

Superposition by interaction

We use our two-spin-system:

$$H = \frac{a\hbar^2}{4}\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right)$$

We can describe every state in this system by four basis vectors:

$$|\overrightarrow{e_1}\rangle, |\overrightarrow{e_2}\rangle, |\overrightarrow{e_3}\rangle, |\overrightarrow{e_4}\rangle$$

At time $t = 0$ from the tensor product space we use the state $|\vec{w}(0)\rangle$:

$$|\vec{w}(0)\rangle := |\overrightarrow{z_+z_-}\rangle = \frac{1}{\sqrt{2}}(|\overrightarrow{e_2}\rangle + |\overrightarrow{e_4}\rangle)$$

By using the eigenvalues $E_1 = \frac{a\hbar^2}{4}$ and $E_2 = -\frac{3a\hbar^2}{4}$ of the Hamiltonian we get:

$$|\vec{w}(t)\rangle = \frac{1}{\sqrt{2}}(|\overrightarrow{e_2}\rangle + |\overrightarrow{e_4}\rangle) = \frac{1}{\sqrt{2}}\left(e^{i\frac{a\hbar^2}{4}t}|\overrightarrow{e_2}\rangle + e^{i\frac{-3a\hbar^2}{4}t}|\overrightarrow{e_4}\rangle\right)$$

To simplify writing we use:

$$\omega_1 := \frac{a\hbar^2}{4};\ \omega_2 := \frac{-3a\hbar^2}{4}$$

We rewrite:

$$|\vec{w}(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{i\omega_1 t}|\vec{e_2}\rangle + e^{i\omega_2 t}|\vec{e_4}\rangle\right)$$

We can factor out the exponentials if:

$$e^{i\omega_1 t} = e^{i\omega_2 t}$$

$$e^{i\omega_1 t - i\omega_2 t} = 1$$

$$i\omega_1 t - i\omega_2 t = 2n\pi$$

Note: $n \in \mathbb{Z}$

$$t(\omega_1 - \omega_2) = 2n\pi$$

$$t = \frac{2n\pi}{\omega_1 - \omega_2}$$

For these times t the system is the state $|\overrightarrow{z_+z_-}\rangle$.

With the basis vectors $|\vec{e_2}\rangle$ and $|\vec{e_4}\rangle$ we can build another state of the tensor product space:

$$|\overrightarrow{z_-z_+}\rangle = \frac{1}{\sqrt{2}}(|\vec{e_2}\rangle - |\vec{e_4}\rangle)$$

For this we need:

$$e^{i\omega_1 t} = -e^{i\omega_2 t}$$

$$e^{i\omega_1 t - i\omega_2 t} = -1$$

$$i\omega_1 t - i\omega_2 t = (2n+1)\pi$$

Note: $n \in \mathbb{Z}$

$$t(\omega_1 - \omega_2) = (2n+1)\pi$$

$$t = \frac{(2n+1)\pi}{\omega_1 - \omega_2}$$

For these times t the system is the state $|\overrightarrow{z_-z_+}\rangle$.

For all other times the system is in a superposition state:

$$|\vec{w}(t)\rangle = a(t)|\overrightarrow{z_+z_-}\rangle + b(t)|\overrightarrow{z_-z_+}\rangle$$

Note: a and b are (complex) constants.