This paper deals with unitarian operators used in quantum mechanics.

Related information you may find at:

https://www.math.tamu.edu/~dallen/m640_03c/lectures/chapter4.pdf

http://dorsett-edu.us/QuantumMechanics/Zwiebach%20Notes%20PDF.pdf

Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German…)

Hope I can help you with learning quantum mechanics.

Contents

Exponential of a matrix

The exponential of a matrix A is defined by power series:

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

We check this with Pauli matrices:

$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We need the powers of the Pauli matrices.

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x{}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x{}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x{}^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\dots$$

We get:

$$\sigma_x{}^{2n} = id, \qquad \sigma_x{}^{2n+1} = \sigma_x$$

The power series:

$$e^{a \cdot \sigma_x} := \sum_{n=0}^{\infty} \frac{a^n}{n!} \sigma_x{}^n =$$

$$\frac{1}{0!} id + \frac{a}{1!}\sigma_x + \frac{a^2}{2!} id + \frac{a^3}{3!}\sigma_x + \frac{a^4}{4!} id + \frac{a^5}{5!}\sigma_x + \cdots =$$

$$\left(\frac{1}{0!} id + \frac{a^2}{2!} id + \frac{a^4}{4!} id + \cdots\right) + \left(\frac{a}{1!}\sigma_x + \frac{a^3}{3!}\sigma_x + \frac{a^5}{5!}\sigma_x + \cdots\right) =$$

$$id\left(\frac{1}{0!} + \frac{a^2}{2!} + \frac{a^4}{4!} + \cdots\right) + \sigma_x\left(\frac{a}{1!} + \frac{a^3}{3!} + \frac{a^5}{5!} + \cdots\right) =$$

$$id \cdot cosh(a) + \sigma_x \sinh(a) =$$

$$\begin{pmatrix} cosh(a) & \sinh(a) \\ \sinh(a) & cosh(a) \end{pmatrix}$$

$cosh$ and $sinh$ are nonperiodic functions, the values for $e^{a \cdot \sigma_x}$ are growing with respect to a.

We try exponentiation of the complex operator and multiply $a \cdot \sigma_x$ by the imaginary unit $-i$:

$$e^{-i \cdot a \cdot \sigma_x} := \sum_{n=0}^{\infty} \frac{(-i)^n a^n}{n!} {\sigma_x}^n =$$

$$\frac{1}{0!} id - i \cdot \frac{a}{1!} \sigma_x - \frac{a^2}{2!} id + i \cdot \frac{a^3}{3!} \sigma_x + \frac{a^4}{4!} id - i \cdot \frac{a^5}{5!} \sigma_x - \frac{a^6}{6!} id \dots =$$

$$\left(\frac{1}{0!} id - \frac{a^2}{2!} id + \frac{a^4}{4!} id - \cdots\right) + \left(-i \cdot \frac{a}{1!} \sigma_x + i \cdot \frac{a^3}{3!} \sigma_x - i \cdot \frac{a^5}{5!} \sigma_x + \cdots\right) =$$

$$id \left(\frac{1}{0!} id - \frac{a^2}{2!} id + \frac{a^4}{4!} id - \cdots\right) - i \cdot \sigma_x \left(\frac{a}{1!} - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\right) =$$

$$id \cdot cos(a) - i \cdot \sigma_x \sin(a) =$$

$$\begin{pmatrix} cos(a) & -\mathrm{i} \cdot \sin(a) \\ -\mathrm{i} \cdot \sin(a) & cos(a) \end{pmatrix}$$

The same for σ_y:

$$\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$${\sigma_y}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$${\sigma_x}^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$${\sigma_x}^4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\dots$$

We get:

$${\sigma_y}^{2n} = id, \qquad {\sigma_y}^{2n+1} = \sigma_y$$

We calculate:

$$e^{-i \cdot a \cdot \sigma_y} := \sum_{n=0}^{\infty} \frac{(-i)^n a^n}{n!} {\sigma_y}^n =$$

$$\frac{1}{0!} id - i \cdot \frac{a}{1!} \sigma_y - \frac{a^2}{2!} id + i \cdot \frac{a^3}{3!} \sigma_y + \frac{a^4}{4!} id - i \cdot \frac{a^5}{5!} \sigma_y - \frac{a^6}{6!} id \dots =$$

$$\left(\frac{1}{0!} id - \frac{a^2}{2!} id + \frac{a^4}{4!} id - \cdots\right) + \left(-i \cdot \frac{a}{1!} \sigma_y + i \cdot \frac{a^3}{3!} \sigma_y - i \cdot \frac{a^5}{5!} \sigma_y + \cdots\right) =$$

$$id \left(\frac{1}{0!} - \frac{a^2}{2!} + \frac{a^4}{4!} - \cdots\right) - i \cdot \sigma_y \left(\frac{a}{1!} - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\right) =$$

$$id \cdot cos(a) - i \cdot \sigma_y \sin(a) =$$

$$\begin{pmatrix} cos(a) & -\sin(a) \\ \sin(a) & cos(a) \end{pmatrix}$$

Note:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The same for σ_z.

$$\sigma_z \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$${\sigma_z}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$${\sigma_x}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$${\sigma_x}^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\dots$$

We get:

$${\sigma_z}^{2n} = id, \qquad {\sigma_z}^{2n+1} = \sigma_z$$

We calculate:

$$e^{-i\cdot a\cdot \sigma_z} \coloneqq \sum_{n=0}^{\infty} \frac{(-i)^n a^n}{n!} {\sigma_z}^n =$$

$$\frac{1}{0!} id - i \cdot \frac{a}{1!}\sigma_z - \frac{a^2}{2!} id + i \cdot \frac{a^3}{3!}\sigma_z + \frac{a^4}{4!} id - i \cdot \frac{a^5}{5!}\sigma_z - \frac{a^6}{6!} id \dots =$$

$$\left(\frac{1}{0!} id - \frac{a^2}{2!} id + \frac{a^4}{4!} id - \cdots\right) + \left(-i \cdot \frac{a}{1!}\sigma_z + i \cdot \frac{a^3}{3!}\sigma_z - i \cdot \frac{a^5}{5!}\sigma_z + \cdots\right) =$$

$$id\left(\frac{1}{0!} id - \frac{a^2}{2!} id + \frac{a^4}{4!} id - \cdots\right) - i \cdot \sigma_z\left(\frac{a}{1!} - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\right) =$$

$$id \cdot cos(a) - i \cdot \sigma_z \sin(a) =$$

$$\begin{pmatrix} cos(a) & 0 \\ 0 & cos(a) \end{pmatrix} - i \cdot \begin{pmatrix} \sin(a) & 0 \\ 0 & -\sin(a) \end{pmatrix} =$$

$$\begin{pmatrix} cos(a) - i \cdot sin(a) & 0 \\ 0 & cos(a) + i \cdot sin(a) \end{pmatrix} =$$

$$\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}$$

Note: for diagonal matrices A we have:

$$e^{\begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n \end{pmatrix}} = \begin{pmatrix} e^a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^n \end{pmatrix}$$

The exponential of a diagonal matrix a_{ii} is the diagonal matrix of the exponentials $e^{a_{ii}}$.

The process of exponentiation gives new matrices. We name them:

$$U_x(a) = e^{-i \cdot a \cdot \sigma_x} = \begin{pmatrix} cos(a) & -\mathrm{i} \cdot \sin(a) \\ -\mathrm{i} \cdot \sin(a) & cos(a) \end{pmatrix}$$

$$U_y(a) = e^{-i \cdot a \cdot \sigma_y} = \begin{pmatrix} cos(a) & -\sin(a) \\ \sin(a) & cos(a) \end{pmatrix}$$

$$U_z(a) = e^{-i \cdot a \cdot \sigma_z} = \begin{pmatrix} cos(a) - i \cdot sin(a) & 0 \\ 0 & cos(a) + i \cdot sin(a) \end{pmatrix} = \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}$$

Unitary matrix

A matrix U is called unitary:

$$UU^\dagger = U^\dagger U = id$$

Note: † denotes the matrix is transposed and complex conjugated.

The matrices $U_x(a), U_y(a), U_z(a)$ are unitary. We check this:

$$U_x(a)U_x(a)^\dagger =$$

$$\begin{pmatrix} cos(a) & -\mathrm{i} \cdot \sin(a) \\ -\mathrm{i} \cdot \sin(a) & cos(a) \end{pmatrix} \begin{pmatrix} cos(a) & \mathrm{i} \cdot \sin(a) \\ \mathrm{i} \cdot \sin(a) & cos(a) \end{pmatrix} =$$

$$\begin{pmatrix} cos^2(a) + sin^2(a) & \mathrm{i} \cdot \cos(\mathrm{a}) \cdot \sin(a) - \mathrm{i} \cdot \cos(\mathrm{a}) \cdot \sin(a) \\ \mathrm{i} \cdot \cos(\mathrm{a}) \cdot \sin(a) - \mathrm{i} \cdot \cos(\mathrm{a}) \cdot \sin(a) & sin^2(a) + cos^2(a) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_y(a)U_y(a)^\dagger =$$

$$\begin{pmatrix} cos(a) & -\sin(a) \\ \sin(a) & cos(a) \end{pmatrix} \begin{pmatrix} cos(a) & \sin(a) \\ -\sin(a) & cos(a) \end{pmatrix} =$$

$$\begin{pmatrix} cos^2(a) + sin^2(a) & \cos(\mathrm{a}) \cdot \sin(a) - \cos(\mathrm{a}) \cdot \sin(a) \\ \cos(\mathrm{a}) \cdot \sin(a) - \cos(\mathrm{a}) \cdot \sin(a) & sin^2(a) + cos^2(a) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_z(a)U_z(a)^\dagger =$$

$$\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix} \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: if H is a Hermitian matrix then e^{iH} is unitary.

Dot product

Unitarian matrices don't change the dot product:

We set $|u'\rangle = U|u\rangle$ and $|v'\rangle = U|v\rangle$.

We need the bra:

$$|u'\rangle = U|u\rangle \rightarrow \langle u'| = \langle u|U^\dagger$$

Note: switching the matrix from left to right needs transposition, switching from ket to bra needs complex conjugation.

We build the dot product:

$$\langle u'|v'\rangle = \langle u|U^\dagger U|v\rangle = \langle u|v\rangle$$

Note: unitarian matrices map orthonormal bases to new orthonormal bases.

Note: for each two orthonormal bases exists an appropriate unitarian matrix.

Let $|e_i\rangle$ be an orthonormal basis:

$$U|e_i\rangle = |f_i\rangle$$

$$U^\dagger|f_i\rangle = |e_i\rangle$$

Note: unitary matrices are the complex equivalent to rotation matrices.

Unitary matrices and spin vectors

We remember:

matrix	eigenvalues	Normalized eigenvectors	Written as
$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$+1, -1$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\|x+\rangle, \|x-\rangle$
$\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$+1, -1$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$	$\|y+\rangle, \|y-\rangle$
$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$+1, -1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\|z+\rangle, \|z-\rangle$

We take basis vectors $|e_1\rangle = |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |e_2\rangle = |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We use the unitary matrix:

$$U_y(a) = e^{-ia\sigma_y} = \begin{pmatrix} cos(a) & -sin(a) \\ sin(a) & cos(a) \end{pmatrix}$$

We try $a := \frac{\pi}{4}$:

$$U_y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

We check the effect of U_y onto the basis vectors:

$U_y\left(\frac{\pi}{4}\right)$ onto $\|x+\rangle$:	$U_y\left(\frac{\pi}{4}\right)$ onto $\|x-\rangle$:
$U_y\left(\frac{\pi}{4}\right)\|x+\rangle =$	$U_y\left(\frac{\pi}{4}\right)\|x-\rangle =$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \|z-\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \|z+\rangle$

$U_y\left(\frac{\pi}{4}\right)$ onto $\|y+\rangle$:	$U_y\left(\frac{\pi}{4}\right)$ onto $\|y-\rangle$:
$U_y\left(\frac{\pi}{4}\right)\|y+\rangle =$	$U_y\left(\frac{\pi}{4}\right)\|y-\rangle =$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$

$U_y\left(\frac{\pi}{4}\right)$ onto $\|z+\rangle$:	$U_y\left(\frac{\pi}{4}\right)$ onto $\|z-\rangle$:
$U_y\left(\frac{\pi}{4}\right)\|z+\rangle =$	$U_y\left(\frac{\pi}{4}\right)\|z-\rangle =$
$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \|x+\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = -\|x-\rangle$

Note: U_y swaps the z-basis onto the x-basis and vice versa.

We examine the effect of U_y onto the y-basis.

We change from column to row vector:

$$\frac{1}{2}\begin{pmatrix}1-i\\ 1+i\end{pmatrix} \rightarrow \frac{1}{2}\big((1+i)(1-i)\big)$$

We build the dot product:

$$\frac{1}{2}\big((1+i)(1-i)\big)\frac{1}{2}\begin{pmatrix}1+i\\ 1-i\end{pmatrix} = \frac{1}{4}\big((1+i)(1+i)+(1-i)(1-i)\big) =$$

$$\frac{1}{4}(2i-2i) = 0$$

The vectors are orthogonal.

We check the size:

$$\frac{1}{2}\big((1+i)(1-i)\big)\frac{1}{2}\begin{pmatrix}1-i\\ 1+i\end{pmatrix} =$$

$$\frac{1}{4}\big((1+i)(1-i)+(1-i)(1+i)\big) =$$

$$\frac{1}{4}(1+1+1+1) = 1$$

The vectors $\frac{1}{2}\begin{pmatrix}1-i\\ 1+i\end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix}1+i\\ 1-i\end{pmatrix}$ build an orthonormal basis.

In general, we have:

$$U_y(a) = \begin{pmatrix}cos(a) & -sin(a)\\ sin(a) & cos(a)\end{pmatrix}$$

The effect of $U_y(a)$ onto $\|y+\rangle$:	The effect of $U_y(a)$ onto $\|y-\rangle$:
$U_y(a)\|y+\rangle = \begin{pmatrix}cos(a) & -sin(a)\\ sin(a) & cos(a)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix}cos(a)-i\cdot sin(a)\\ sin(a)+i\cdot cos(a)\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix}cos(a)-i\cdot sin(a)\\ i\big(cos(a)-i\cdot sin(a)\big)\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\big(cos(a)-i\cdot sin(a)\big)\begin{pmatrix}1\\ i\end{pmatrix} =$ $e^{-ia}\|y+\rangle$	$U_y(a)\|y-\rangle = \begin{pmatrix}cos(a) & -sin(a)\\ sin(a) & cos(a)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix}cos(a)+i\cdot sin(a)\\ sin(a)-i\cdot cos(a)\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix}cos(a)+i\cdot sin(a)\\ -i\big(cos(a)+i\cdot sin(a)\big)\end{pmatrix} =$ $\frac{1}{\sqrt{2}}\big(cos(a)+i\cdot sin(a)\big)\begin{pmatrix}1\\ -i\end{pmatrix} =$ $e^{ia}\|y-\rangle$

Note: up to a constant phase $U_y(a)$ maintains $|y+\rangle$ and $|y-\rangle$.

Note: Rotation by $\frac{\pi}{4}$ in the complex system corresponds to a rotation by $\frac{\pi}{2}$ in real space.

We check the same effect of $U_x(a)$ and $U_z(a)$.

$$U_x(a) = e^{-ia\sigma_x} = \begin{pmatrix} cos(a) & -isin(a) \\ -isin(a) & cos(a) \end{pmatrix}$$

The effect of $U_x(a)$ onto $\|x+\rangle$:	The effect of $U_x(a)$ onto $\|x-\rangle$:
$U_x(a)\|x+\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} cos(a) - i\cdot sin(a) \\ -i\cdot sin(a) + cos(a) \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\left(cos(a) - i\cdot sin(a)\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$ $e^{-ia}\|x+\rangle$	$U_x(a)\|x-\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} cos(a) + i\cdot sin(a) \\ -i\cdot sin(a) - cos(a) \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\left(cos(a) + i\cdot sin(a)\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$ $e^{ia}\|x-\rangle$

The effect of $U_x(a)$ onto $\|y+\rangle$:	The effect of $U_x(a)$ onto $\|y-\rangle$:
$U_x(a)\|y+\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} cos(a) + sin(a) \\ i\left(cos(a) - sin(a)\right) \end{pmatrix} =;$ We set $a \coloneqq \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ i\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$ $\|z+\rangle$	$U_x(a)\|y-\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} cos(a) - sin(a) \\ -i\left(sin(a) + cos(a)\right) \end{pmatrix} =;$ We set $a \coloneqq \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ -i\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i\frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix} = -i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $-i\cdot\|z-\rangle$ Note: $-i$ is a constant phase.

The effect of $U_x(a)$ onto $\|z+\rangle$:	The effect of $U_x(a)$ onto $\|z-\rangle$:
$U_x(a)\|z+\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$ $\begin{pmatrix} cos(a) \\ -i\cdot sin(a) \end{pmatrix} =;$ We set $a \coloneqq \frac{\pi}{4}$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i\cdot\frac{1}{\sqrt{2}} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$ $\|y-\rangle$	$U_x(a)\|z-\rangle =$ $\begin{pmatrix} cos(a) & -i\cdot sin(a) \\ -i\cdot sin(a) & cos(a) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$ $\begin{pmatrix} -i\cdot sin(a) \\ cos(a) \end{pmatrix} =;$ We set $a \coloneqq \frac{\pi}{4}$ $\begin{pmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} =$ $-i\cdot\|y+\rangle$ Note: $-i$ is a constant phase.

$$U_z(a) = e^{-ia\sigma_z} = \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}$$

The effect of $U_z(a)$ onto $\vert x+\rangle$:	The effect of $U_z(a)$ onto $\vert x-\rangle$:
$U_z(a)\vert x+\rangle =$ $\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ e^{ia} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}e^{-ia}\begin{pmatrix} 1 \\ e^{2ia} \end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\begin{pmatrix} 1 \\ e^{i\frac{\pi}{2}} \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\vert y+\rangle$ Note: $e^{-i\frac{\pi}{4}}$ is a constant phase	$U_z(a)\vert x-\rangle =$ $\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ -e^{ia} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}e^{-ia}\begin{pmatrix} 1 \\ -e^{2ia} \end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\begin{pmatrix} 1 \\ -e^{i\frac{\pi}{2}} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\vert y-\rangle$

The effect of $U_z(a)$ onto $\vert y+\rangle$:	The effect of $U_z(a)$ onto $\vert y-\rangle$:
$U_z(a)\vert y+\rangle =$ $\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ i \cdot e^{ia} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ e^{i\frac{\pi}{2}} \cdot e^{ia} \end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{4}} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{4}} \\ e^{i\frac{3\pi}{4}} \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\frac{4\pi}{4}} \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\vert x-\rangle$ Note: $e^{-i\frac{\pi}{4}}$ is a constant phase	$U_z(a)\vert y-\rangle =$ $\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ -i \cdot e^{ia} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ia} \\ e^{i\frac{3\pi}{2}} \cdot e^{ia} \end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{4}} \\ e^{i\frac{3\pi}{2}} \cdot e^{i\frac{\pi}{4}} \end{pmatrix} =$ $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\pi}{4}} \\ e^{i\frac{7\pi}{4}} \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\frac{8\pi}{4}} \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$ $e^{-i\frac{\pi}{4}}\vert x+\rangle$

The effect of $U_z(a)$ onto $\|z+\rangle$:	The effect of $U_z(a)$ onto $\|z-\rangle$:
$U_z(a)\|z+\rangle = \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-ia}\\0\end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\begin{pmatrix}e^{-i\frac{\pi}{4}}\\0\end{pmatrix} = e^{-i\frac{\pi}{4}}\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\frac{\pi}{4}}\|z+\rangle$ Note: $e^{-i\frac{\pi}{4}}$ is a constant phase	$U_z(a)\|z-\rangle = \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\e^{ia}\end{pmatrix} =;$ We set $a := \frac{\pi}{4}$ $\begin{pmatrix}0\\e^{i\frac{\pi}{4}}\end{pmatrix} = e^{i\frac{\pi}{4}}\begin{pmatrix}0\\1\end{pmatrix} = e^{i\frac{\pi}{4}}\|z-\rangle$

We summarize the results:

The effect of $U_x\left(\frac{\pi}{4}\right)$			The effect of $U_y\left(\frac{\pi}{4}\right)$			The effect of $U_z\left(\frac{\pi}{4}\right)$	
onto $\|x+\rangle$:	$e^{-i\frac{\pi}{4}}\|x+\rangle$		onto $\|x+\rangle$:	$\|z-\rangle$		onto $\|x+\rangle$:	$e^{-i\frac{\pi}{4}}\|y+\rangle$
onto $\|x-\rangle$:	$e^{i\frac{\pi}{4}}\|x-\rangle$		onto $\|x-\rangle$:	$\|z+\rangle$		onto $\|x-\rangle$:	$e^{-i\frac{\pi}{4}}\|y-\rangle$
onto $\|y+\rangle$:	$\|z+\rangle$		onto $\|y+\rangle$:	$e^{-i\frac{\pi}{4}}\|y+\rangle$		onto $\|y+\rangle$:	$e^{-i\frac{\pi}{4}}\|x-\rangle$
onto $\|y-\rangle$:	$-i\cdot\|z-\rangle$		onto $\|y-\rangle$:	$e^{i\frac{\pi}{4}}\|y-\rangle$		onto $\|y-\rangle$:	$e^{-i\frac{\pi}{4}}\|x+\rangle$
onto $\|z+\rangle$:	$\|y-\rangle$		onto $\|z+\rangle$:	$\|x+\rangle$		onto $\|z+\rangle$:	$e^{-i\frac{\pi}{4}}\|z+\rangle$
onto $\|z-\rangle$:	$-i\cdot\|y+\rangle$		onto $\|z-\rangle$:	$\|x-\rangle$		onto $\|z-\rangle$:	$e^{i\frac{\pi}{4}}\|z-\rangle$

If we omit the constant phases we get:

The effect of $U_x\left(\frac{\pi}{4}\right)$			The effect of $U_y\left(\frac{\pi}{4}\right)$			The effect of $U_z\left(\frac{\pi}{4}\right)$	
onto $\|x+\rangle$:	$\|x+\rangle$		onto $\|x+\rangle$:	$\|z-\rangle$		onto $\|x+\rangle$:	$\|y+\rangle$
onto $\|x-\rangle$:	$\|x-\rangle$		onto $\|x-\rangle$:	$\|z+\rangle$		onto $\|x-\rangle$:	$\|y-\rangle$
onto $\|y+\rangle$:	$\|z+\rangle$		onto $\|y+\rangle$:	$\|y+\rangle$		onto $\|y+\rangle$:	$\|x-\rangle$
onto $\|y-\rangle$:	$\|z-\rangle$		onto $\|y-\rangle$:	$\|y-\rangle$		onto $\|y-\rangle$:	$\|x+\rangle$
onto $\|z+\rangle$:	$\|y-\rangle$		onto $\|z+\rangle$:	$\|x+\rangle$		onto $\|z+\rangle$:	$\|z+\rangle$
onto $\|z-\rangle$:	$\|y+\rangle$		onto $\|z-\rangle$:	$\|x-\rangle$		onto $\|z-\rangle$:	$\|z-\rangle$

Unitary matrices and 3D rotation

We take the Pauli matrices and build a vector:

$$\begin{pmatrix}\sigma_x\\\sigma_y\\\sigma_z\end{pmatrix}$$

We combine this with vector $\vec{b} \in \mathbb{R}^3$:

$$\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}$$

We build the dot product:

$$(b_x\ b_y\ b_z)\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z$$

This results in a matrix B:

$$B \coloneqq b_x\sigma_x + b_y\sigma_y + b_z\sigma_z$$

B is a 2D matrix with complex coefficients:

$$B = b_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + b_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + b_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} =$$

$$\begin{pmatrix}0 & b_x\\ b_x & 0\end{pmatrix} + \begin{pmatrix}0 & -i\cdot b_y\\ i\cdot b_y & 0\end{pmatrix} + \begin{pmatrix}b_z & 0\\ 0 & -b_z\end{pmatrix} =$$

$$\begin{pmatrix}b_z & b_x - i\cdot b_y\\ b_x + i\cdot b_y & -b_z\end{pmatrix}$$

Note: $b_i \in \mathbb{R}$

Applying the unitary matrix $U^{\dagger}(a)BU(a)$ results in a rotation by double angle a of vector $\vec{b}$:

$U^{\dagger}{}_x(a)BU_x(a) \cong R_x(2a)\vec{b}$	$U^{\dagger}{}_y(a)BU_y(a) \cong R_y(2a)\vec{b}$	$U^{\dagger}{}_z(a)BU_z(a) \cong R_z(2a)\vec{b}$

Note: This calculation not performed here.

Note: Pauli matrices represent basis vectors in $\mathbb{R}^3$.

The rotation matrices:

$R_x(a) = \begin{pmatrix}1 & 0 & 0\\ 0 & cos(a) & sin(a)\\ 0 & -sin(a) & cos(a)\end{pmatrix}$	$R_y(a) = \begin{pmatrix}cos(a) & 0 & -sin(a)\\ 0 & 1 & 0\\ sin(a) & 0 & cos(a)\end{pmatrix}$	$R_z(a) = \begin{pmatrix}cos(a) & sin(a) & 0\\ -sin(a) & cos(a) & 0\\ 0 & 0 & 1\end{pmatrix}$

Remark: It is not astonishing that rotation in 3D corresponds to a 2D complex matrix. Rotation essentially is a 2D operation leaving the rotation axis unchanged.

Remark: The correspondence between 2D complex space and 3D spatial space gives easy access to measuring the spin in any direction. We combine any arbitrary direction in 2D complex space by help of the Pauli matrices and this way get the operator for measuring the 3D spatial direction.

Example

To measure the spin in spatial direction $\frac{1}{\sqrt{2}}(e_x + e_y)$ we build the corresponding operator:

$$\frac{\hbar}{2}\frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

By finding eigenvectors and eigenvalues of this operator, we will know the possible outcomes of a measurement along the direction $(e_x + e_y)$.

By combining the basis vectors e_x, e_y, e_z we can measure the spin in any spatial direction.