This paper deals with the topic that any Hermitian matrix can be written as a diagonal matrix.

Every normal matrix is diagonalizable. A matrix M is called normal if it commutes with its Hermitian conjugate:

$$[M^{\dagger}, M] = 0$$

The Hermitian conjugate of a matrix is its transposed and complex conjugated counterpart.

Every Hermitian, every unitary matrix is diagonalizable.

We show this explicitly with a $2x2$-Hermitian matrix.

For more Information see Griffiths, appendix A.5, Eigenvectors and Eigenvalues.

An example for a $3x3$-matrix you find at:

https://www.math.tamu.edu/~dallen/DistanceEd/Math640/chapter5/node8.html

Hope I can help you with learning quantum mechanics.

We use the common method of transforming the target matrix to the final diagonal form and parallel apply these transformations to the identity matrix.

Note: a and c are real numbers, complex conjugation of b expressed as b^*.

Here we begin $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

multiply first line by $\frac{b^*}{a}$ gives $\begin{pmatrix} b^* & \frac{bb^*}{a} \\ b^* & c \end{pmatrix} | \begin{pmatrix} \frac{b^*}{a} & 0 \\ 0 & 1 \end{pmatrix}$

subtract first line from second line gives $\begin{pmatrix} b^* & \frac{bb^*}{a} \\ 0 & c - \frac{bb^*}{a} \end{pmatrix} | \begin{pmatrix} \frac{b^*}{a} & 0 \\ -\frac{b^*}{a} & 1 \end{pmatrix}$

transform the fraction in the second line $\begin{pmatrix} b^* & \frac{bb^*}{a} \\ 0 & \frac{ac-bb^*}{a} \end{pmatrix} | \begin{pmatrix} \frac{b^*}{a} & 0 \\ -\frac{b^*}{a} & 1 \end{pmatrix}$

multiply second line with $\frac{bb^*}{ac-bb^*}$ $\begin{pmatrix} b^* & \frac{bb^*}{a} \\ 0 & \frac{bb^*}{a} \end{pmatrix} | \begin{pmatrix} \frac{b^*}{a} & 0 \\ -\frac{b^*}{a} \cdot \frac{bb^*}{ac-bb^*} & \frac{bb^*}{ac-bb^*} \end{pmatrix}$

subtract second line from first line $\begin{pmatrix} b^* & 0 \\ 0 & \frac{bb^*}{a} \end{pmatrix} | \begin{pmatrix} \left(\frac{b^*}{a} + \frac{b^*}{a} \cdot \frac{bb^*}{ac-bb^*}\right) & -\frac{bb^*}{ac-bb^*} \\ -\frac{b^*}{a} \cdot \frac{bb^*}{ac-bb^*} & \frac{bb^*}{ac-bb^*} \end{pmatrix} =$

$$\begin{pmatrix} b^* & 0 \\ 0 & \frac{bb^*}{a} \end{pmatrix} | \begin{pmatrix} \frac{b^*}{a}\left(1 + \frac{bb^*}{ac-bb^*}\right) & -\frac{bb^*}{ac-bb^*} \\ -\frac{b^*}{a} \cdot \frac{bb^*}{ac-bb^*} & \frac{bb^*}{ac-bb^*} \end{pmatrix}$$

divide first line by b^* $\begin{pmatrix} 1 & 0 \\ 0 & \frac{bb^*}{a} \end{pmatrix} | \begin{pmatrix} \frac{1}{a}\left(1 + \frac{bb^*}{ac-bb^*}\right) & -\frac{b}{ac-bb^*} \\ -\frac{b^*}{a} \cdot \frac{bb^*}{ac-bb^*} & \frac{bb^*}{ac-bb^*} \end{pmatrix}$

multiply the whole matrix by a $\begin{pmatrix} a & 0 \\ 0 & bb^* \end{pmatrix} | \begin{pmatrix} \left(1 + \frac{bb^*}{ac-bb^*}\right) & -\frac{ab}{ac-bb^*} \\ -\frac{b^*bb^*}{ac-bb^*} & \frac{abb^*}{ac-bb^*} \end{pmatrix}$

On the left there is a diagonal matrix with real values only, bb^* is the square of the absolute value of b. On the right we have the transformation matrix.

To have these transformations possible, a, b^* and $ac - bb^*$ must not be zero.

> If a is zero, we can transform the matrix into an anti-diagonal matrix $\begin{pmatrix} 0 & c \\ b^* & 0 \end{pmatrix}$.
>
> If b^* is zero, then also b is zero and we already have a diagonal matrix.
>
> If $ac - bb^*$ is zero, the determinant of the matrix is zero and we cannot transform it into a diagonal matrix.

We assume that these conditions are fulfilled and check whether we did the correct calculation and multiply the transformation matrix with our Hermitian matrix above:

$$\begin{pmatrix} \left(1+\frac{bb^*}{ac-bb^*}\right) & -\frac{ab}{ac-bb^*} \\ -\frac{b^*bb^*}{ac-bb^*} & \frac{abb^*}{ac-bb^*} \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{ac-bb^*}{ac-bb^*}+\frac{bb^*}{ac-bb^*}\right) & -\frac{ab}{ac-bb^*} \\ -\frac{b^*bb^*}{ac-bb^*} & \frac{abb^*}{ac-bb^*} \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} =$$

$$\begin{pmatrix} \frac{ac}{ac-bb^*} & -\frac{ab}{ac-bb^*} \\ -\frac{b^*bb^*}{ac-bb^*} & \frac{abb^*}{ac-bb^*} \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} =$$

$$\frac{1}{ac-bb^*}\begin{pmatrix} ac & -ab \\ -b^*bb^* & abb^* \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} =;$$

We do this in parts.

Entry $(1,1)$

$$\frac{aca-abb^*}{ac-bb^*} = a\cdot\frac{ca-bb^*}{ac-bb^*} = a$$

Entry $(1,2)$

$$\frac{acb-abc}{ac-bb^*} = 0$$

Entry $(2,1)$

$$\frac{-b^*bb^*a+abb^*b^*}{ac-bb^*} = 0$$

Entry $(2,2)$

$$\frac{-b^*bb^*b+abb^*c}{ac-bb^*} =$$

$$bb^*\frac{-b^*b+ac}{ac-bb^*} = bb^*$$

Result:

$$\begin{pmatrix} \frac{1}{a}\left(1+\frac{bb^*}{ac-bb^*}\right) & -\frac{b}{ac-bb^*} \\ -\frac{b^*}{a}\cdot\frac{bb^*}{ac-bb^*} & \frac{bb^*}{ac-bb^*} \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & bb^* \end{pmatrix}$$

This is the general case.

For the general 3×3 case the solution is possible in the same way but not as handy as the 2×2 one.

We take the general matrix:

$$\begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix}$$

We triangularize it:

$$\begin{pmatrix} a & b & c \\ 0 & ad-bb^* & ae-b^*c \\ 0 & 0 & (ad-bb^*)f+(b^*c-ae)e^*+(be-cd)c^* \end{pmatrix}$$

This is formally correct if the diagonal has only real values. We check this:

- a is a real number
- $ad-bb^*$ is a real number
- $(ad-bb^*)f+(b^*c-ae)e^*+(be-cd)c^*$?
    - $(ad-bb^*)f$ is a real number
    - $(b^*c-ae)e^*+(be-cd)c^* = b^*ce^*-aee^*+bec^*-cc^*d =$;
      aee^* and cc^*d are real numbers
      $b^*ce^*+bc^*e$ is a real number too.

It is possible to cancel out entries $(1,2)$, $(1,3)$ and $(2,3)$ and this way to diagonalize the matrix, but we get no simple scheme.