This short paper shows how to calculate angular momentum in cartesian coordinates. We will do all calculations explicitly – if you need results only, you either skip the calculations or maybe you find other sources better suitable for your purpose.

Hope I can help you with learning quantum mechanics.

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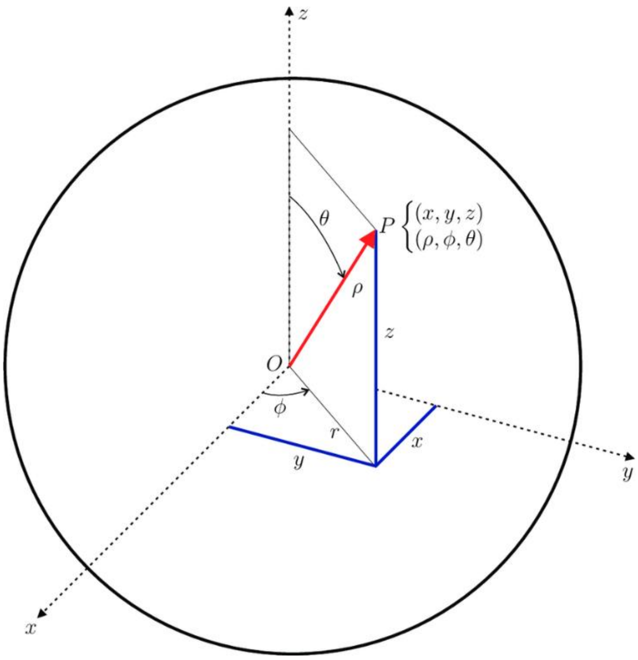
## Spherical and cartesian coordinates




https://www.researchgate.net/profile/Werner-Dahm

Note: $\theta$ and $\phi$ often are reversed in literature.

## Angular momentum classical

| Linear momentum: $$\vec{p} = m \cdot \vec{v} = m \cdot \dot{\vec{x}}$$ | Angular momentum: $$\vec{L} = m \cdot \vec{r} \times \vec{v}$$ |
|---|---|
| with unit $\left[\frac{kg \cdot m}{s}\right]$. | with unit $\left[\frac{kg \cdot m^2}{s}\right]$. |

Note that the unit of angular momentum is different from the unit of linear momentum.

In 3D we have:

| Linear momentum | Angular momentum cartesian |
|---|---|
| $\vec{p} = m \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ | $\vec{L} = m \cdot \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix} = \begin{pmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{pmatrix} := \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ |

| Coordinates spherical | Coordinates cartesian |
|---|---|
| $r = \sqrt{x^2 + y^2 + z^2}$<br>$\theta = arccos\left(\frac{z}{r}\right)$<br>$\phi = arctan\left(\frac{y}{x}\right)$ | $x = r \cdot sin(\theta) \cdot cos(\phi)$<br>$y = r \cdot sin(\theta) \cdot sin(\phi)$<br>$z = r \cdot cos(\theta)$ |

## $L_x, L_y, L_z$ in terms of spherical coordinates

Note: for moving objects we have $x(t), y(t), z(t), r(t), \theta(t), \phi(t)$.

We need the derivatives.

$$\dot{x} = \frac{d}{dt}x = \frac{d}{dt}\big(r \cdot sin(\theta) \cdot cos(\phi)\big) =$$

$$\dot{r} \cdot sin(\theta) \cdot cos(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}$$

$$\dot{y} = \frac{d}{dt}y = \frac{d}{dt}\big(r \cdot sin(\theta) \cdot sin(\phi)\big) =$$

$$\dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}$$

$$\dot{z} = \frac{d}{dt}z = \frac{d}{dt}\big(r \cdot cos(\theta)\big) =$$

$$\dot{r} \cdot cos(\theta) - r \cdot sin(\theta) \cdot \dot{\theta}$$

We get:

$$L_x = m \cdot (y\dot{z} - z\dot{y}) =$$

$$m \cdot \Big(\big(r \cdot sin(\theta) \cdot sin(\phi)\big)\big(\dot{r} \cdot cos(\theta) - r \cdot sin(\theta) \cdot \dot{\theta}\big) - \big(r \cdot cos(\theta)\big)\big(\dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big)\Big) =$$

$$m \cdot \Big(\big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos(\theta) - r \cdot sin(\theta) \cdot sin(\phi) \cdot r \cdot sin(\theta) \cdot \dot{\theta}\big) - \big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot cos(\theta) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot cos(\theta) \cdot r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big)\Big) =$$

$$m \cdot \Big(\big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos(\theta) - r^2 \cdot sin^2(\theta) \cdot sin(\phi) \cdot \dot{\theta}\big) - \big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi) + r^2 \cdot cos^2(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big)\Big) =$$

$$m \cdot \big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos\theta - r^2 \cdot sin^2(\theta) \cdot sin(\phi) \cdot \dot{\theta} - r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi) - r^2 \cdot cos^2(\theta) \cdot \dot{\theta} \cdot sin(\phi) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big) =$$

$$m \cdot \big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos(\theta) - r^2 \cdot sin^2(\theta) \cdot sin(\phi) \cdot \dot{\theta} - r^2 \cdot cos^2(\theta) \cdot \dot{\theta} \cdot sin(\phi) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi} - r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi)\big) =$$

$$m \cdot \big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos(\theta) - r^2 \cdot sin(\phi) \cdot \dot{\theta}\big(sin^2(\theta) + cos^2(\theta)\big) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi} - r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi)\big) =$$

$$m \cdot \big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot cos(\theta) - r^2 \cdot sin(\phi) \cdot \dot{\theta} - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi} - r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi)\big) =$$

$$m \cdot \big(-r^2 \cdot sin(\phi) \cdot \dot{\theta} - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big) =$$

$$-m \cdot r^2 \cdot \big(sin(\phi) \cdot \dot{\theta} + cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big)$$

$L_y = m \cdot (z\dot{x} - x\dot{z}) =$

$$m \cdot \Big(\big(r \cdot cos(\theta)\big)\big(\dot{r} \cdot sin(\theta) \cdot cos(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big) - \big(r \cdot sin(\theta) \cdot cos(\phi)\big)\big(\dot{r} \cdot cos(\theta) - r \cdot sin(\theta) \cdot \dot{\theta}\big)\Big) =$$

$$m \cdot \Big(\big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r \cdot cos(\theta) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r \cdot cos(\theta) \cdot r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big) - \big(r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot cos(\theta) - r \cdot sin(\theta) \cdot cos(\phi) \cdot r \cdot sin\theta \cdot \dot{\theta}\big)\Big) =$$

$$m \cdot \big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r^2 \cdot cos^2(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi} - r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot cos(\theta) + r^2 \cdot sin^2(\theta) \cdot cos(\phi) \cdot r \cdot \dot{\theta}\big) =$$

$$m \cdot \Big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r^2 \cdot cos^2(\theta) \cdot \dot{\theta} \cdot cos(\phi) + r^2 \cdot sin^2(\theta) \cdot cos(\phi) \cdot r \cdot \dot{\theta} - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi} - r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot cos(\theta)\Big) =$$

$$m \cdot \Big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r^2 \cdot \dot{\theta} \cdot cos(\phi)\big(cos^2(\theta) + sin^2(\theta)\big) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi} - r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot cos(\theta)\Big) =$$

$$m \cdot \Big(r \cdot cos(\theta) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r^2 \cdot \dot{\theta} \cdot cos(\phi) - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi} - r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot cos(\theta)\Big) =$$

$$m \cdot \big(r^2 \cdot cos(\phi) \cdot \dot{\theta} - r^2 \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big) =$$

$$m \cdot r^2 \cdot \big(cos(\phi) \cdot \dot{\theta} - cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big)$$

$L_z = m \cdot (x\dot{y} - y\dot{x}) =$

$$m \cdot \Big(\big(r \cdot sin(\theta) \cdot cos(\phi)\big)\big(\dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big) - \big(r \cdot sin(\theta) \cdot sin(\phi)\big)\big(\dot{r} \cdot sin(\theta) \cdot cos(\phi) + r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big)\Big) =$$

$$m \cdot \Big(\big(r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\big) - \big(r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) + r \cdot sin(\theta) \cdot sin(\phi) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) - r \cdot sin(\theta) \cdot sin(\phi) \cdot r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big)\Big) =$$

$$m \cdot \big(r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{r} \cdot sin(\theta) \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot sin(\phi) + r \cdot sin(\theta) \cdot cos(\phi) \cdot r \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi} - r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{r} \cdot sin(\theta) \cdot cos(\phi) - r \cdot sin(\theta) \cdot sin(\phi) \cdot r \cdot cos(\theta) \cdot \dot{\theta} \cdot cos(\phi) + r \cdot sin(\theta) \cdot sin(\phi) \cdot r \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\big) =$$

$$m \cdot \big(r \cdot \dot{r} \cdot sin^2(\theta) \cdot sin(\phi) \cdot cos(\phi) + r^2 \cdot sin(\theta) \cdot sin(\phi) \cdot cos(\phi) \cdot cos(\theta) \cdot \dot{\theta} + r^2 \cdot sin^2(\theta) \cdot cos^2(\phi) \cdot \dot{\phi} - r \cdot \dot{r} \cdot sin^2(\theta) \cdot sin(\phi) \cdot cos(\phi) - r^2 \cdot sin(\theta) \cdot sin(\phi) \cdot cos(\phi) \cdot cos(\theta) \cdot \dot{\theta} + r^2 \cdot sin^2(\theta) \cdot sin^2(\phi) \cdot \dot{\phi}\big) =$$

$$m \cdot \left(r^2 \cdot sin(\theta) \cdot sin(\phi) \cdot cos(\phi) \cdot cos(\theta) \cdot \dot{\theta} + r^2 \cdot sin^2(\theta) \cdot cos^2(\phi) \cdot \dot{\phi} - r^2 \cdot sin(\theta) \cdot sin(\phi) \cdot cos(\theta) \cdot cos(\phi) \cdot \dot{\theta} + r^2 \cdot sin^2(\theta) \cdot sin^2(\phi) \cdot \dot{\phi}\right) =$$

$$m \cdot \left(r^2 \cdot sin^2(\theta) \cdot cos^2(\phi) \cdot \dot{\phi} + r^2 \cdot sin^2(\theta) \cdot sin^2(\phi) \cdot \dot{\phi}\right) =$$

$$m \cdot r^2 \cdot sin^2(\theta) \cdot \dot{\phi}\left(cos^2(\phi) + sin^2(\phi)\right) =$$

$$m \cdot r^2 \cdot sin^2(\theta) \cdot \dot{\phi}$$

We collect the results:

$$L_x = m \cdot (y\dot{z} - z\dot{y}) = -m \cdot r^2 \cdot \left(sin(\phi) \cdot \dot{\theta} + cos(\theta) \cdot sin(\theta) \cdot cos(\phi) \cdot \dot{\phi}\right)$$

$$L_y = m \cdot (z\dot{x} - x\dot{z}) = m \cdot r^2 \cdot \left(cos(\phi) \cdot \dot{\theta} - cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \cdot \dot{\phi}\right)$$

$$L_z = m \cdot (x\dot{y} - y\dot{x}) = m \cdot r^2 \cdot sin^2(\theta) \cdot \dot{\phi}$$

## Applications

### Rotation in the $x - y$ plane

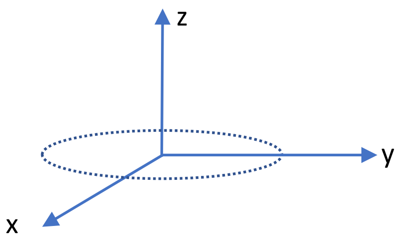
An object rotating in the $x - y$ plane has $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$.

We get the angular momenta:

$$L_x = -m \cdot r^2 \cdot \left(sin(\phi) \cdot 0 + 0 \cdot 1 \cdot cos(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_y = m \cdot r^2 \cdot \left(cos(\phi) \cdot 0 - 0 \cdot 1 \cdot sin(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_z = m \cdot r^2 \cdot \dot{\phi}$$




### Linear movement through the origin

An object linear moving through the origin has $\dot{\theta} = 0$ and $\dot{\phi} = 0$

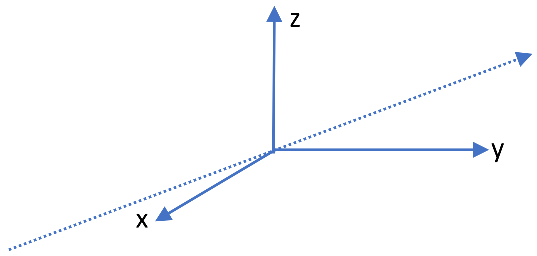
$\theta$ and $\phi$ are constant but switches to $\phi + \pi$ and $\theta + \pi$ if the particle crosses the origin.

We get the angular momenta outside the origin:

$$L_x = -m \cdot r^2 \cdot \left(sin(\phi) \cdot 0 + 0 \cdot 1 \cdot cos(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_y = m \cdot r^2 \cdot \left(cos(\phi) \cdot 0 - 0 \cdot 1 \cdot sin(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_z = m \cdot r^2 \cdot 0 = 0$$




During crossing the origin, the derivatives $\dot{\theta}$ and $\dot{\phi}$ become delta functions.

### Linear movement bypassing the origin

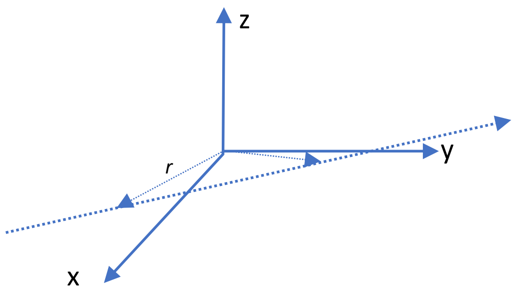
An object moving in the $x - y$ plane, bypassing the origin is constantly changing $r$ and $\phi$.

We have:

$$x(t) = v_x \cdot t + x_0$$

$$y(t) = v_y \cdot t + y_0$$

$$r(t) = \sqrt{(v_x \cdot t + x_0)^2 + \left(v_y \cdot t + y_0\right)^2}$$




Note: $v_x, v_y, x_0, y_0$ are constants.

We calculate $\dot{r}$:

$$\dot{r} = \frac{d}{dt}r(t) = \frac{2\cdot(v_x\cdot t+x_0)\cdot v_x + 2\cdot\left(v_y\cdot t+y_0\right)\cdot v_y}{2\cdot\sqrt{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}} =$$

$$\frac{(v_x\cdot t+x_0)\cdot v_x + \left(v_y\cdot t+y_0\right)\cdot v_y}{\sqrt{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}} =$$

$$\frac{t\cdot\left({v_x}^2+{v_y}^2\right)+x_0\cdot v_x+y_0\cdot v_y}{\sqrt{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}}$$

We calculate $\phi(t)$:

$$\phi(t) = \arctan\left(\frac{y(t)}{x(t)}\right) = \arctan\left(\frac{v_y\cdot t+y_0}{v_x\cdot t+x_0}\right)$$

We remember:

$$\frac{d}{du}arctan(u) = \frac{1}{1+u^2}$$

We calculate $\dot{\phi}$ by use of the chain rule:

$$\dot{\phi} = \frac{d}{dt}\left(\arctan\left(\frac{v_y\cdot t+y_0}{v_x\cdot t+x_0}\right)\right) =$$

$$\frac{1}{1+\left(\frac{v_y\cdot t+y_0}{v_x\cdot t+x_0}\right)^2}\cdot\frac{\left(v_y\right)(v_x\cdot t+x_0)-\left(v_y\cdot t+y_0\right)(v_x)}{(v_x\cdot t+x_0)^2} =$$

$$\frac{1}{\frac{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}{(v_x\cdot t+x_0)^2}}\cdot\frac{\left(v_y\right)(v_x\cdot t+x_0)-\left(v_y\cdot t+y_0\right)(v_x)}{(v_x\cdot t+x_0)^2} =$$

$$\frac{(v_x\cdot t+x_0)^2}{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}\cdot\frac{\left(v_y\right)(v_x\cdot t+x_0)-\left(v_y\cdot t+y_0\right)(v_x)}{(v_x\cdot t+x_0)^2} =$$

$$\frac{\left(v_y\right)(v_x\cdot t+x_0)-\left(v_y\cdot t+y_0\right)(v_x)}{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2} =$$

$$\frac{\left(v_y\cdot v_x\cdot t+v_y\cdot x_0\right)-\left(v_x\cdot v_y\cdot t+v_x\cdot y_0\right)}{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2} =$$

$$\frac{v_y\cdot v_x\cdot t+v_y\cdot x_0-v_x\cdot v_y\cdot t-v_x\cdot y_0}{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2} =$$

$$\frac{v_y\cdot x_0-v_x\cdot y_0}{(v_x\cdot t+x_0)^2+\left(v_y\cdot t+y_0\right)^2}$$

So far, we have:

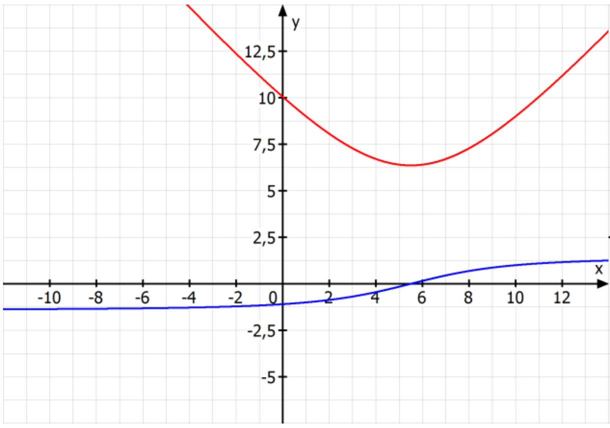
$$r(t) = \sqrt{(v_x \cdot t + x_0)^2 + (v_y \cdot t + y_0)^2}$$

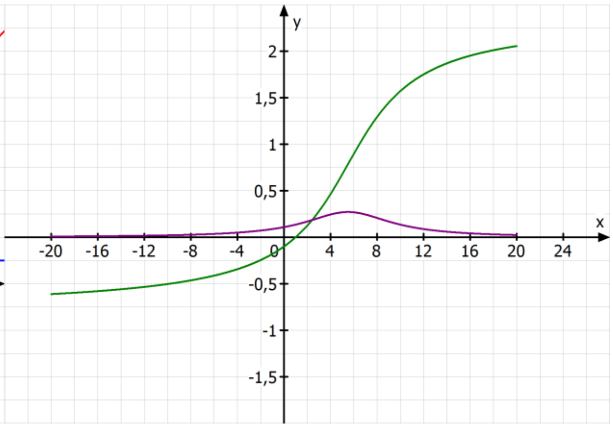
$$\dot{r} = \frac{t \cdot (v_x{}^2 + v_y{}^2) + x_0 \cdot v_x + y_0 \cdot v_y}{\sqrt{(v_x \cdot t + x_0)^2 + (v_y \cdot t + y_0)^2}}$$

$$\phi(t) = \arctan\left(\frac{v_y \cdot t + y_0}{v_x \cdot t + x_0}\right)$$

$$\dot{\phi} = \frac{v_y \cdot x_0 - v_x \cdot y_0}{(v_x \cdot t + x_0)^2 + (v_y \cdot t + y_0)^2} =$$

We check choosing printable values $v_x = -1, v_y = +1, x_0 = 10, y_0 = -1$:




*red:* $r(t)$ *blue:* $\dot{r}$ 

*green:* $\phi(t)$ *purple:* $\dot{\phi}$

Note: $\arctan\left(\frac{v_y \cdot t + y_0}{v_x \cdot t + x_0}\right)$ produces a discontinuity if $v_x \cdot t + x_0 = 0$. We corrected this by using:

$$\arctan\left(\frac{v_y \cdot t + y_0}{v_x \cdot t + x_0}\right) \; for \; (v_x \cdot t + x_0) > 0$$

$$\arctan\left(\frac{v_y \cdot t + y_0}{v_x \cdot t + x_0}\right) + \pi \; for \; (v_x \cdot t + x_0) < 0$$

$$\frac{\pi}{2} \; for \; (v_x \cdot t + x_0) = 0$$

We get the angular momentum:

$$L_x = -m \cdot r^2 \cdot \left(sin(\phi) \cdot 0 + 0 \cdot 1 \cdot cos(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_y = m \cdot r^2 \cdot \left(cos(\phi) \cdot 0 - 0 \cdot 1 \cdot sin(\phi) \cdot \dot{\phi}\right) = 0$$

$$L_z = m \cdot r^2 \cdot \dot{\phi}$$

We take a closer look at $L_z$. We use $r(t)$ and $\dot{\phi}$:

$$r(t) = \sqrt{(v_x \cdot t + x_0)^2 + (v_y \cdot t + y_0)^2}$$

$$\dot{\phi} = \frac{v_y \cdot x_0 - v_x \cdot y_0}{(v_x \cdot t + x_0)^2 + \left(v_y \cdot t + y_0\right)^2}$$

We calculate $L_z$:

$$L_z(t) = m \cdot r(t)^2 \cdot \dot{\phi} =$$

$$m \cdot \left((v_x \cdot t + x_0)^2 + \left(v_y \cdot t + y_0\right)^2\right) \cdot \frac{v_y \cdot x_0 - v_x \cdot y_0}{(v_x \cdot t + x_0)^2 + \left(v_y \cdot t + y_0\right)^2} =$$

$$m \cdot \left(v_y \cdot x_0 - v_x \cdot y_0\right) =;$$

$v_x, v_y, x_0, y_0$ are constants depending on speed of the particle $v_x, v_y$ and the starting point $x_0, y_0$ chosen.

We conclude $L_z(t)$ is constant in time:

$$L_z(t) = m \cdot c$$

We can choose an inertial system with $L_z(t) = 0$:

$$v_y \cdot x_0 - v_x \cdot y_0 = 0$$

$$v_y \cdot x_0 = v_x \cdot y_0$$

$$\frac{v_y}{v_x} = \frac{y_0}{x_0}$$

A system moving with constant speed $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ will measure no angular momentum because the particle is on rest.

## Angular momentum quantum

We remember angular momentum classic in 3D:

| Angular momentum classic | Angular momentum quantum |
|---|---|
| $\vec{L} = m \cdot \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix} = \begin{pmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{pmatrix} := \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ | $\hat{\vec{L}} = \begin{pmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{pmatrix} := \begin{pmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{pmatrix}$ |

| Coordinates spherical | Coordinates cartesian |
|---|---|
| $r = \sqrt{x^2 + y^2 + z^2}$ <br> $\theta = arccos\left(\frac{z}{r}\right)$ <br> $\phi = arctan\left(\frac{y}{x}\right)$ | $x = r \cdot sin(\theta) \cdot cos(\phi)$ <br> $y = r \cdot sin(\theta) \cdot sin(\phi)$ <br> $z = r \cdot cos(\theta)$ |

## $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in terms of spherical coordinates

We remember:

| | |
|---|---|
| $\frac{d}{du} arccos(u) = -\frac{1}{\sqrt{1-u^2}}$ | $\frac{d}{du} arctan(u) = \frac{1}{1+u^2}$ |

## Partial derivatives

| | | |
|---|---|---|
| $\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$ |
| $\frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{z}{r}\right)^2}} \cdot \frac{zx}{r^3}$ | $\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{z}{r}\right)^2}} \cdot \frac{zy}{r^3}$ | $\frac{\partial \theta}{\partial z} = \frac{1}{\sqrt{1-\left(\frac{z}{r}\right)^2}} \cdot \frac{z^2}{r^3}$ |
| $\frac{\partial \phi}{\partial x} = -\frac{y}{x^2+y^2}$ | $\frac{\partial \phi}{\partial y} = \frac{1}{x \cdot \left(1+\left(\frac{y}{x}\right)^2\right)} =$ | $\frac{\partial \phi}{\partial z} = 0$ |

We replace $x, y, z$ by its corresponding angular expressions. These are lengthy calculations, we show it by $\frac{1}{\sqrt{1-\left(\frac{z}{r}\right)^2}} \cdot \frac{zx}{r^3}$ as an example.

$$\frac{1}{\sqrt{1-\left(\frac{z}{r}\right)^2}} \cdot \frac{zx}{r^3} \rightarrow$$

$$\frac{1}{\sqrt{1-\left(cos(\theta)\right)^2}} \cdot \frac{r \cdot cos(\theta) \cdot x}{r^3} = \frac{1}{\sqrt{1-\left(cos(\theta)\right)^2}} \cdot \frac{cos(\theta) \cdot r \cdot sin(\theta) \cdot cos(\phi)}{r^2} =$$

$$\frac{1}{\sqrt{1-\left(cos(\theta)\right)^2}} \cdot \frac{cos(\theta) \cdot sin(\theta) \cdot cos(\phi)}{r} = \frac{1}{\sqrt{1-1+\left(sin(\theta)\right)^2}} \cdot \frac{cos(\theta) \cdot sin(\theta) \cdot cos(\phi)}{r} =$$

$$\frac{1}{sin(\theta)} \cdot \frac{cos(\theta) \cdot sin(\theta) \cdot cos(\phi)}{r} =$$

$$\frac{cos(\theta) \cdot cos(\phi)}{r}$$

Result:

| | | |
|---|---|---|
| $\frac{\partial r}{\partial x} = sin(\theta) \cdot cos(\phi)$ | $\frac{\partial r}{\partial y} = sin(\theta) \cdot sin(\phi)$ | $\frac{\partial r}{\partial z} = cos(\theta)$ |
| $\frac{\partial \theta}{\partial x} = \frac{cos(\theta) \cdot cos(\phi)}{r}$ | $\frac{\partial \theta}{\partial y} = \frac{cos(\theta) \cdot sin(\phi)}{r}$ | $\frac{\partial \theta}{\partial z} = -\frac{sin(\theta)}{r}$ |
| $\frac{\partial \phi}{\partial x} = -\frac{sin(\phi)}{r \cdot sin(\theta)}.$ | $\frac{\partial \phi}{\partial y} = \frac{cos(\phi)}{r \cdot sin(\theta)}$ | $\frac{\partial \phi}{\partial z} = 0$ |

Each column gives us a partial derivative:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \rightarrow$$

$$\frac{\partial}{\partial x} = sin(\theta) \cdot cos(\phi) \frac{\partial}{\partial r} + \frac{cos(\theta) \cdot cos(\phi)}{r} \frac{\partial}{\partial \theta} - \frac{sin(\phi)}{r \cdot sin(\theta)} \frac{\partial}{\partial \phi}$$

Similar:

$$\frac{\partial}{\partial y} = sin(\theta) \cdot sin(\phi) \frac{\partial}{\partial r} + \frac{cos(\theta) \cdot sin(\phi)}{r} \frac{\partial}{\partial \theta} + \frac{cos(\phi)}{r \cdot sin(\theta)} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = cos(\theta) \frac{\partial}{\partial r} - \frac{sin(\theta)}{r} \frac{\partial}{\partial \theta}$$

We write angular momenta in terms of spherical components.

## Angular momentum $\hat{L}_x$

$$\begin{pmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{pmatrix} := \begin{pmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{pmatrix}$$

We remember operators:

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \hat{p}_z = -i\hbar \frac{\partial}{\partial z}$$

$$\hat{x} = x, \hat{y} = y, \hat{z} = z$$

We calculate:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) =$$

$$-i\hbar \left( r \cdot sin(\theta) \cdot sin(\phi) \frac{\partial}{\partial z} - r \cdot cos(\theta) \frac{\partial}{\partial y} \right) =$$

$$-i\hbar \left( r \cdot sin(\theta) \cdot sin(\phi) \left( cos(\theta) \frac{\partial}{\partial r} - \frac{sin(\theta)}{r} \frac{\partial}{\partial \theta} \right) - r \cdot cos(\theta) \left( sin(\theta) \cdot sin(\phi) \frac{\partial}{\partial r} + \frac{cos(\theta) \cdot sin(\phi)}{r} \frac{\partial}{\partial \theta} + \frac{cos(\phi)}{r \cdot sin(\theta)} \frac{\partial}{\partial \phi} \right) \right) =$$

$$-i\hbar \left( r \cdot sin(\theta) \cdot sin(\phi) \cdot cos(\theta) \frac{\partial}{\partial r} - r \cdot sin(\theta) \cdot sin(\phi) \cdot \frac{sin(\theta)}{r} \frac{\partial}{\partial \theta} - r \cdot cos(\theta) \cdot sin(\theta) \cdot sin(\phi) \frac{\partial}{\partial r} - r \cdot cos(\theta) \frac{cos(\theta) \cdot sin(\phi)}{r} \frac{\partial}{\partial \theta} - r \cdot cos(\theta) \frac{cos(\phi)}{r \cdot sin(\theta)} \frac{\partial}{\partial \phi} \right) =$$

$$-i\hbar \left( -r \cdot sin(\theta) \cdot sin(\phi) \cdot \frac{sin(\theta)}{r} \frac{\partial}{\partial \theta} - r \cdot cos(\theta) \frac{cos(\theta) \cdot sin(\phi)}{r} \frac{\partial}{\partial \theta} - r \cdot cos(\theta) \frac{cos(\phi)}{r \cdot sin(\theta)} \frac{\partial}{\partial \phi} \right) =$$

$$i\hbar \left( sin(\theta) \cdot sin(\phi) \cdot sin(\theta) \frac{\partial}{\partial \theta} + cos(\theta) cos(\theta) \cdot sin(\phi) \frac{\partial}{\partial \theta} + cos(\theta) \frac{cos(\phi)}{sin(\theta)} \frac{\partial}{\partial \phi} \right) =$$

$$i\hbar \left( sin^2(\theta) \cdot sin(\phi) \frac{\partial}{\partial \theta} + cos^2(\theta) \cdot sin(\phi) \frac{\partial}{\partial \theta} + cos(\theta) \frac{cos(\phi)}{sin(\theta)} \frac{\partial}{\partial \phi} \right) =$$

$$i\hbar \left( \left( sin^2(\theta) + cos^2(\theta) \right) \cdot sin(\phi) \frac{\partial}{\partial \theta} + cos(\theta) \frac{cos(\phi)}{sin(\theta)} \frac{\partial}{\partial \phi} \right) =$$

$$i\hbar\left(sin(\phi)\frac{\partial}{\partial\theta}+cos(\phi)\frac{cos(\theta)}{sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$i\hbar\left(sin(\phi)\frac{\partial}{\partial\theta}+cos(\phi)cot(\theta)\frac{\partial}{\partial\phi}\right)$$

Note that $\hat{L}_x$ is independent of the radial component.

## Angular momentum $\hat{L}_y$

$$\hat{L}_y=\hat{z}\hat{p}_x-\hat{x}\hat{p}_z=-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)=$$

$$-i\hbar\left(r\cdot cos(\theta)\frac{\partial}{\partial x}-r\cdot sin(\theta)\cdot cos(\phi)\frac{\partial}{\partial z}\right)=$$

$$-i\hbar\cdot r\left(cos(\theta)\left(\left(sin(\theta)\cdot cos(\phi)\right)\frac{\partial}{\partial r}+\frac{cos(\theta)\cdot cos(\phi)}{r}\frac{\partial}{\partial\theta}-\frac{sin(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)-sin(\theta)\right.$$
$$\left.\cdot cos(\phi)\left(cos(\theta)\frac{\partial}{\partial r}-\frac{sin(\theta)}{r}\frac{\partial}{\partial\theta}\right)\right)=$$

$$-i\hbar\cdot r\left(cos(\theta)\cdot sin(\theta)\cdot cos(\phi)\frac{\partial}{\partial r}+cos(\theta)\frac{cos(\theta)\cdot cos(\phi)}{r}\frac{\partial}{\partial\theta}-cos(\theta)\frac{sin(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}-sin(\theta)\right.$$
$$\left.\cdot cos(\phi)cos(\theta)\frac{\partial}{\partial r}+sin(\theta)\cdot cos(\phi)\frac{sin(\theta)}{r}\frac{\partial}{\partial\theta}\right)=$$

$$-i\hbar\cdot r\left(cos(\theta)\frac{cos(\theta)\cdot cos(\phi)}{r}\frac{\partial}{\partial\theta}-cos(\theta)\frac{sin(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}+sin(\theta)\cdot cos(\phi)\frac{sin(\theta)}{r}\frac{\partial}{\partial\theta}\right)=$$

$$-i\hbar\left(cos(\theta)\cdot cos(\theta)\cdot cos(\phi)\frac{\partial}{\partial\theta}-cos(\theta)\frac{sin(\phi)}{sin(\theta)}\frac{\partial}{\partial\phi}+sin(\theta)\cdot cos(\phi)\cdot sin(\theta)\frac{\partial}{\partial\theta}\right)=$$

$$-i\hbar\left(cos^2(\theta)\cdot cos(\phi)\frac{\partial}{\partial\theta}-cos(\theta)\frac{sin(\phi)}{sin(\theta)}\frac{\partial}{\partial\phi}+sin^2(\theta)\cdot cos(\phi)\frac{\partial}{\partial\theta}\right)=$$

$$-i\hbar\left(\left(cos^2(\theta)+sin^2(\theta)\right)\cdot cos(\phi)\frac{\partial}{\partial\theta}-cos(\theta)\frac{sin(\phi)}{sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$-i\hbar\left(cos(\phi)\frac{\partial}{\partial\theta}-sin(\phi)\frac{cos(\theta)}{sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$-i\hbar\left(cos(\phi)\frac{\partial}{\partial\theta}-sin(\phi)cot(\theta)\frac{\partial}{\partial\phi}\right)$$

Note that $\hat{L}_y$ is independent of the radial component.

## Angular momentum $\hat{L}_z$

$$\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x=-i\hbar\left(r\cdot sin(\theta)\cdot cos(\phi)\frac{\partial}{\partial y}-r\cdot sin(\theta)\cdot sin(\phi)\frac{\partial}{\partial x}\right)=$$

$$-i\hbar\left(r\cdot sin(\theta)\cdot cos(\phi)\left(\left(sin(\theta)\cdot sin(\phi)\right)\frac{\partial}{\partial r}+\frac{cos(\theta)\cdot sin(\phi)}{r}\frac{\partial}{\partial\theta}+\frac{cos(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)-r\cdot sin(\theta)\right.$$

$$\left.\cdot\, sin(\phi)\left(\left(sin(\theta)\cdot cos(\phi)\right)\frac{\partial}{\partial r}+\frac{cos(\theta)\cdot cos(\phi)}{r}\frac{\partial}{\partial\theta}-\frac{sin(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)\right)=$$

$$-i\hbar\cdot r\left(sin^2(\theta)\cdot sin(\phi)\cdot cos(\phi)\frac{\partial}{\partial r}+\frac{sin(\theta)\cdot cos(\phi)\cdot cos(\theta)\cdot sin(\phi)}{r}\frac{\partial}{\partial\theta}+\frac{sin(\theta)\cdot cos^2(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right.$$

$$-\,sin^2(\theta)\cdot sin(\phi)\cdot cos(\phi)\frac{\partial}{\partial r}-\frac{sin(\theta)\cdot cos(\phi)\cdot cos(\theta)\cdot sin(\phi)}{r}\frac{\partial}{\partial\theta}$$

$$\left.+\frac{sin^2(\phi)\cdot sin(\theta)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$-i\hbar\cdot r\left(sin^2(\theta)\cdot sin(\phi)\cdot cos(\phi)\frac{\partial}{\partial r}+\frac{sin(\theta)\cdot cos^2(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}-sin^2(\theta)\cdot sin(\phi)\cdot cos(\phi)\frac{\partial}{\partial r}\right.$$

$$\left.+\frac{sin^2(\phi)\cdot sin(\theta)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$-i\hbar\cdot r\left(\frac{sin(\theta)\cdot cos^2(\phi)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}+\frac{sin^2(\phi)\cdot sin(\theta)}{r\cdot sin(\theta)}\frac{\partial}{\partial\phi}\right)=$$

$$-i\hbar\left(\frac{sin(\theta)\cdot cos^2(\phi)}{sin(\theta)}+\frac{sin^2(\phi)\cdot sin(\theta)}{sin(\theta)}\right)\frac{\partial}{\partial\phi}=$$

$$-i\hbar\left(\frac{sin(\theta)\cdot cos^2(\phi)+sin^2(\phi)\cdot sin(\theta)}{sin(\theta)}\right)\frac{\partial}{\partial\phi}=$$

$$-i\hbar\left(\frac{sin(\theta)}{sin(\theta)}\right)\frac{\partial}{\partial\phi}=$$

$$-i\hbar\frac{\partial}{\partial\phi}$$

Note that $\hat{L}_z$ depends on $\frac{\dot{\partial}}{\partial\phi}$ only.

We collect the results:

| Angular momentum quantum | Angular momentum classic |
|---|---|
| $\hat{L}_x=-i\hbar\left(sin(\phi)\frac{\partial}{\partial\theta}+cos(\phi)cot(\theta)\frac{\partial}{\partial\phi}\right)$<br>$\hat{L}_y=-i\hbar\left(cos(\phi)\frac{\partial}{\partial\theta}-sin(\phi)cot(\theta)\frac{\partial}{\partial\phi}\right)$<br>$\hat{L}_z=-i\hbar\frac{\partial}{\partial\phi}$ | $L_x=-m\cdot r^2\cdot\left(sin(\phi)\cdot\dot{\theta}+cos(\theta)\cdot sin(\theta)\cdot cos(\phi)\cdot\dot{\phi}\right)$<br>$L_y=m\cdot r^2\cdot\left(cos(\phi)\cdot\dot{\theta}-cos(\theta)\cdot sin(\theta)\cdot sin(\phi)\cdot\dot{\phi}\right)$<br>$L_z=m\cdot(x\dot{y}-y\dot{x})=m\cdot r^2\cdot sin^2(\theta)\cdot\dot{\phi}$ |