Bloch sphere representation of general two-level system

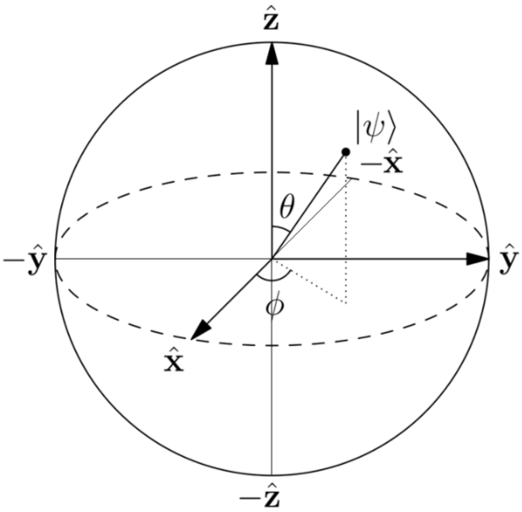
Note: More information you may find at https://davidmeyer.github.io/qc/bloch_sphere.pdf

The state $|\psi\rangle = a|0\rangle + b|1\rangle$ may be represented as a point on the surface of a unit sphere (the "Bloch sphere"), provided $|a|^2 + |b|^2 = 1$.

Because the overall phase of the state is unobservable we can parameterize this general two-level system state with a pure real factor at $|0\rangle$ as:

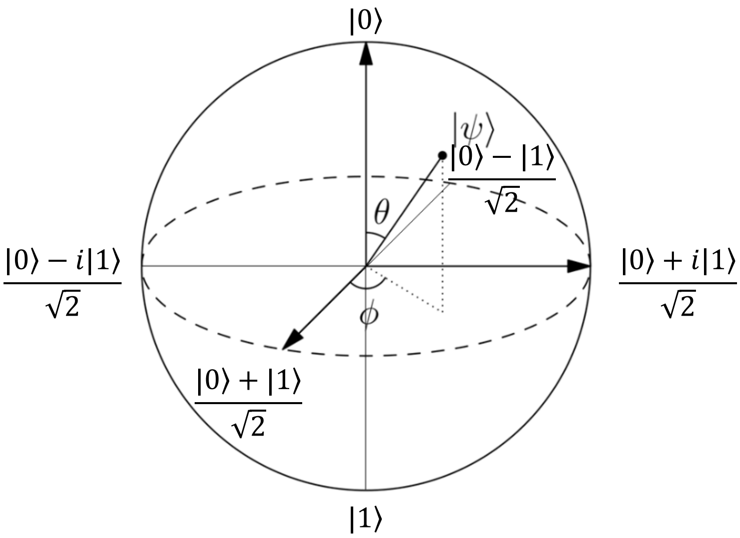
$$|\psi\rangle = cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}sin\left(\frac{\theta}{2}\right)|1\rangle$$

We use the usual spherical coordinate conventions for θ and ϕ:



The three axes in the sphere gives us three special points and their counterparts.

We can describe all points on the Bloch sphere by the standard basis $|0\rangle$ and $|1\rangle$:



Sometimes it might be necessary to rename resp. to change the basis.

We call $\frac{|0\rangle+|1\rangle}{\sqrt{2}} := |+\rangle$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}} := |-\rangle$, the plus/minus-basis. This is an orthonormal basis.

We calculate:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle - i|1\rangle}{\sqrt{2}} = \frac{\frac{|+\rangle + |-\rangle}{\sqrt{2}} - i\frac{|+\rangle - |-\rangle}{\sqrt{2}}}{\sqrt{2}} =$$

$$\frac{1}{2}(|+\rangle + |-\rangle - i|+\rangle + i|-\rangle) =$$

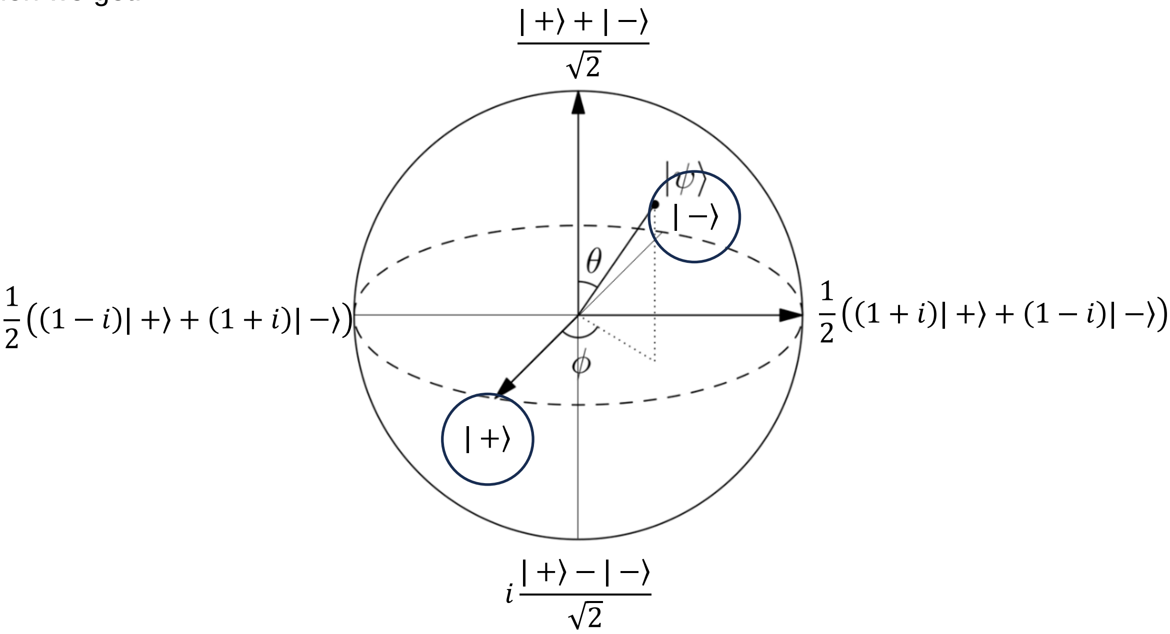
$$\frac{1}{2}\big((1-i)|+\rangle + (1+i)|-\rangle\big)$$

$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{\frac{|+\rangle + |-\rangle}{\sqrt{2}} + i\frac{|+\rangle - |-\rangle}{\sqrt{2}}}{\sqrt{2}} =$$

$$\frac{1}{2}(|+\rangle + |-\rangle + i|+\rangle - i|-\rangle) =$$

$$\frac{1}{2}\big((1+i)|+\rangle + (1-i)|-\rangle\big)$$

Then we get:



We can do the same with the third possibility.

We call $\frac{|0\rangle+i|1\rangle}{\sqrt{2}} \coloneqq |i\rangle$ and $\frac{|0\rangle-i|1\rangle}{\sqrt{2}} \coloneqq |-i\rangle$, the $i/{-i}$-basis. This is an orthonormal basis too.

We calculate:

$$|i\rangle + |-i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} + \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = 2\frac{|0\rangle}{\sqrt{2}} = \sqrt{2}|0\rangle \rightarrow$$

$$\frac{|i\rangle + |-i\rangle}{\sqrt{2}} = |0\rangle$$

$$|i\rangle - |-i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} - \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = 2\frac{|1\rangle}{\sqrt{2}} = \sqrt{2}|1\rangle \rightarrow$$

$$\frac{|i\rangle - |-i\rangle}{\sqrt{2}} = |1\rangle$$

We calculate:

$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{\frac{|i\rangle + |-i\rangle}{\sqrt{2}} + i\frac{|i\rangle - |-i\rangle}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}(|i\rangle + |-i\rangle + i|i\rangle - i|-i\rangle) =$$

$$\frac{1}{2}\big((1+i)|i\rangle + (1-i)|-i\rangle\big)$$

$$\frac{|0\rangle - i|1\rangle}{\sqrt{2}} = \frac{\frac{|i\rangle + |-i\rangle}{\sqrt{2}} - i\frac{|i\rangle - |-i\rangle}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}(|i\rangle + |-i\rangle - i|i\rangle + i|-i\rangle) =$$

$$\frac{1}{2}\big((1-i)|i\rangle + (1+i)|-i\rangle\big)$$

Then we get: