Two observables are simultaneous measurable if and only if the respective operators are commutating. If the operators do not commutate, we get an uncertainty in the measurement.

Related information you may find at:

https://www.reed.edu/physics/courses/P342.S10/Physics342/page1/files/Lecture.20.pdf

Griffiths, 3.5 The Uncertainty Principle

Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German…)

Hope I can help you with learning quantum mechanics.

Contents

Commutator

The commutator of two matrices A and B:

$$[A,B] = AB - BA$$

Note: the product of two matrices normally is not commutative:

$$[A,B] = AB - BA \neq 0$$

We call a matrix H hermitian if it has the following shape:

$$H \coloneqq \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

Note: $a, b, c \in \mathbb{R}$, $d, e, f \in \mathbb{C}$.

Note: for a Hermitian matrix we have: $a_{ij} = a_{ji}{}^*$.

Note: the diagonal entries are real numbers.

Note: A Hermitian matrix is diagonalizable.

If we transpose a matrix M and complex conjugate it, we call the result the adjoint matrix $M^\dagger$.

Hermitian matrices survive this process unchanged:

$$H \coloneqq \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} \to H^\dagger = \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

Note: a real number does not change by complex conjugation.

We build the commutator of two Hermitian matrices A, B:

$$A \coloneqq \begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}$$

$$B \coloneqq \begin{pmatrix} k & n & o \\ n^* & l & p \\ o^* & p^* & m \end{pmatrix}$$

Note: $a, b, c, k, l, m \in \mathbb{R}$, $d, e, f, n, o, p \in \mathbb{C}$.

We build the commutator:

$$[A,B] = AB - BA =$$

$$\begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix}\begin{pmatrix} k & n & o \\ n^* & l & p \\ o^* & p^* & m \end{pmatrix} - \begin{pmatrix} k & n & o \\ n^* & l & p \\ o^* & p^* & m \end{pmatrix}\begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} =;$$

$$AB = \begin{pmatrix} ak + dn^* + eo^* & an + dl + ep^* & ao + dp + em \\ d^*k + bn^* + fo^* & d^*n + bl + fp^* & d^*o + bp + fm \\ e^*k + f^*n^* + co^* & e^*n + f^*l + cp^* & e^*o + f^*p + cm \end{pmatrix}$$

$$BA = \begin{pmatrix} ka + nd^* + oe^* & kd + nb + of^* & ke + nf + oc \\ n^*a + ld^* + pe^* & n^*d + lb + pf^* & n^*e + lf + pc \\ o^*a + p^*d^* + me^* & o^*d + p^*b + mf^* & o^*e + p^*f + mc \end{pmatrix}$$

$$AB - BA =$$

$$\begin{pmatrix} ak + dn^* + eo^* - ka - nd^* - oe^* & an + dl + ep^* - kd - nb - of^* & ao + dp + em - ke - nf - oc \\ d^*k + bn^* + fo^* - n^*a - ld^* - pe^* & d^*n + bl + fp^* - n^*d - lb - pf^* & d^*o + bp + fm - n^*e - lf - pc \\ e^*k + f^*n^* + co^* - o^*a - p^*d^* - me^* & e^*n + f^*l + cp^* - o^*d - p^*b - mf^* & e^*o + f^*p + cm - o^*e - p^*f - mc \end{pmatrix} =$$

$$\begin{pmatrix} dn^* + eo^* - nd^* - oe^* & an + dl + ep^* - kd - nb - of^* & ao + dp + em - ke - nf - oc \\ d^*k + bn^* + fo^* - n^*a - ld^* - pe^* & d^*n + fp^* - n^*d - pf^* & d^*o + bp + fm - n^*e - lf - pc \\ e^*k + f^*n^* + co^* - o^*a - p^*d^* - me^* & e^*n + f^*l + cp^* - o^*d - p^*b - mf^* & e^*o + f^*p - o^*e - p^*f \end{pmatrix} =$$

$$\begin{pmatrix} dn^* - nd^* + eo^* - oe^* & an + dl + ep^* - kd - nb - of^* & ao + dp + em - ke - nf - oc \\ -an^* - d^*l - e^*p + kd^* + n^*b + o^*f & d^*n - n^*d + fp^* - pf^* & d^*o + bp + fm - n^*e - lf - pc \\ -ao^* - d^*p^* - e^*m + ke^* + n^*f^* + o^*c & -do^* - bp^* - f^*m + ne^* + lf^* + p^*c & e^*o - o^*e + f^*p - p^*f \end{pmatrix}$$

Note: a, b, c, k, l, m are real numbers and identic with their complex conjugated counterparts.

Note: $(dn^* - nd^*)$ are pure imaginary numbers.

Result:

We get a matrix of type:

$$[A, B] = \begin{pmatrix} i \cdot x & u & v \\ -u^* & i \cdot y & w \\ -v^* & -w^* & i \cdot z \end{pmatrix}$$

Note: x, y, z are real numbers, so ix, iy, iz are pure imaginary.

Note: a matrix of this type is called anti-Hermitian.

Note: The diagonal values of an anti-Hermitian matrix are purely imaginary or zero.

We build the adjoint, we transpose and complex conjugate:

$$\begin{pmatrix} i \cdot x & u & v \\ -u^* & i \cdot y & w \\ -v^* & -w^* & i \cdot z \end{pmatrix}^\dagger = \begin{pmatrix} -i \cdot x & -u & -v \\ u^* & -i \cdot y & -w \\ v^* & w^* & -i \cdot z \end{pmatrix} = -\begin{pmatrix} i \cdot x & u & v \\ -u^* & i \cdot y & w \\ -v^* & -w^* & i \cdot z \end{pmatrix}$$

If we build the commutator of two Hermitian matrices, we get an anti-Hermitian matrix:

$$[A, B]^\dagger = -[A, B]$$

Note: The commutator of two anti-Hermitian matrices is anti-Hermitian too.

The commutators of the Pauli matrices:

$\sigma_x \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$[\sigma_x, \sigma_y] = 2i\sigma_z$	$[\sigma_x, \sigma_z] = 2i\sigma_x$	$[\sigma_x, \sigma_z] = -2i\sigma_y$

Properties of commutators:

$$[B, A] = -[A, B]$$

$$[A, B + C] = [A, B] + [A, C]$$

$$[A, BC] = [A, B]C + B[A, C]$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \qquad \text{Jacobi-identity}$$

Observables

Two observables are simultaneous measurable if and only if the respective matrices are simultaneous diagonalizable.

Two matrices are simultaneous diagonalizable if and only if they commute.

We check this with an example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

We build the commutator $[A, B]$:

$$[A, B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 8 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 8 & 0 \end{pmatrix} = 0$$

We calculate the eigenvalues of matrix A:

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)^2 \rightarrow$$

$$\lambda_1 = 1, \lambda_2 = 2$$

Note: both eigenvalues have double multiplicity.

We calculate the eigenvectors to eigenvalue $\lambda_1 = 1$:

$$\begin{pmatrix} 1-1 & 0 & 0 & 0 \\ 0 & 1-1 & 0 & 0 \\ 0 & 0 & 2-1 & 0 \\ 0 & 0 & 0 & 2-1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow c = d = 0$$

Eigenspace to eigenvalue $\lambda_1 = 1$:

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

We calculate the eigenvectors to eigenvalue $\lambda_2 = 2$:

$$\begin{pmatrix} 1-2 & 0 & 0 & 0 \\ 0 & 1-2 & 0 & 0 \\ 0 & 0 & 2-2 & 0 \\ 0 & 0 & 0 & 2-2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow a = b = 0$$

Eigenspace to eigenvalue $\lambda_2 = 2$:

$$\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$$

Note: we can calculate an orthonormal basis of the eigenspaces by the Gram-Schmidt procedure. In this case we can choose the canonical basis vectors as an orthonormal basis of the two eigenspaces:

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; and \; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; and \; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We calculate the eigenvalues of matrix B:

$$0 = \begin{vmatrix} 0-\lambda & 3 & 0 & 0 \\ 3 & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 4 \\ 0 & 0 & 4 & 0-\lambda \end{vmatrix} =$$

$$(0-\lambda)(0-\lambda)\big((0-\lambda)(0-\lambda)-16\big) - 3\cdot 3\cdot\big((0-\lambda)(0-\lambda)-16\big) =$$

$$(-\lambda)(-\lambda)\big((-\lambda)(-\lambda)-16\big) - 3\cdot 3\cdot\big((-\lambda)(-\lambda)-16\big) =$$

$$\lambda^4 - 16\lambda^2 - 9\lambda^2 + 144 = 0$$

$$\lambda^4 - 25\lambda^2 + 144 = 0$$

We substitute $\lambda^2 \coloneqq x$

We have the quadratic equation:

$$x^2 - 25x + 144 = 0$$

$$x_{1/2} = \frac{25 \mp \sqrt{25^2 - 4\cdot 144}}{2}$$

$$x_{1/2} = \frac{25 \mp \sqrt{49}}{2}$$

$$x_{1/2} = \frac{25 \mp 7}{2}$$

$$x_1 = 9$$

$$x_2 = 16$$

We get:

$$\lambda_1 = -3, \lambda_2 = 3, \lambda_3 = -4, \lambda_4 = 4$$

We calculate the eigenvectors to eigenvalue $\lambda_1 = -3$:

$$\begin{pmatrix} 0-(-3) & 3 & 0 & 0 \\ 3 & 0-(-3) & 0 & 0 \\ 0 & 0 & 0-(-3) & 4 \\ 0 & 0 & 4 & 0-(-3) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3a+3b \\ 3a+3b \\ 3c+4d \\ 4c+3d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We have three equations:

$$3a + 3b = 0 \rightarrow a = -b$$

$$3c + 4d = 0 \rightarrow c = -\frac{4}{3}d$$

$$4c + 3d = 0 \rightarrow c = -\frac{3}{4}d$$

From equation 2 and 3 we get the only solution $c = d = 0$.

We get eigenvector to eigenvalue -3:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The other eigenvectors are:

eigenvector to eigenvalue 3:	eigenvector to eigenvalue -4:	eigenvector to eigenvalue 4:
$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

The two matrices A and B have the following eigenvalues/eigenvectors:

Matrix A $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$		Matrix B $\begin{pmatrix}0&3&0&0\\3&0&0&0\\0&0&0&4\\0&0&4&0\end{pmatrix}$		
Eigenvalue	Eigenvector	Eigenvalue	Eigenvector	
1	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	-3	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	is also eigenvector to matrix A with eigenvalue 1
1	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	3	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	is also eigenvector to matrix A with eigenvalue 1
2	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	-4	$\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	is also eigenvector to matrix A with eigenvalue 2
2	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	4	$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	is also eigenvector to matrix A with eigenvalue 2

Note: The eigenvectors to matrix B also are eigenvectors to matrix A but not vice versa.

We normalize them and get the orthonormal basis:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

Result: We found a common orthonormal basis to both matrices from eigenvectors of both matrices.

Quantum mechanics claims that the result of a simultaneous measurement with both operators A and B determines the state of the system after the measurement without uncertainty. There exists an orthonormal basis common to both matrices. If this is the case, we call the operators A and B a complete set of commutating operators.

Diagonalizing both matrices

From the eigenvalues of the matrices, we can calculate a set of basis vectors so that both matrices have a diagonal representation.

Matrix A is diagonal, matrix B has eigenvalues $-3,3,-4,4$. The corresponding eigenvectors are

$$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

Matrix B with respect to the new basis vectors has diagonal shape:

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

We check this. We build the matrix P from the eigenvalues of matrix B:

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

We calculate the inverse and transpose it because we need it from the left:

$$P^{-1T} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We calculate:

$$P^{-1T}BP =$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

If we apply the same procedure to matrix A, we get:

$$P^{-1T}AP =$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Result: We can diagonalize both matrices simultaneously.

Anti-commutator

In contrast to the commutator $[A, B] = AB - BA$ we define the anti-commutator:

$$\{A, B\} = AB + BA$$

The anticommutators of the Pauli-matrices are zero:

$\sigma_x \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\{\sigma_x, \sigma_y\} = 0$	$\{\sigma_x, \sigma_z\} = 0$	$\{\sigma_x, \sigma_z\} = 0$

Remarks:

- The commutator of two Hermitian matrices is anti-Hermitian and thus does not describe an observable.
- The commutator of anti-Hermitian matrices is anti-Hermitian.
- The anti-commutator of Hermitian matrices is Hermitian.
- The anti-commutator of Hermitian matrices has pure real eigenvalues.
- The commutator of Hermitian matrices has pure imaginary eigenvalues.

Note: we can get back the product of two operators by combination of commutator and anti-commutator:

$$AB = \frac{1}{2}(\{A, B\} + [A, B])$$

Note: this is a kind of resolving the identity.

Uncertainty

We remember that the uncertainty is defined as the square root of the variance, the mean quadratic deviation from the average:

$$(\Delta A)_v = \sqrt{\langle v|(A - \langle A\rangle_v)^2|v\rangle}$$

Note: the index v remembers that all calculations are made with respect to a certain basis $|v\rangle$ is expressed in.

Note: A is an observable → the matrix A is Hermitian.

We define:

$$\tilde{A} \coloneqq A - \langle A\rangle_v$$

We get:

$$(\Delta A)_v = \sqrt{\langle v|(A - \langle A\rangle_v)^2|v\rangle} =$$

$$\sqrt{\langle v|\tilde{A}^2|v\rangle} = \sqrt{\langle \tilde{A}v|\tilde{A}v\rangle} = \|\tilde{A}v\|$$

The same holds for a second observable B.

We build the product:

$$(\Delta A)_v(\Delta B)_v = \|\tilde{A}v\|\|\tilde{B}v\|$$

We apply the Schwarz inequality:

$$|\langle u|v\rangle| \leq \|u\|\|v\|$$

We get:

$$(\Delta A)_v(\Delta B)_v = \|\tilde{A}v\|\|\tilde{B}v\| \geq |\langle \tilde{A}v|\tilde{B}v\rangle| = |\langle v|\tilde{A}\tilde{B}|v\rangle|$$

We resolve the identity and get:

$$(\Delta A)_v(\Delta B)_v = |\langle v|\tilde{A}\tilde{B}|v\rangle| = \left|\left\langle v\left|\frac{1}{2}\left(\{\tilde{A},\tilde{B}\} + [\tilde{A},\tilde{B}]\right)\right|v\right\rangle\right| =$$

$$\frac{1}{2}|\langle v|\{\tilde{A},\tilde{B}\}|v\rangle| + |\langle v|[\tilde{A},\tilde{B}]|v\rangle|$$

We remember:

$\{\tilde{A},\tilde{B}\}$ has real eigenvalues only

$[\tilde{A},\tilde{B}]$ has pure imaginary eigenvalues only

We calculate the commutator:

$$[\tilde{A},\tilde{B}] = (A - \langle A\rangle_v)(B - \langle B\rangle_v) - (B - \langle B\rangle_v)(A - \langle A\rangle_v) =$$

$$\cancel{AB} - \cancel{A\langle B\rangle_v} - \cancel{\langle A\rangle_v B} + \langle A\rangle_v\langle B\rangle_v - (\cancel{BA} - \cancel{B\langle A\rangle_v} - \cancel{\langle B\rangle_v A} + \langle B\rangle_v\langle A\rangle_v) =$$

$$\langle A\rangle_v\langle B\rangle_v - \langle B\rangle_v\langle A\rangle_v = [A,B]$$

Result:

$$[\tilde{A},\tilde{B}] = [A,B]$$

Note: A, B are Hermitian.

We try the anti-commutator:

$$\{\tilde{A},\tilde{B}\} = (A - \langle A\rangle_v)(B - \langle B\rangle_v) + (B - \langle B\rangle_v)(A - \langle A\rangle_v) =$$

$$AB - A\langle B\rangle_v - \langle A\rangle_v B + \langle A\rangle_v\langle B\rangle_v + BA - B\langle A\rangle_v - \langle B\rangle_v A + \langle B\rangle_v\langle A\rangle_v =$$

$$2(AB - A\langle B\rangle_v - \langle A\rangle_v B + \langle A\rangle_v\langle B\rangle_v) =$$

$$2\big(A(B - \langle B\rangle_v) - \langle A\rangle_v(B - \langle B\rangle_v)\big) =$$

$$2\big((A - \langle A\rangle_v)(B - \langle B\rangle_v)\big) =$$

$$2\tilde{A}\cdot\tilde{B} \neq 0$$

Note: A, B are Hermitian.

The eigenvectors of $\tilde{A}, \tilde{B}$ build an orthonormal basis → we can represent $|v\rangle$ in this basis.

What we get is:

$$\frac{1}{2}\left|\langle v|\{\tilde{A},\tilde{B}\}|v\rangle\right| + \left|\langle v|[\tilde{A},\tilde{B}]|v\rangle\right|$$

This is essentially a complex number: $x + i \cdot y$.

We are interested in the absolute value:

$$\frac{1}{2}\left|\langle v|\{\tilde{A},\tilde{B}\}|v\rangle + \langle v|[\tilde{A},\tilde{B}]|v\rangle\right| = \frac{1}{2}\sqrt{\langle v|\{\tilde{A},\tilde{B}\}|v\rangle^2 + \langle v|[\tilde{A},\tilde{B}]|v\rangle^2}$$

Result:

$$(\Delta A)_v(\Delta B)_v \geq \frac{1}{2}\sqrt{\langle v|\{\tilde{A},\tilde{B}\}|v\rangle^2 + \langle v|[A,B]|v\rangle^2} \geq \frac{1}{2}\sqrt{\langle v|[A,B]|v\rangle^2}$$

Note: $\langle v|\{\tilde{A},\tilde{B}\}|v\rangle^2$ is a real number ≥ 0.

We omit the anti-commutator and get the Heisenberg uncertainty relation:

$$(\Delta A)_v(\Delta B)_v \geq \frac{1}{2}\sqrt{\langle v|[A,B]|v\rangle^2} = \frac{1}{2}|\langle v|[A,B]|v\rangle|$$

Remark

Omitting the anti-commutator makes the uncertainty relation easier to calculate but less exact. In some cases, e. g. the Pauli matrices, the anti-commutator becomes zero.

Approach according to Griffiths

An alternative approach you find in Griffiths, 3.5. We compare.

For any observable A, we have:

$$\sigma_A^2 = \langle(\hat{A} - \langle A\rangle)\Psi|(\hat{A} - \langle A\rangle)\Psi\rangle = \langle f|f\rangle$$

Note: $f := (\hat{A} - \langle A\rangle)$

$$\sigma_B^2 = \langle(\hat{B} - \langle B\rangle)\Psi|(\hat{B} - \langle B\rangle)\Psi\rangle = \langle g|g\rangle$$

Note: g:= $(\hat{B} - \langle B\rangle)$

The Schwarz inequality:

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

For any complex number holds:

$$|z|^2 = |Re(z)|^2 + |Im(z)|^2 \geq |Im(z)| =$$

$$\left[\frac{1}{2i}(z - z^*)\right]^2$$

Applied to $z := \langle f|g\rangle$:

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$$

We calculate $\langle f|g\rangle$:

$$\langle f|g\rangle = \left\langle \left(\hat{A} - \langle A\rangle\right)\Psi \middle| \left(\hat{B} - \langle B\rangle\right)\Psi \right\rangle =;$$

Note: $\left(\hat{A} - \langle A\rangle\right)$ is Hermitian

$$\left\langle \Psi \middle| \left(\hat{A} - \langle A\rangle\right)\left(\hat{B} - \langle B\rangle\right)\Psi \right\rangle =$$

$$\left\langle \Psi \middle| \left(\hat{A}\hat{B} - \hat{A}\langle B\rangle - \langle A\rangle\hat{B} + \langle A\rangle\langle B\rangle\right)\Psi \right\rangle =$$

$$\left\langle \Psi \middle| \left(\hat{A}\hat{B}\right)\Psi \right\rangle - \left\langle \Psi \middle| \left(\hat{A}\langle B\rangle\right)\Psi \right\rangle - \left\langle \Psi \middle| \left(\langle A\rangle\hat{B}\right)\Psi \right\rangle + \langle \Psi|(\langle A\rangle\langle B\rangle)\Psi\rangle =$$

$$\left\langle \Psi \middle| \left(\hat{A}\hat{B}\right)\Psi \right\rangle - \langle B\rangle\left\langle \Psi \middle| \left(\hat{A}\right)\Psi \right\rangle - \langle A\rangle\left\langle \Psi \middle| \left(\hat{B}\right)\Psi \right\rangle + \langle A\rangle\langle B\rangle\langle \Psi|\Psi\rangle =$$

$$\langle \hat{A}\hat{B}\rangle - \langle B\rangle\langle \hat{A}\rangle - \langle A\rangle\langle \hat{B}\rangle + \langle A\rangle\langle B\rangle =;$$

Note: $\langle A\rangle = \langle \hat{A}\rangle, \langle B\rangle = \langle \hat{B}\rangle$

$$\langle \hat{A}\hat{B}\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle =;$$

Note: $\langle A\rangle, \langle B\rangle$ are numbers

$$\langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

Analog we get:

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle B\rangle\langle A\rangle$$

We build the commutator:

$$\langle f|g\rangle - \langle g|f\rangle =$$

$$\langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \left(\langle \hat{B}\hat{A}\rangle - \langle B\rangle\langle A\rangle\right) =$$

$$\langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle - \langle \hat{B}\hat{A}\rangle + \langle B\rangle\langle A\rangle =$$

$$\langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle = \left[\hat{A}, \hat{B}\right]$$

Griffiths Conclusion:

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2 \rightarrow$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left[\hat{A}, \hat{B}\right]\right)^2$$

Note: The commutator of two Hermitian matrices has pure imaginary eigenvalues, the i cancels out. $\left(\frac{1}{2i}\left[\hat{A}, \hat{B}\right]\right)$ is a positive real number.

We compare with our solution:

$$(\Delta A)_v(\Delta B)_v \geq \frac{1}{2}\langle v|[A, B]|v\rangle$$

Both solutions have in common that they drop the real part of a complex number.

The main difference is that we explicitly work with vectors in a finite dimensional vector space. We use matrices and bases of vector spaces. Griffiths presents the general solution, valid for any wave function.