This paper deals with commutators, using functions of operators. It is based on "Commutation relations for functions of operators", Mark K. Transtrum and Jean-Francois S. Van Huele.

The original paper can be found at:
https://scholarsarchive.byu.edu/cgi/viewcontent.cgi?article=1371&context=facpub

Hope I can help you with learning quantum mechanics.

List of often used operators:

$Symbol$	Name	Effect
$\hat{x}$	Position operator	Multiplication by x
	The position operator can be expressed in terms of lowering operator $\hat{a}$ and raising operator $\hat{a}^\dagger$:	$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$
$\hat{x}^2$	Position operator squared	Multiplication by x^2
$\hat{x}^n$	Position operator to n-th power	Multiplication by x^n
$\hat{p}$	Momentum operator	$-i\hbar\frac{\partial}{\partial x}$
	The momentum operator can be expressed in terms of lowering operator $\hat{a}$ and raising operator $\hat{a}^\dagger$:	$\hat{p} = \sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$
$\hat{p}^2$	Momentum operator squared	$-\hbar^2\frac{\partial^2}{\partial x^2}$
$\hat{p}^n$	Momentum operator to n-th power	$(-i)^n\hbar^n\frac{\partial^2}{\partial x^2}$ Note: $(-i)^1 = -i, (-i)^2 = -1$ $(-i)^3 = i, (-i)^4 = +1$ $(-i)^5 = -i, \ldots$
$\hat{T} := \frac{\hat{p}^2}{2m}$	Kinetic energy	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
$\hat{V}(x)$	Potential energy	Multiplication by $V(x)$
$\hat{H}$	Hamiltonian, total energy, kinetic plus potential energy	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$
$\hat{a}$	Lowering operator	Reduces the n-th state wave function of the harmonic oscillator by one: $\hat{a}\psi_n = \sqrt{n}\,\psi_{n-1}$
	The lowering operator can be expressed in terms of $\hat{x}$ and $\hat{p}$:	$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{1}{m\omega}\hat{p}\right)$
$\hat{a}^\dagger$	Raising operator	Increments the n-th state wave function of the harmonic oscillator by one: $\hat{a}^\dagger\psi_n = \sqrt{n+1}\,\psi_{n+1}$
	The raising operator can be expressed in terms of $\hat{x}$ and $\hat{p}$:	$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{1}{m\omega}\hat{p}\right)$
$\hat{N}$	Number operator: $\hat{N} = \hat{a}^\dagger\hat{a}$	Gives back the n-th state wave function of the harmonic oscillator multiplied by n: $\hat{N}\psi_n = n\,\psi_n$

Note: The ground state wave function of the harmonic oscillator:

$$\psi_0 = N_0 e^{-\frac{m\omega x^2}{2\hbar}}$$

Note: $N_0, N_1, N_2, \ldots$ are normalizing constants.

Commuting Operators describe quantities that can simultaneously be measured exactly.

Operators that do not commute describe quantities that cannot be measured simultaneously exact.

In one-dimensional problems the position operator $\hat{x}$ and the momentum operator $\hat{p}$ are an example.

We have the commutation relation:

$$[\hat{x}, \hat{p}_x] = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) = i\hbar$$

The position operator $\hat{x}$ is realized by multiplying with x, the momentum operator $\hat{p}_x$ by applying the partial derivation $-i\hbar\frac{\partial}{\partial x}$.

For 3-dimensional spatial systems the commutation relations are generalized to:

$$[\hat{x}_i, \hat{p}_j] = \delta_{ij} i\hbar$$

Note: δ_{ij} is the Kronecker delta.

Independent dimensions are simultaneously measurable because $\delta_{ij} = 0$ for $i \neq j$.

If we do not have the plain operators $\hat{x}$ and $\hat{p}$, but reasonable functions of them, then the commutators change.

If we have $[\hat{x}, \hat{p}] = i\hbar$, then we get:

$$[\hat{x}, f(\hat{p})] = i\hbar \frac{df(\hat{p})}{d\hat{p}}$$

$$[\hat{p}, f(\hat{x})] = -i\hbar \frac{df(\hat{x})}{d\hat{x}}$$

In general:

If:

$$[\hat{x}_1, \hat{x}_2] = c$$

Then:

$$[\hat{x}_1, f(\hat{x}_2)] = c\frac{df(\hat{x}_2)}{d\hat{x}_2}$$

Note: c is a constant.

Note: $f(\hat{x}_2)$ must be an analytic function that allows Taylor expansion, so it must be infinitely differentiable.

Examples with lowering and raising operators.

$$[\hat{a}, \hat{a}^\dagger] = \hat{1}$$

Note: $\hat{1}$ often written as 1 is the unit operator.

$$[\hat{N}, \hat{a}] = -\hat{a}$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$$

$$[\hat{a}, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^{k-1}$$

$$[\hat{a}^\dagger, (\hat{a})^k] = -k(\hat{a})^{k-1}$$

$$\left[\hat{N}, (\hat{a})^k\right] = -k(\hat{a})^k$$

$$\left[\hat{N}, \left(\hat{a}^\dagger\right)^k\right] = k\left(\hat{a}^\dagger\right)^k$$

Linear example

We shift the operator $\hat{x}_2$ by a linear function:

$$f(\hat{x}_2) = a \cdot \hat{x}_2 + \hat{b}$$

Note: a is a real number, $\hat{b}$ an appropriate operator.

By assumption $[\hat{x}_1, \hat{x}_2] = c$ we get:

$$[\hat{x}_1, f(\hat{x}_2)] = c\frac{df(\hat{x}_2)}{d\hat{x}_2} = c \cdot a$$

Power example

We shift the second operator to its second power:

$$f(\hat{x}_2) = (\hat{x}_2)^2 = \hat{x}_2 \cdot \hat{x}_2$$

By assumption $[\hat{x}_1, \hat{x}_2] = c$ we get:

$$[\hat{x}_1, f(\hat{x}_2)] = c \cdot \frac{df(\hat{x}_2)}{d\hat{x}_2} = c \cdot \frac{d}{d\hat{x}_2}(\hat{x}_2)^2 = c \cdot 2 \cdot \hat{x}_2$$

The general case $[f(\hat{x}_1), g(\hat{x}_2)]$ leads to complex results – please look at the original paper if you need to deal with.

Mixed example

By assumption $[\hat{x}_1, \hat{x}_2] = c$ we get:

$$[(\hat{x}_1)^2, f(\hat{x}_2)] = 2 \cdot c \cdot \hat{x}_1 \frac{df(\hat{x}_2)}{d\hat{x}_2} - c^2 \cdot \frac{d^2 f(\hat{x}_2)}{d{\hat{x}_2}^2}$$

This can be generalized to any power of $(\hat{x}_1)$:

$$[(\hat{x}_1)^n, f(\hat{x}_2)] = \sum_{k=1}^{n} (-1)^{k+1} \cdot \binom{n}{k} \cdot c^k \cdot {\hat{x}_1}^{n-k} \cdot \frac{d^k f(\hat{x}_2)}{d{\hat{x}_2}^k}$$