This paper works with differential equations of first and second order.

We solve the following differential equations:

$$y' - 2 \cdot x \cdot y = 0$$

$$t \cdot \dot{y} + 2 \cdot y = t^5$$

$$x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$$

$$y'' + 2 \cdot y' + 65 \cdot y = -1 + x^2$$

$$y'' + 2 \cdot y' + 65 \cdot y = e^{2 \cdot x}$$

We do this explicit. If you are interested in results only, you may skip the calculations.

If you search for an elaborated theory of differential equations, you might find it on other places in the internet, e. g.:

https://math.libretexts.org/Courses/Monroe_Community_College/MTH_211_Calculus_II/Chapter_8%3A_Introduction_to_Differential_Equations/8.1%3A_Basics_of_Differential_Equations

In the second part we will deal with a more physical system, the spring-dashboard system. In this part we introduce the operator-notation for differential equations.

Hope I can help you with learning quantum mechanics.

Contents

Part I

Differential equation of first order

Differential equations bring together functions and their derivatives.

Examples of first order differential equations:

$$I)\ y' - 2 \cdot x \cdot y = 0$$

$$II\ t \cdot \dot{y} + 2 \cdot y = t^5$$

$$III)\ x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$$

Note:

$$y' = \frac{d}{dx} y = \frac{dy}{dx}$$

dy, dx are called differentials. They can be treated like ordinary variables:

$$y' = \frac{dy}{dx} \rightarrow y' \cdot dx = dy$$

Note: this is an often-discussed behavior, more information you may e. g. find at Wikipedia:
https://en.wikipedia.org/wiki/Differential_(mathematics)

Example $I)$ separation of variables

	$y' - 2 \cdot x \cdot y = 0$
We rewrite:	$y' = 2 \cdot x \cdot y$
	$\frac{dy}{dx} = 2 \cdot x \cdot y$
(*) We separate the variables. Note that we divide by y so we must check in the end that this causes no problems.	$\frac{dy}{y} = 2 \cdot x \cdot dx$
We integrate both sides:	$\int \frac{1}{y} dy = \int 2 \cdot x\, dx$
We get:	$ln(y) = x^2 + c_1$ $y = e^{x^2+C} = e^{x^2} \cdot e^{c_1} = e^{x^2} \cdot c_2$
We check (*): As $y(x) \neq 0$ we have no problems with (*).	$\frac{d}{dx}\left(e^{x^2} \cdot c_2\right) = 2 \cdot x \cdot e^{x^2} \cdot c_2 = 2 \cdot x \cdot y$

Separation of variables works in all cases like:

$$\frac{dy}{dx} = p(x) \cdot g(y)$$

General solution:

$$\int \frac{1}{g(y)} dy = \int p(x) dx + C$$

Differential equations of this type are called homogeneous first order linear ordinary differential equations.

Example II variation of parameters

$$t \cdot \dot{y} + 2 \cdot y = t^5$$

Note: This is an inhomogeneous first order linear ordinary differential equation like the third one. We solve this by "variation of parameters".

Note: We assume $t > 0$

Step 1: We solve the homogeneous equation:

$$t \cdot \dot{y} + 2 \cdot y = 0$$

$$\frac{d}{dt} y = -\frac{2 \cdot y}{t}$$

$$\frac{1}{y} dy = -\frac{2}{t}$$

$$\int \frac{1}{y} dy = -2 \cdot \int \frac{1}{t} dt$$

$$ln|y| = -2 \cdot ln(t) + c$$

$$e^{ln|y|} = y$$

$$e^{-2 \cdot ln(t) + c} = c \cdot e^{-2 \cdot ln(t)} = c \cdot e^{ln(t^{-2})} = c \cdot t^{-2}$$

$$y = c \cdot t^{-2}$$

$$\dot{y} = -2 \cdot c \cdot t^{-3}$$

Note: This is the solution of the homogeneous equation. We modify the parameter c and replace it by $c = u(t)$.

We get a new y:

$$y = u \cdot t^{-2} \rightarrow \dot{y} = \dot{u} \cdot t^{-2} - 2 \cdot u \cdot t^{-3}$$

We put this solution into the inhomogeneous equation:

$$t \cdot \dot{y} + 2 \cdot y = t^5$$

$$t \cdot (\dot{u} \cdot t^{-2} - 2 \cdot u \cdot t^{-3}) + 2 \cdot u \cdot t^{-2} = t^5$$

$$\dot{u} \cdot t^{-1} - 2 \cdot u \cdot t^{-2} + 2 \cdot u \cdot t^{-2} = t^5$$

Note: If this doesn't happen you have a computational error …

$$\dot{u} \cdot t^{-1} = t^5$$

$$\dot{u} = t^6$$

$$u = \frac{1}{7} t^7 + c$$

Now we got $u(t)$. We insert this and get the new y:

$$y = u \cdot t^{-2} = \frac{1}{7} t^5 + c \cdot t^{-2}$$

$$\dot{y} = \frac{5}{7} t^4 - 2 \cdot c \cdot t^{-3}$$

We check whether the differential equation holds for this y:

$$t \cdot \dot{y} + 2 \cdot y = t^5$$

$$t \cdot \left(\frac{5}{7}t^4 - 2 \cdot c \cdot t^{-3}\right) + 2 \cdot \left(\frac{1}{7}t^5 + c \cdot t^{-2}\right) =$$

$$\frac{5}{7}t^5 - 2 \cdot c \cdot t^{-2} + \frac{2}{7}t^5 + 2 \cdot c \cdot t^{-2} =$$

$$\frac{5}{7}t^5 + \frac{2}{7}t^5 =$$

$$t^5$$

This is the correct differential equation. We solved by using the method of "variation of parameters".

The next example we solve by a method called "integrating factor".

Example III) integrating factor

$$x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$$

$$a(x) \cdot y' + b(x) \cdot y = c(x)$$

Note: $\frac{dy}{dx} = y'$

Note: The equation is called linear because y' and y do not appear in mixed terms like $y \cdot y'$ or ${y'}^2$ or something like this.

Note: If $c(x) = 0$ we call the equation homogeneous.

We try separation of variables:

$$x^2 \cdot \frac{dy}{dx} + 3 \cdot x \cdot y = 6 \cdot x - 11$$

$$x^2 \cdot dy + (3 \cdot x \cdot y)dx = (6 \cdot x - 11)dx\ ??$$

Separation of variables obviously does not work.

We call equations of this kind inhomogeneous linear first order differential equations.

We take a look at:

$$x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$$

$$a(x) \cdot y' + b(x) \cdot y = c(x)$$

We bring this into the standard form:

$$y' + \frac{b(x)}{a(x)} \cdot y = \frac{c(x)}{a(x)}$$

Note: We divide by $a(x)$ so we must check in the end that this causes no problems: $a(x) = 0$.

We rewrite the equation in its standard form:

$$y' + p(x) \cdot y = q(x)$$

$$y' + \frac{3 \cdot x}{x^2} \cdot y = \frac{6 \cdot x - 11}{x^2}$$

$$y' + \frac{3}{x} \cdot y = \frac{6}{x} - \frac{11}{x^2}$$

We have $p(x) = \frac{3}{x}$ and $q(x) = \frac{6}{x} - \frac{11}{x^2}$.

We remember the product rule of differentiation:

$$\frac{d}{dx}\big(f(x) \cdot g(x)\big) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

The left side of $y' + p(x) \cdot y = q(x)$ is not perfect. We take a function $r(x)$ and multiply the equation in order to make it perfect in the sense of having the complete product rule of differentiation We take a look at the left side:

$$(y' + p \cdot y) \cdot r$$

$$y' \cdot r + y \cdot p \cdot r$$

We adjust $r(x)$ in such way that $p(x) \cdot r(x) = r'(x)$ and get:

$$y' \cdot r + y \cdot p \cdot r \rightarrow y' \cdot r + y \cdot r' = \frac{d}{dx}(y \cdot r)$$

This is the complete differentiation procedure. With this we rewrite the differential equation:

$$\frac{d}{dx}(y \cdot r) = q \cdot r$$

But before we can deal with this equation we need to solve the separable first order differential equation:

$$p(x) \cdot r(x) = \frac{d}{dx} r(x)$$

$$pdx = \frac{1}{r} dr$$

Note: For better readability we write:

$$p(x)dx = \frac{1}{r} dr$$

Solution:

$$\int p(x)dx = \int \frac{1}{r} dr$$

The right side gives:

$$\int \frac{1}{r} dr = ln\big(r(x)\big)$$

The left side gives, using $p(x) = \frac{3}{x}$:

$$\int p(x)dx = 3 \cdot \int \frac{1}{x} dx = 3 \cdot ln(x) + C = ln(x^3) + C$$

We get:

$$ln(x^3) + C = ln\big(r(x)\big)$$

$$r(x) = e^{ln(x^3)+C} = x^3 \cdot e^C = c_1 \cdot x^3$$

$$r'(x) = 3 \cdot x^2 \cdot c_1$$

We check:

$$p(x) \cdot r(x) = r'(x)?$$

$$\frac{3}{x} \cdot c_1 \cdot x^3 = 3 \cdot x^2 \cdot c_1 \surd$$

We go back to the equation:

$$y' + \frac{3}{x} \cdot y = \frac{6}{x} - \frac{11}{x^2}$$

We multiply by $r(x)$:

$$y' \cdot r(x) + y \cdot p(x) \cdot r(x) = q(x) \cdot r(x)$$

We insert the values:

$$y' \cdot (c_1 \cdot x^3) + y \cdot \left(\frac{3}{x}\right) \cdot (c_1 \cdot x^3) = \left(\frac{6}{x} - \frac{11}{x^2}\right) \cdot (c_1 \cdot x^3)$$

$$\frac{d}{dx}(y \cdot c_1 \cdot x^3) = \left(\frac{6}{x} - \frac{11}{x^2}\right) \cdot (c_1 \cdot x^3)$$

$$\frac{d}{dx}(y \cdot c_1 \cdot x^3) = 6 \cdot c_1 \cdot x^2 - 11 \cdot c_1 \cdot x$$

$$d(y \cdot c_1 \cdot x^3) = \int 6 \cdot c_1 \cdot x^2 - 11 \cdot c_1 \cdot x \, dx$$

$$y \cdot c_1 \cdot x^3 = 2 \cdot c_1 \cdot x^3 + c_2 - \frac{11}{2} \cdot c_1 \cdot x^2 + c_3$$

$$y \cdot c_1 \cdot x^3 = 2 \cdot c_1 \cdot x^3 - \frac{11}{2} \cdot c_1 \cdot x^2 + c_4$$

We get:

$$y = \frac{2 \cdot c_1 \cdot x^3 - \frac{11}{2} \cdot c_1 \cdot x^2 + c_4}{c_1 \cdot x^3} = 2 - \frac{11}{2 \cdot x} + \frac{c_5}{x^3}$$

We check the result. We need the derivative:

$$y' = \frac{11}{2 \cdot x^2} - 3 \cdot \frac{c_5}{x^4}$$

We check:

$$x^2 \cdot y' + 3 \cdot x \cdot y - 6 \cdot x + 11 = 0\,?$$

$$x^2 \cdot \left(\frac{11}{2 \cdot x^2} - 3 \cdot \frac{c_5}{x^4}\right) + 3 \cdot x \cdot \left(2 - \frac{11}{2 \cdot x} + \frac{c_5}{x^3}\right) - 6 \cdot x + 11 = 0\,?$$

$$\frac{11}{2} - 3 \cdot \frac{c_5}{x^2} + 6 \cdot x - \frac{33}{2} + 3 \cdot \frac{c_5}{x^2} - 6 \cdot x + 11 = 0 \surd$$

We divided by x. We see that $\frac{1}{x}$ and $\frac{1}{x^3}$ are part of $y(x)$ – our solution has definition gap at $x = 0$.

Geometric representation

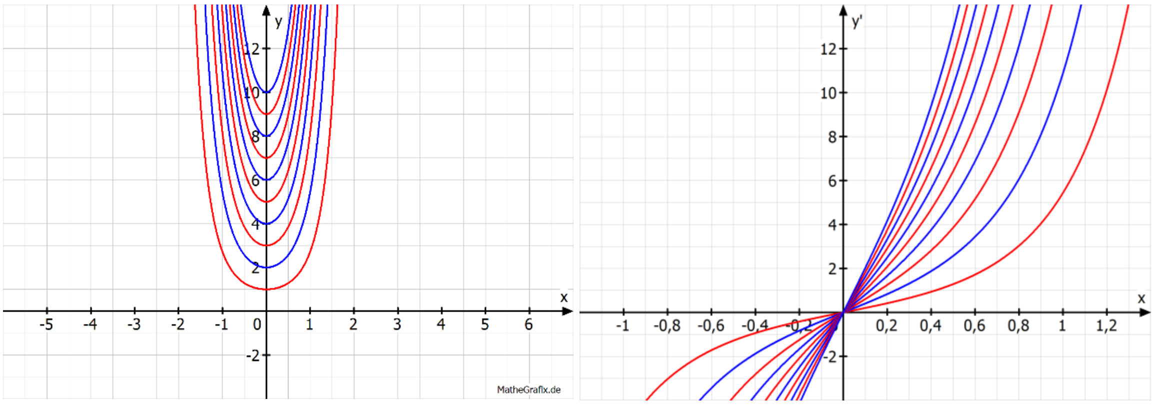
We plot the solutions of our differential equations. We plot the constant c in the range of 1 … 10.

$$I)\ y' - 2 \cdot x \cdot y = 0$$

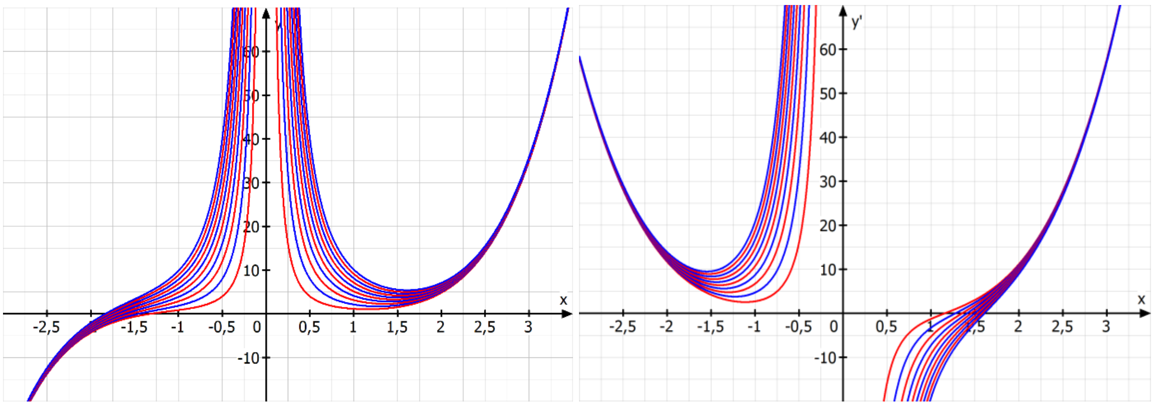
$$II\ t \cdot \dot{y} + 2 \cdot y = t^5$$

$$III)\ x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$$

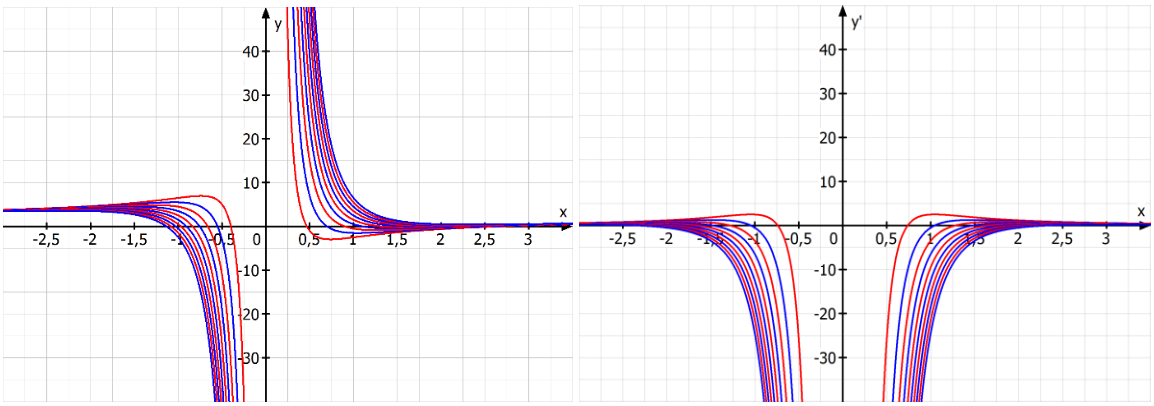
$I\ y' - 2 \cdot x \cdot y = 0$ with solution $y = e^{x^2} \cdot c$



$II\ t \cdot \dot{y} + 2 \cdot y = t^5$ with solution $y = \frac{1}{7}t^5 + c \cdot t^{-2}$



$III\ x^2 \cdot y' + 3 \cdot x \cdot y = 6 \cdot x - 11$ with solution $y = 2 - \frac{11}{2 \cdot x} + \frac{c_5}{x^3}$



Although we will not go through the proofs the following list may be important if you deal with first order linear ordinary differential equations (ODE), homogeneous or inhomogeneous:

- Every first-order linear ODE can be written in standard linear form:
  - Homogeneous: $\dot{y} + p(t) \cdot y = 0$
  - Inhomogeneous: $\dot{y} + p(t) \cdot y = q(t)$
- All first order homogeneous equations can be solved using separation of variables.
- All first order inhomogeneous equations can be solved by either variation of parameters or the use of integrating factors.
- A linear combination of a list of functions is any function that can be built by using scalar multiplication and addition:
  - $f(t) \to c \cdot f(t), c \in \mathbb{R}$
  - $f(t) + g(t) \to c_1 \cdot f(t) + c_2 \cdot g(t),\ c_1, c_2 \in \mathbb{R}$
- We have the superposition principle:
  - If $y(t)$ is a solution of $\dot{y} + p(t) \cdot y = q(t)$ then
    $a \cdot y(t)$ is a solution of $\dot{y} + p(t) \cdot y = a \cdot q(t)$
  - If
    $y_1(t)$ is a solution of $\dot{y} + p(t) \cdot y = q_1(t)$ and
    $y_2(t)$ is a solution of $\dot{y} + p(t) \cdot y = q_2(t)$
    then
    $y_1(t) + y_2(t)$ is a solution of $\dot{y} + p(t) \cdot y = q_1(t) + q_2(t)$
- If $p(t)$ and $q(t)$ are continuous functions on an open interval $I \in \mathbb{R}$, $a \in I$, $b \in \mathbb{R}$.
  Then there exists a unique solution to the first order linear ODE $\dot{y} + p(t) \cdot y = q(t)$ satisfying the initial condition $y(a) = b$.

Differential equation of second order

The differential equation of second represent e. g. the motion of the harmonic oscillator:

$$m \cdot \ddot{x} = -k \cdot x$$

Another example is the height x of a falling object due to air resistance:

$$m \cdot \ddot{x} = -m \cdot g - b \cdot \dot{x}$$

Homogeneous differential equation of second order

Notes:

- Differential equations of second order can be arbitrarily complicated.
- We will work with constant coefficients only:

$$y'' + a \cdot y' + b \cdot y = g(x)$$

- Any factor of y'' could be eliminated by dividing the equation with this factor.

We begin with the homogeneous differential equation:

$$y'' + a \cdot y' + b \cdot y = 0$$

We use the exponential ansatz:

$$y = e^{r \cdot x}$$

We differentiate:

$$y' = r \cdot e^{r \cdot x}$$

$$y'' = r^2 \cdot e^{r \cdot x}$$

We try:

$$r^2 \cdot e^{r \cdot x} + a \cdot r \cdot e^{r \cdot x} + b \cdot e^{r \cdot x} = 0$$

$$e^{r \cdot x} \cdot (r^2 + a \cdot r + b) = 0$$

$$r^2 + a \cdot r + b = 0$$

This polynomial is the characteristic equation, a quadratic equation with solution:

$$r_{1/2} = \frac{-a \pm \sqrt{a^2 - 4 \cdot b}}{2}$$

Possible solutions:

Case 1: $D = (a^2 - 4 \cdot b) > 0 \rightarrow r_1 \neq r_2 \in \mathbb{R}$

Case 2: $D = (a^2 - 4 \cdot b) = 0 \rightarrow r_1 = r_2 \in \mathbb{R}$

Case 3: $D = (a^2 - 4 \cdot b) < 0 \rightarrow z, z^* \in \mathbb{C}$

Note: z^* is the complex conjugated of z.

Note: D is called the determinant of the quadratic equation.

Case 1: $r_1 \neq r_2 \in \mathbb{R}$	Case 2: $r_1 = r_2 \in \mathbb{R}$	Case 3: $z, z^* \in \mathbb{C}$
General solution: $y_1(x) = e^{r_1 \cdot x}$ $y_2(x) = e^{r_2 \cdot x}$ $y = C_1 \cdot e^{r_1 \cdot x} + C_2 \cdot e^{r_2 \cdot x}$	We have: $r = -\frac{b}{2 \cdot a}$ General solution: $y_1(x) = y_2(x) = e^{r \cdot x}$ We get the second solution by variation of parameters. We try the ansatz: $y = C(x) \cdot e^{r \cdot x} = (C_1 \cdot x + C_2) \cdot e^{r \cdot x}$ We get two solutions: $y_1(x) = C_1 \cdot x \cdot e^{r \cdot x}$ $y_2(x) = C_2 \cdot e^{r \cdot x}$	The general solution: $y_1(x) = e^{z \cdot x}$ $y_2(x) = e^{z^* \cdot x}$ $y = C_1 \cdot e^{z \cdot x} + C_2 \cdot e^{z^* \cdot x}$

Wronski determinant

$$W\big(y_1(x), y_2(x)\big) := \left|\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}\right|$$

If you find two solutions $y_1(x)$, $y_2(x)$ of a homogeneous differential equation, you can check whether those solutions are linear independent by help of the Wronski determinant. If the determinant is different from zero, the solutions are linear independent.

We check the Wronski determinant:

$$\left|\begin{pmatrix} e^{z \cdot x} & e^{z^* \cdot x} \\ z \cdot e^{z \cdot x} & z^* \cdot e^{z^* \cdot x} \end{pmatrix}\right| = e^{z \cdot x} \cdot z^* \cdot e^{z^* \cdot x} - e^{z \cdot x} \cdot z \cdot e^{z^* \cdot x} =$$

$$z^* \cdot e^{(z+z^*) \cdot x} - e^{(z+z^*) \cdot x} \cdot z =$$

$$e^{(z+z^*) \cdot x} \cdot (z^* - z)$$

The Wronski determinant is zero in case $z^* = z$ giving z is a real number. This was case 2 where we used the variation of constants.

By writing $z := a + i \cdot b$ we can rewrite the solution:

$$y_1(x) = e^{(a - i \cdot b) \cdot x} = e^{a \cdot x} e^{-i \cdot b \cdot x}$$

$$y_2(x) = e^{(a + i \cdot b) \cdot x} = e^{a \cdot x} e^{+i \cdot b \cdot x}$$

Note: The solution is symmetric to the real axis of the complex plane.

Inhomogeneous differential equation of second order

$$y'' + a \cdot y' + b \cdot y = g(x)$$

Note: $g(x)$ is called perturbing function.

The complete solution of this equation is built by the sum of

- the solution of the associated homogeneous differential equation $y'' + a \cdot y' + b \cdot y = 0$, this solution is called $y_0(x)$
- one particular solution of inhomogeneous differential equation $y'' + a \cdot y' + b \cdot y = g(x)$, the solution is called $y_p(x)$

Case I, $g(x)$ is a polynomial of order n:

$$g(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n$$

Let $Q_n(x)$ be a polynomial of order n, then we use the ansatz:

Case $a \neq 0, b \neq 0$:	Case $a \neq 0, b = 0$:	Case $a = 0, b \neq 0$
$y_p(x) = Q_n(x)$	$y_p(x) = x \cdot Q_n(x)$	$y_p(x) = x^2 \cdot Q_n(x)$

Note: if $a = 0$ we get a complex solution $y = e^{i \cdot \sqrt{b} \cdot x}$ for the homogeneous equation.

Example:

$$y'' + 2 \cdot y' + 65 \cdot y = -1 + x^2$$

We solve the homogeneous equation:

$$y'' + 2 \cdot y' + 65 \cdot y = 0$$

We set:

$$y = e^{r \cdot x}$$

We get the characteristic equation:

$$r^2 + 2 \cdot r + 65 = 0$$

Solution:

$$r_{1/2} = \frac{-2 \pm \sqrt{4 - 260}}{2} = \frac{-2 \pm i \cdot 16}{2} = -1 \pm i \cdot 8$$

$$r_1 = -1 - i \cdot 8, r_2 = -1 + i \cdot 8$$

Note: We get a complex solution of the homogeneous differential equation:

$$y_0(x) = C_1 \cdot e^{(-1 - i \cdot 8) \cdot x} + C_2 \cdot e^{(-1 + i \cdot 8) \cdot x} =$$

We need a particular solution of the inhomogeneous differential equation, case $a \neq 0, b \neq 0$:

$$y_p(x) = Q_2(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2$$

We calculate the derivatives:

$$y_p'(x) = a_1 + 2 \cdot a_2 \cdot x$$

$$y_p''(x) = 2 \cdot a_2$$

We insert into the inhomogeneous differential equation:

$$y'' + 2 \cdot y' + 65 \cdot y = -1 + x^2 \rightarrow$$

$$2 \cdot a_2 + 2 \cdot (a_1 + 2 \cdot a_2 \cdot x) + 65 \cdot (a_0 + a_1 \cdot x + a_2 \cdot x^2) = -1 + x^2$$

$$2 \cdot a_2 + 2 \cdot a_1 + 4 \cdot a_2 \cdot x + 65 \cdot a_0 + 65 \cdot a_1 \cdot x + 65 \cdot a_2 \cdot x^2 = -1 + x^2$$

$$(2 \cdot a_2 + 2 \cdot a_1 + 65 \cdot a_0) + (4 \cdot a_2 + 65 \cdot a_1) \cdot x + 65 \cdot a_2 \cdot x^2 = -1 + x^2$$

$$-1 \qquad + 0 \cdot x \qquad + 1 \cdot x^2 = -1 + x^2$$

By comparing the coefficients, we get:

$$2 \cdot a_2 + 2 \cdot a_1 + 65 \cdot a_0 = -1$$

$$4 \cdot a_2 + 65 \cdot a_1 = 0$$

$$65 \cdot a_2 = 1$$

We solve:

$a_2 = \frac{1}{65}$	$\frac{4}{65} + 65 \cdot a_1 = 0 \rightarrow a_1 = -\frac{4}{4225}$	$2 \cdot \frac{1}{65} - 2 \cdot \frac{4}{4225} + 65 \cdot a_0 = -1$ $\frac{2}{65} - \frac{8}{4225} + 65 \cdot a_0 = -1$ $\frac{122}{4225} + 65 \cdot a_0 = -1$ $65 \cdot a_0 = -1 - \frac{122}{4225}$ $a_0 = -\frac{4347}{274625}$

We assemble the solution:

$$y(x) = y_0(x) + y_p(x)$$

$$y(x) = C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{4347}{274625} - \frac{4}{4225} \cdot x + \frac{1}{65} \cdot x^2 =$$

We check by calculating the derivatives:

$$y'(x) = (-1 - i \cdot 8) \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{4}{4225} + \frac{2}{65} \cdot x$$

$$y''(x) = (-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{2}{65}$$

We insert into the differential equation:

$$y'' + 2 \cdot y' + 65 \cdot y + 1 - x^2 = 0$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{2}{65} +$$
$$2 \cdot \left((-1 - i \cdot 8) \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{4}{4225} + \frac{2}{65} \cdot x \right) +$$
$$65 \cdot \left(C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{4347}{274625} - \frac{4}{4225} \cdot x + \frac{1}{65} \cdot x^2 \right) + 1 - x^2 =$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{2}{65} +$$
$$2 \cdot (-1 - i \cdot 8) \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + 2 \cdot (-1 + i \cdot 8) \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{8}{4225} + \frac{4}{65} \cdot x +$$
$$65 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + 65 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} - \frac{65 \cdot 4347}{274625} - \frac{65 \cdot 4}{4225} \cdot x + \frac{65}{65} \cdot x^2 + 1 - x^2 =$$

$$(-1-i\cdot 8)^2\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+(-1+i\cdot 8)^2\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+$$
$$2\cdot(-1-i\cdot 8)\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+2\cdot(-1+i\cdot 8)\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+$$
$$65\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+65\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}$$
$$+\frac{2}{65}-\frac{8}{4225}+\frac{4}{65}\cdot x-\frac{65\cdot 4347}{274625}-\frac{65\cdot 4}{4225}\cdot x+\frac{65}{65}\cdot x^2+1-x^2=$$

$$(-1-i\cdot 8)^2\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+(-1+i\cdot 8)^2\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+$$
$$2\cdot(-1-i\cdot 8)\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+2\cdot(-1+i\cdot 8)\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+$$
$$65\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+65\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}$$
$$+\frac{130-8-4347+4225}{4225}=$$

$$(-1-i\cdot 8)^2\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+(-1+i\cdot 8)^2\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+2\cdot(-1-i\cdot 8)\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+$$
$$2\cdot(-1+i\cdot 8)\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}+65\cdot C_1\cdot e^{(-1-i\cdot 8)\cdot x}+65\cdot C_2\cdot e^{(-1+i\cdot 8)\cdot x}=$$

$$C_1\cdot e^{(-1-i\cdot 8)\cdot x}((-1-i\cdot 8)^2+2\cdot(-1-i\cdot 8)+65)+$$
$$C_2\cdot e^{(-1+i\cdot 8)\cdot x}((-1+i\cdot 8)^2+2\cdot(-1+i\cdot 8)+65)=$$

$$C_1\cdot e^{(-1-i\cdot 8)\cdot x}(1+i\cdot 16-64-2-i\cdot 16+65)+$$
$$C_2\cdot e^{(-1+i\cdot 8)\cdot x}(1-i\cdot 16-64-2+i\cdot 16+65)=$$

$$C_1\cdot e^{(-1-i\cdot 8)\cdot x}\cdot(0)+C_2\cdot e^{(-1+i\cdot 8)\cdot x}\cdot(0)=0$$

Our calculation was correct.

We take a look at the solution:

$$y(x)=C_1\cdot e^{(-1-i\cdot 8)\cdot x}+C_2\cdot e^{(-1+i\cdot 8)\cdot x}-\frac{4347}{274625}-\frac{4}{4225}\cdot x+\frac{1}{65}\cdot x^2$$

We rewrite the complex part:

$$C_1\cdot e^{(-1-i\cdot 8)\cdot x}+C_2\cdot e^{(-1+i\cdot 8)\cdot x}=$$

$$C_1\cdot e^{-x}e^{-i\cdot 8x}+C_2\cdot e^{-x}e^{i\cdot 8x}=$$

$$C_1\cdot e^{-x}\left(e^{-i\cdot 8x}+\frac{C_2}{C_1}e^{i\cdot 8x}\right)$$

We interpret this as an exponential decay ($C_1\cdot e^{-x}$), modulated by $\left(e^{-i\cdot 8x}+\frac{C_2}{C_1}e^{i\cdot 8x}\right)$.

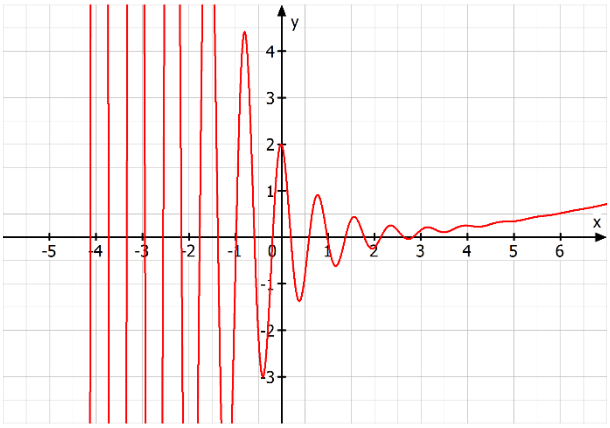
The perturbing function is a parabola that dominates for big x-values.

We set $C_1=C_2=1$. The function becomes real:

$$y(x)=e^{-x}\left(e^{-i\cdot 8x}+e^{i\cdot 8x}\right)-0{,}016-0{,}001\cdot x+0{,}015\cdot x^2$$

We plot:

$$f(x) = 2 \cdot exp(-x) \cdot cos(8 \cdot x) - 0{,}016 - 0{,}001 \cdot x + 0{,}015 \cdot x^2$$

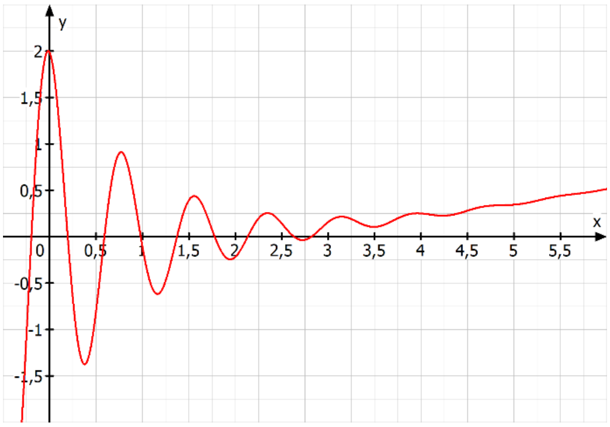


To the left we find a function heavily oscillating in combination with exponential growth.

To the right we find a function becoming a smooth parabola.

Near the origin we see damped oscillation.

The last zero-crossing $x \cong 2{,}083$:



Case II, $g(x)$ is an exponential:

$$g(x) = e^{c \cdot x}$$

We remember: the complete solution of this equation differential is built by the sum of

- the solution of the associated homogeneous differential equation
$$y'' + a \cdot y' + b \cdot y = 0$$
We write this as $y_0(x)$
- plus, one particular solution of inhomogeneous differential equation
$$y'' + a \cdot y' + b \cdot y = g(x)$$
We write this as $y_p(x)$

We solve the characteristic equation of the homogeneous linear differential equation and get either r_1, r_2 or $r_1 = r_2$ or z, z^*. Depending on this result we use the ansatz:

Case a is not part of the solution, we use ansatz 1:	Case a is either r_1 or r_2, we use ansatz 2:	Case $a = r_1 = r_2$, we use ansatz 3:
$y_p(x) = c_3 \cdot e^{c \cdot x}$	$y_p(x) = c_3 \cdot x \cdot e^{c \cdot x}$	$y_p(x) = c_3 \cdot x^2 \cdot e^{c \cdot x}$

We take the previous example and replace the polynomial by an exponential:

$$y'' + 2 \cdot y' + 65 \cdot y = e^{2 \cdot x}$$

We solve the homogeneous equation:

$$y'' + 2 \cdot y' + 65 \cdot y = 0$$

Like above we proceed: We set: $y = e^{r \cdot x}$ and get the characteristic equation for the homogeneous linear differential equation $r^2 + 2 \cdot r + 65 = 0$, with solutions $r_1 = -1 - i \cdot 8$ and $r_2 = -1 + i \cdot 8$

We remember from above the solution of the homogeneous linear differential equation:

$$y_0(x) = C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + C_2 \cdot e^{(-1+i \cdot 8) \cdot x}$$

We notice that $2 \notin \{r_1, r_2\} \rightarrow$ we need ansatz 1:

$$y_p(x) = c_3 \cdot e^{2 \cdot x}$$

We differentiate:

$$y_p'(x) = 2 \cdot c_3 \cdot e^{2 \cdot x}$$

$$y_p''(x) = 4 \cdot c_3 \cdot e^{2 \cdot x}$$

We insert into the inhomogeneous differential equation:

$$y'' + 2 \cdot y' + 65 \cdot y = c_3 \cdot e^{2 \cdot x} \rightarrow$$

$$4 \cdot c_3 \cdot e^{2 \cdot x} + 2 \cdot 2 \cdot c_3 \cdot e^{2 \cdot x} + 65 \cdot c_3 \cdot e^{2 \cdot x} = e^{2 \cdot x}$$

$$(4 + 4 + 65) \cdot c_3 \cdot e^{2 \cdot x} = e^{2 \cdot x}$$

$$c_3 = \frac{1}{73}$$

Particular solution:

$$y_p(x) = \frac{1}{73} \cdot e^{2 \cdot x}$$

We get the general solution:

$$y(x) = y_0(x) + y_p(x) =$$

$$C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{1}{73} \cdot e^{2 \cdot x}$$

We check by calculating the derivatives:

$$y'(x) = (-1 - i \cdot 8) \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{2}{73} \cdot e^{2 \cdot x}$$

$$y''(x) = (-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{4}{73} \cdot e^{2 \cdot x}$$

We insert into the differential equation:

$$y'' + 2 \cdot y' + 65 \cdot y - e^{2 \cdot x} = 0?$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{4}{73} \cdot e^{2 \cdot x} + 2 \cdot \left((-1 - i \cdot 8) \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{2}{73} \cdot e^{2 \cdot x}\right) + 65 \cdot \left(C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{1}{73} \cdot e^{2 \cdot x}\right) - e^{2 \cdot x} =$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{4}{73} \cdot e^{2 \cdot x} + (-1 - i \cdot 8) \cdot 2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot 2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{4}{73} \cdot e^{2 \cdot x} + 65 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + 65 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + \frac{65}{73} \cdot e^{2 \cdot x} - e^{2 \cdot x} =$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + (-1 - i \cdot 8) \cdot 2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot 2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + 65 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + 65 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} =$$

$$(-1 - i \cdot 8)^2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 - i \cdot 8) \cdot 2 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + 65 \cdot C_1 \cdot e^{(-1-i \cdot 8) \cdot x} + (-1 + i \cdot 8)^2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + (-1 + i \cdot 8) \cdot 2 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} + 65 \cdot C_2 \cdot e^{(-1+i \cdot 8) \cdot x} =$$

$$C_1 \cdot e^{(-1-i \cdot 8) \cdot x} \cdot ((-1 - i \cdot 8)^2 + (-1 - i \cdot 8) \cdot 2 + 65) + C_2 \cdot e^{(-1+i \cdot 8) \cdot x}((-1 + i \cdot 8)^2 + (-1 + i \cdot 8) \cdot 2 + 65) =$$

$$C_1 \cdot e^{(-1-i \cdot 8) \cdot x} \cdot (1 + i \cdot 16 - 64 - 2 - i \cdot 16 + 65) + C_2 \cdot e^{(-1+i \cdot 8) \cdot x}(1 - i \cdot 16 - 64 - 2 + i \cdot 16 + 65) =$$

$$C_1 \cdot e^{(-1-i \cdot 8) \cdot x} \cdot (1 - 64 - 2 + 65) + C_2 \cdot e^{(-1+i \cdot 8) \cdot x} \cdot (1 - 64 - 2 + 65) = 0$$

Our calculation was correct.

We take the solution:

$$y(x) = C_1 \cdot e^{(-1-i\cdot 8)\cdot x} + C_2 \cdot e^{(-1+i\cdot 8)\cdot x} + \frac{1}{73} \cdot e^{2\cdot x}$$

We rewrite the complex part:

$$C_1 \cdot e^{(-1-i\cdot 8)\cdot x} + C_2 \cdot e^{(-1+i\cdot 8)\cdot x} \to C_1 \cdot e^{-x} \cdot e^{-i\cdot 8x}\left(1 + \frac{C_2}{C_1} e^{i\cdot 16x}\right)$$

We interpret this as an exponential decay ($C_1 \cdot e^{-x}$), modulated by $e^{-i\cdot 8x}\left(1 + \frac{C_2}{C_1} e^{i\cdot 16x}\right)$.

The perturbing function is an exponential that dominates for big x-values.

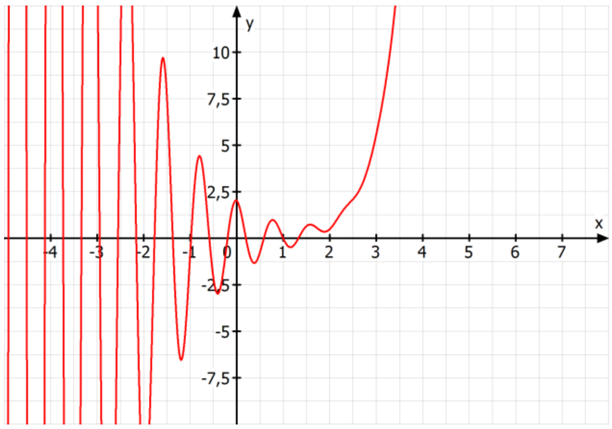
We set $C_1 = C_2 = 1$.

The function then becomes real:

$$y(x) = e^{-x} \cdot \left(e^{-i\cdot 8x} + e^{i\cdot 8x}\right) + \frac{1}{73} \cdot e^{2\cdot x}$$

We plot:

$$f(x) = 2 \cdot exp(-x) \cdot cos(8 \cdot x) + \frac{1}{73} \cdot e^{2\cdot x}$$



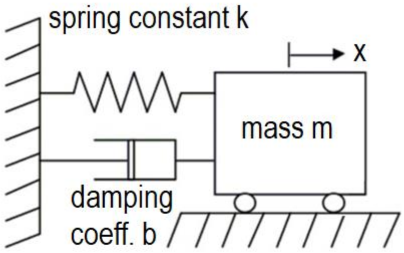
To the left we find a function heavily oscillating in combination with exponential growth.

To the right we find exponential growth.

Part II

Spring Dashboard system

The spring dashboard system is a physical application.



We describe the system by the forces acting:

$$m\ddot{x} = -kx - b\dot{x}$$

Standard differential equation:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Note: $m, b, k \in \mathbb{R}$

We rename:

$$\frac{b}{m} \coloneqq A, \frac{k}{m} \coloneqq B$$

Note: $A, B \in \mathbb{R}$

This is a linear homogeneous differential equation of second order.

It has two independent solutions:

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

We try the ansatz:

$$x(t) = e^{rt}$$

We differentiate:

$$\dot{x} = re^{rt}$$

$$\ddot{x} = r^2 e^{rt}$$

We insert this into the equation:

$$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0$$

We divide by e^{rt}:

$$r^2 + Ar + B = 0$$

Note: $e^{rt} \neq 0 \; \forall r, t$

Note: This is the characteristic equation of the differential equation.

The equation has two (complex) solutions:

$$r_{1/2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

The function $x(t) = e^{rt}$ is a solution if r is a root of the characteristic polynomial.

We get:

$$r_1 = \frac{-A + \sqrt{A^2 - 4B}}{2}$$

$$r_2 = \frac{-A - \sqrt{A^2 - 4B}}{2}$$

We get the solutions:

$x_1(t) = c_1 \exp\left(\frac{-A + \sqrt{A^2 - 4B}}{2} t\right)$	$x_2(t) = c_2 \exp\left(\frac{-A - \sqrt{A^2 - 4B}}{2} t\right)$

Note: c_1 and c_2 are arbitrary constants.

We have three possibilities for the roots:

- The roots can be real and distinct
- The roots can be complex conjugated
- The roots can be a single real number

Case 1: real distinct roots r_1 and r_2, $r_1 \neq r_2$.

Solution:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Numeric example one:

$$\ddot{x} + 4\dot{x} + 3x = 0$$

We get the characteristic equation:

$$r^2 + 4r + 3 = 0$$

We get r_1 and r_2:

$$r_1 = \frac{-4 + \sqrt{16 - 12}}{2} = -1$$

$$r_2 = \frac{-4 - \sqrt{16 - 12}}{2} = -3$$

We get the solution:

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

$$\dot{x}(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$$

$$\ddot{x}(t) = c_1 e^{-t} + 9c_2 e^{-3t}$$

We set the initial conditions. We drag the object to position 1 and then release it:

$$x(0) = 1, \dot{x}(0) = 0$$

From the first condition we get:

$$x(t = 0) = 1$$

$$c_1 e^{-t} + c_2 e^{-3t} = 1$$

$$c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

From the second condition we get:

$$\dot{x}(t = 0) = 0$$

$$-c_1 e^{-t} - 3c_2 e^{-3t} = 0$$

$$-c_1 - 3c_2 = 0$$

$$-c_1 - 3(1 - c_1) = 0$$

$$-c_1 - 3 + 3c_1 = 0$$

$$2c_1 = 3$$

$$c_1 = \frac{3}{2} \rightarrow c_2 = -\frac{1}{2}$$

Result:

$$x(t) = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

We plot:

This is an overdamped “oscillation”.

Case 2: complex conjugated roots r_1 and r_2.

Method 1: working with sin and cos.

We use:

If $r = u + iv$ is a solution to the differential equation $\ddot{x} + A\dot{x} + Bx = 0$ then u is a solution and v is a solution provided A and B are real numbers.

Solution:

$$x(t) = e^{(u+iv)t} = e^{ut}e^{ivt} \rightarrow$$

$$x(t) = c_1 e^{ut} cos(vt) + c_2 e^{ut} sin(vt)$$

Note: The complex conjugated $r = u - iv$ gives the same result.

Numeric example:

$$\ddot{x} + 3\dot{x} + \frac{25}{4}x = 0$$

We get the characteristic equation:

$$r^2 + 3r + \frac{25}{4} = 0$$

We get r_1 and r_2:

$$r_{1/2} = \frac{-3 \pm \sqrt{9 - 25}}{2} = -\frac{3}{2} \pm i \cdot 2$$

We get the solution:

$$x(t) = e^{-\frac{3}{2}t}\left(c_1 cos(2t) + c_2 sin(2t)\right)$$

$$\dot{x}(t) = -\frac{3}{2}e^{-\frac{3}{2}t}\left(c_1 cos(2t) + c_2 sin(2t)\right) + 2e^{-\frac{3}{2}t}\left(c_2 cos(2t) - c_1 sin(2t)\right) =$$

$$-\frac{3}{2}e^{-\frac{3}{2}t}c_1 cos(2t) - \frac{3}{2}e^{-\frac{3}{2}t}c_2 sin(2t) + 2e^{-\frac{3}{2}t}c_2 cos(2t) - 2e^{-\frac{3}{2}t}c_1 sin(2t) =$$

$$-\frac{3}{2}c_1 e^{-\frac{3}{2}t}cos(2t) + 2e^{-\frac{3}{2}t}c_2 cos(2t) - \frac{3}{2}e^{-\frac{3}{2}t}c_2 sin(2t) - 2e^{-\frac{3}{2}t}c_1 sin(2t) =$$

$$e^{-\frac{3}{2}t}\left(cos(2t)\left(2c_2 - \frac{3}{2}c_1\right) - sin(2t)\left(2c_1 + \frac{3}{2}c_2\right)\right)$$

We rename:

$$\left(2c_2 - \frac{3}{2}c_1\right) \coloneqq c_3$$

$$\left(2c_1 + \frac{3}{2}c_2\right) \coloneqq c_4$$

We get:

$$\dot{x}(t) = e^{-\frac{3}{2}t}\left(c_3 cos(2t) - c_4 sin(2t)\right)$$

$$\dot{x}(t) = e^{-\frac{3}{2}t}\left(\left(2c_2 - \frac{3}{2}c_1\right)cos(2t) - \left(2c_1 + \frac{3}{2}c_2\right)sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(2c_2 cos(2t) - \frac{3}{2}c_1 cos(2t) - 2c_1 sin(2t) - \frac{3}{2}c_2 sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(c_2\left(2cos(2t) - \frac{3}{2}sin(2t)\right) - c_1\left(\frac{3}{2}cos(2t) + 2sin(2t)\right)\right)$$

We calculate $\ddot{x}(t)$:

$$\ddot{x}(t) = -\frac{3}{2}e^{-\frac{3}{2}t}\left(c_3 cos(2t) - c_4 sin(2t)\right) + 2e^{-\frac{3}{2}t}\left(-c_3 sin(2t) - c_4 cos(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(\frac{3}{2}\left(c_4 sin(2t) - c_3 cos(2t)\right) - 2\left(c_3 sin(2t) + c_4 cos(2t)\right)\right) =$$

$$e^{-\frac{3}{2}t}\left(\frac{3}{2}c_4 sin(2t) - \frac{3}{2}c_3 cos(2t) - 2c_3 sin(2t) - 2c_4 cos(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(\frac{3}{2}c_4 - 2c_3\right) - cos(2t)\left(\frac{3}{2}c_3 + 2c_4\right)\right)$$

We go back to c_1 and c_2.

$$\ddot{x}(t) = e^{-\frac{3}{2}t}\left(sin(2t)\left(\frac{3}{2}\left(2c_1 + \frac{3}{2}c_2\right) - 2\left(2c_2 - \frac{3}{2}c_1\right)\right) - cos(2t)\left(\frac{3}{2}\left(2c_2 - \frac{3}{2}c_1\right) + 2\left(2c_1 + \frac{3}{2}c_2\right)\right)\right) =$$

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(3c_1 + \frac{9}{4}c_2 - 4c_2 + 3c_1\right) - cos(2t)\left(3c_2 - \frac{9}{4}c_1 + 4c_1 + 3c_2\right)\right) =$$

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(6c_1 - \frac{7}{4}c_2\right) - cos(2t)\left(6c_2 + \frac{7}{4}c_1\right)\right)$$

We check the differential equation $\ddot{x} + 3\dot{x} + \frac{25}{4}x = 0$:

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(6c_1 - \frac{7}{4}c_2\right) - cos(2t)\left(6c_2 + \frac{7}{4}c_1\right)\right) + 3e^{-\frac{3}{2}t}\left(cos(2t)\left(2c_2 - \frac{3}{2}c_1\right) - sin(2t)\left(2c_1 + \frac{3}{2}c_2\right)\right) + \frac{25}{4}e^{-\frac{3}{2}t}\left(c_1 cos(2t) + c_2 sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(6c_1 - \frac{7}{4}c_2\right) - cos(2t)\left(6c_2 + \frac{7}{4}c_1\right) + 3\left(cos(2t)\left(2c_2 - \frac{3}{2}c_1\right) - sin(2t)\left(2c_1 + \frac{3}{2}c_2\right)\right) + \frac{25}{4}\left(c_1 cos(2t) + c_2 sin(2t)\right)\right) =$$

$$e^{-\frac{3}{2}t}\left(sin(2t)\left(6c_1 - \frac{7}{4}c_2\right) - cos(2t)\left(6c_2 + \frac{7}{4}c_1\right) + \left(cos(2t)\left(6c_2 - \frac{9}{2}c_1\right) - sin(2t)\left(6c_1 + \frac{9}{2}c_2\right)\right) + \frac{25}{4}c_1 cos(2t) + \frac{25}{4}c_2 sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(6c_1 sin(2t) - \frac{7}{4}c_2 sin(2t) - 6c_2 cos(2t) - \frac{7}{4}c_1 cos(2t) + 6c_2 cos(2t) - \frac{9}{2}c_1 cos(2t) - 6c_1 sin(2t) - \frac{9}{2}c_2 sin(2t) + \frac{25}{4}c_1 cos(2t) + \frac{25}{4}c_2 sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(-\frac{7}{4}c_2 sin(2t) - \frac{7}{4}c_1 cos(2t) - \frac{9}{2}c_1 cos(2t) - \frac{9}{2}c_2 sin(2t) + \frac{25}{4}c_1 cos(2t) + \frac{25}{4}c_2 sin(2t)\right) =$$

$$e^{-\frac{3}{2}t}\left(-\frac{7}{4}c_2 sin(2t) - \frac{7}{4}c_1 cos(2t) - \frac{18}{4}c_1 cos(2t) - \frac{18}{4}c_2 sin(2t) + \frac{25}{4}c_1 cos(2t) + \frac{25}{4}c_2 sin(2t)\right) = 0$$

We set the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

From the first condition we get:

$$x(t=0) = \left(c_1 cos(0) + c_2 sin(0)\right)$$

$$x(t=0) = c_1 \rightarrow c_1 = 1$$

From the second condition we get:

$$\dot{x}(t=0) = 0$$

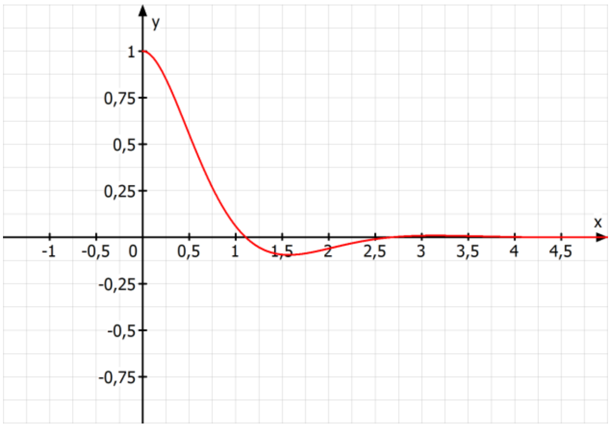
$$\dot{x}(t=0) = \left(c_2 \left(2cos(0) - \frac{3}{2} sin(0) \right) - \left(\frac{3}{2} cos(0) + 2sin(0) \right) \right) =$$

$$2c_2 - \frac{3}{2} = 0 \rightarrow c_2 = \frac{3}{4}$$

We get the solution with this initial condition:

$$x(t) = e^{-\frac{3}{2}t} \left(cos(2t) + \frac{3}{4} sin(2t) \right)$$

We plot:



This is an oscillation near the aperiodic limit.

Case 2: complex conjugated roots r_1 and r_2.

Method 2: working with the exponentials $e^{u \pm iv}$.

Ansatz:

$$x(t) = C_1 e^{(u+iv)t} + C_2 e^{(u-iv)t}$$

We need real solutions only so we must get rid of the i. We take C_1 and C_2 being complex numbers. We change $i \rightarrow -i$:

$$C^*{}_1 e^{(u-iv)t} + C^*{}_2 e^{(u+iv)t}$$

If $C_1 e^{(u+iv)t} + C_2 e^{(u-iv)t} = C^*{}_1 e^{(u-iv)t} + C^*{}_2 e^{(u+iv)t}$ then we have only real numbers on both sides of the equation (because the imaginary part plays no role).

The condition for this:

$$C^*{}_1 = C_2 \text{ } and \text{ } C^*{}_2 = C_1$$

We note that $C^*{}_1 = C_2 \rightarrow C^*{}_2 = C_1$.

We have:

$$C^*{}_1 = C_2$$

We set:

$$C_1 := a + ib \rightarrow C_2 = a - ib$$

We get:

$$x(t) = (a + ib)e^{(u+iv)t} + (a - ib)e^{(u-iv)t}$$

We calculate:

$$(a + ib)e^{(u+iv)t} + (a - ib)e^{(u-iv)t} =$$

$$ae^{(u+iv)t} + ibe^{(u+iv)t} + ae^{(u-iv)t} - ibe^{(u-iv)t} =$$

$$ae^{ut}e^{ivt} + ibe^{ut}e^{ivt} + ae^{ut}e^{-ivt} - ibe^{ut}e^{-ivt} =$$

$$e^{ut}\left(a\left(e^{ivt} + e^{-ivt}\right) + ib\left(e^{ivt} - e^{-ivt}\right)\right)$$

We can now replace this by sin and cos:

$$e^{ut}\left(a\left(2cos(vt)\right) + ib\left(2isin(vt)\right)\right) =$$

$$2e^{ut}\left(acos(vt) - bsin(vt)\right)$$

This is a real number. We use the same numeric example as before:

$$\ddot{x} + 3\dot{x} + \frac{25}{4}x = 0$$

We get the characteristic equation:

$$r^2 + 3r + \frac{25}{4} = 0$$

We get r_1 and r_2:

$$r_{1/2} = \frac{-3 \pm \sqrt{9 - 25}}{2} = -\frac{3}{2} \pm i \cdot 2$$

We get the solution:

$$x(t) = (a + ib)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)e^{\left(-\frac{3}{2}-i2\right)t}$$

Note: The complex constants $(a + ib)$ and $(a - ib)$ need to be determined by the initial conditions.

$$x(t) = (a + ib)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)e^{\left(-\frac{3}{2}-i2\right)t}$$

$$\dot{x}(t) = (a + ib)\left(-\frac{3}{2} + i2\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(-\frac{3}{2} - i2\right)e^{\left(-\frac{3}{2}-i2\right)t}$$

$$\ddot{x}(t) = (a + ib)\left(-\frac{3}{2} + i2\right)\left(-\frac{3}{2} + i2\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(-\frac{3}{2} - i2\right)\left(-\frac{3}{2} - i2\right)e^{\left(-\frac{3}{2}-i2\right)t} =$$

$$(a + ib)\left(\frac{9}{4} - 6i - 4\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(\frac{9}{4} + 6i - 4\right)e^{\left(-\frac{3}{2}-i2\right)t} =$$

$$(a + ib)\left(\frac{-7}{4} - 6i\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(\frac{-7}{4} + 6i\right)e^{\left(-\frac{3}{2}-i2\right)t}$$

We check the differential equation $\ddot{x} + 3\dot{x} + \frac{25}{4}x = 0$:

$$(a + ib)\left(\frac{-7}{4} - 6i\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(\frac{-7}{4} + 6i\right)e^{\left(-\frac{3}{2}-i2\right)t} + 3\left((a + ib)\left(-\frac{3}{2} + i2\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(-\frac{3}{2} - i2\right)e^{\left(-\frac{3}{2}-i2\right)t}\right) + \frac{25}{4}\left((a + ib)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)e^{\left(-\frac{3}{2}-i2\right)t}\right) =$$

$$(a + ib)\left(\frac{-7}{4} - 6i\right)e^{\left(-\frac{3}{2}+i2\right)t} + (a - ib)\left(\frac{-7}{4} + 6i\right)e^{\left(-\frac{3}{2}-i2\right)t} + \left(3(a + ib)\left(-\frac{3}{2} + i2\right)e^{\left(-\frac{3}{2}+i2\right)t} + 3(a - ib)\left(-\frac{3}{2} - i2\right)e^{\left(-\frac{3}{2}-i2\right)t}\right) + \left(\frac{25}{4}(a + ib)e^{\left(-\frac{3}{2}+i2\right)t} + \frac{25}{4}(a - ib)e^{\left(-\frac{3}{2}-i2\right)t}\right) =$$

$$(a+ib)\left(\frac{-7}{4}-6i\right)e^{\left(-\frac{3}{2}+i2\right)t}+(a-ib)\left(\frac{-7}{4}+6i\right)e^{\left(-\frac{3}{2}-i2\right)t}+\left((a+ib)\left(-\frac{9}{2}+i6\right)e^{\left(-\frac{3}{2}+i2\right)t}+(a-ib)\left(-\frac{9}{2}-i6\right)e^{\left(-\frac{3}{2}-i2\right)t}\right)+\left(\frac{25}{4}(a+ib)e^{\left(-\frac{3}{2}+i2\right)t}+\frac{25}{4}(a-ib)e^{\left(-\frac{3}{2}-i2\right)t}\right)=$$

$$e^{\left(-\frac{3}{2}+i2\right)t}\left((a+ib)\left(\frac{-7}{4}-6i\right)+(a+ib)\left(-\frac{9}{2}+i6\right)+\frac{25}{4}(a+ib)\right)+e^{\left(-\frac{3}{2}-i2\right)t}\left((a-ib)\left(\frac{-7}{4}+6i\right)+(a-ib)\left(-\frac{9}{2}-i6\right)+\frac{25}{4}(a-ib)\right)=$$

$$e^{\left(-\frac{3}{2}+i2\right)t}\left((a+ib)\left(\frac{-7}{4}-6i-\frac{18}{4}+i6+\frac{25}{4}\right)+\right)+e^{\left(-\frac{3}{2}-i2\right)t}\left((a-ib)\left(\frac{-7}{4}+6i-\frac{18}{4}-i6+\frac{25}{4}\right)\right)=0$$

We set the initial conditions:

$$x(0)=1, \dot{x}(0)=0$$

From the first condition we get:

$$x(t=0)=(a+ib)+(a-ib)$$

$$x(t=0)=1$$

$$(a+ib)+(a-ib)=1$$

$$2a=1\rightarrow a=\frac{1}{2}$$

From the second condition we get:

$$\dot{x}(t=0)=0$$

$$(a+ib)\left(-\frac{3}{2}+i2\right)+(a-ib)\left(-\frac{3}{2}-i2\right)=0$$

$$\left(\frac{1}{2}+ib\right)\left(-\frac{3}{2}+i2\right)+\left(\frac{1}{2}-ib\right)\left(-\frac{3}{2}-i2\right)=0$$

$$-\frac{3}{4}+i-\frac{3ib}{2}-2b-\frac{3}{4}-i+\frac{3ib}{2}-2b=0$$

$$-\frac{3}{4}-2b-\frac{3}{4}-2b=0$$

$$-\frac{6}{4}-4b=0$$

$$b=-\frac{6}{16}$$

Result:

$$x(t)=\left(\frac{1}{2}-\frac{6}{16}i\right)e^{\left(-\frac{3}{2}+i2\right)t}+\left(\frac{1}{2}+\frac{6}{16}i\right)e^{\left(-\frac{3}{2}-i2\right)t}$$

Note: This is of kind $z_1z_2+\overline{z_1z_2}$ and gives a real number.

We expand:

$$x(t)=\left(\frac{1}{2}-\frac{6}{16}i\right)e^{\left(-\frac{3}{2}+i2\right)t}+\left(\frac{1}{2}+\frac{6}{16}i\right)e^{\left(-\frac{3}{2}-i2\right)t}=$$

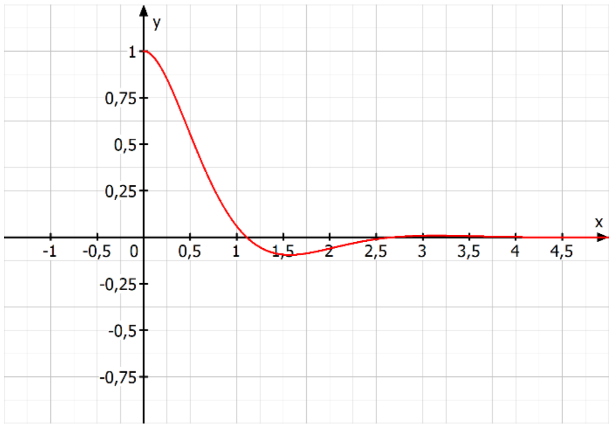
$$e^{-\frac{3}{2}t}\left(\left(\frac{1}{2}-\frac{6}{16}i\right)e^{i2t}+\left(\frac{1}{2}+\frac{6}{16}i\right)e^{-i2t}\right)=$$

$$e^{-\frac{3}{2}t}\left(\frac{1}{2}e^{i2t}-\frac{6}{16}ie^{i2t}+\frac{1}{2}e^{-i2t}+\frac{6}{16}ie^{-i2t}\right)=$$

$$e^{-\frac{3}{2}t}\left(\frac{1}{2}\left(e^{i2t}+e^{-i2t}\right)-\frac{6}{16}i\left(e^{i2t}-e^{-i2t}\right)\right)=$$

$$e^{-\frac{3}{2}t}\left(cos(2t)+\frac{3}{4}sin(2t)\right)$$

We plot:



This is a very heavy damped “oscillation” and the same result we got from working with the $\sin/cos$ variant.

Case 3: single real root.

$$m\ddot{x}+b\dot{x}+kx=0$$

$$\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$$

We get the characteristic equation:

$$r^2+\frac{b}{m}r+\frac{k}{m}=0$$

$$r_{1/2}=\frac{-\frac{b}{m}\pm\sqrt{\left(\frac{b}{m}\right)^2-4\cdot\frac{k}{m}}}{2}$$

We have:

$$\left(\frac{b}{m}\right)^2-4\frac{k}{m}=0\leftrightarrow\left(\frac{b}{m}\right)^2=4\frac{k}{m}$$

$$r_1=-\frac{b}{2m}$$

We get the first solution:

$$e^{-\frac{b}{2m}t}$$

We have only one solution. This is analogous to the case searching basis vectors for a 2-dimensional system. If we have one vector found we can generate a second basis vector by using a linear independent second vector.

In our case we find a second solution by multiplying the first one with t:

$$t \cdot e^{-\frac{b}{2m}t}$$

Note: If we do this correctly we would get $c_1 \cdot t + c_2$ but for our purpose simply t is enough:

We check:

$$x(t) = t \cdot e^{-\frac{b}{2m}t}$$

$$\dot{x}(t) = e^{-\frac{b}{2m}t} + t \cdot \left(-\frac{b}{2m}\right) \cdot e^{-\frac{b}{2m}t} =$$

$$\left(1 - t \cdot \frac{b}{2m}\right) \cdot e^{-\frac{b}{2m}t}$$

$$\ddot{x}(t) = \left(-\frac{b}{2m}\right) \cdot e^{-\frac{b}{2m}t} + \left(1 - t \cdot \frac{b}{2m}\right) \cdot \left(-\frac{b}{2m}\right) \cdot e^{-\frac{b}{2m}t} =$$

$$\left(\left(-\frac{b}{2m}\right) + \left(1 - t \cdot \frac{b}{2m}\right) \cdot \left(-\frac{b}{2m}\right)\right) \cdot e^{-\frac{b}{2m}t} =$$

$$\left(\left(-\frac{b}{2m}\right) + \left(-\frac{b}{2m} + t \cdot \left(\frac{b}{2m}\right)^2\right)\right) \cdot e^{-\frac{b}{2m}t} =$$

$$\left(-\frac{b}{m} + t \cdot \left(\frac{b}{2m}\right)^2\right) \cdot e^{-\frac{b}{2m}t}$$

We insert into the differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$m\left(\left(-\frac{b}{m} + t \cdot \left(\frac{b}{2 \cdot m}\right)^2\right) \cdot e^{-\frac{b}{2m}t}\right) + b\left(\left(1 - t \cdot \frac{b}{2 \cdot m}\right) \cdot e^{-\frac{b}{2m}t}\right) + k\left(t \cdot e^{-\frac{b}{2m}t}\right) = 0$$

$$m\left(-\frac{b}{m} + t \cdot \left(\frac{b}{2 \cdot m}\right)^2\right) + b\left(1 - t \cdot \frac{b}{2 \cdot m}\right) + k \cdot t = 0$$

$$-b + \frac{t \cdot b^2}{4 \cdot m} + b - t \cdot \frac{b^2}{2m} + k \cdot t = 0$$

$$\frac{t \cdot b^2}{4 \cdot m} - \frac{t \cdot b^2}{2m} + k \cdot t = 0$$

$$t \cdot b^2 \left(\frac{1}{4 \cdot m} - \frac{1}{2 \cdot m}\right) + k \cdot t = 0$$

$$t \cdot b^2 \left(\frac{1}{4 \cdot m} - \frac{2}{4 \cdot m}\right) + k \cdot t = 0$$

$$t \cdot b^2 \left(-\frac{1}{4 \cdot m}\right) + k \cdot t = 0$$

$$t \cdot \left(b^2 \left(-\frac{1}{4 \cdot m}\right) + k\right) = 0$$

$$-\frac{b^2}{4 \cdot m} + k = 0$$

$$-\frac{b^2}{4} + m \cdot k = 0$$

$$b^2 = 4 \cdot m \cdot k$$

$$\frac{b^{2}}{m^2} = 4\frac{k}{m}$$

This is our condition we got from the characteristic equation delivering a single solution.

The complete solution:

$$x(t) = c_1 \cdot e^{-\frac{b}{2m}t} + c_2 \cdot t \cdot e^{-\frac{b}{2m}t} =$$

$$(c_1 + c_2 \cdot t) \cdot e^{-\frac{b}{2m}t}$$

Numeric example 1:

$$\ddot{x} + 4\dot{x} + 4x = 0$$

We get the characteristic equation:

$$r^2 + 4r + 4 = 0$$

$$r_{1/2} = \frac{-4 \pm \sqrt{16 - 16}}{2} = -2$$

We have:

$$r_1 = -2$$

We get the first solution:

$$e^{-2t}$$

We use the second solution:

$$t \cdot e^{-2t}$$

The complete solution:

$$x(t) = c_1 \cdot e^{-2t} + c_2 \cdot t \cdot e^{-2t}$$

$$\dot{x}(t) = -2 \cdot t \cdot c_1 \cdot e^{-2t} + c_2 \cdot e^{-2t} - 2 \cdot c_2 \cdot t^2 \cdot e^{-2t} =$$

$$-2 \cdot t \cdot c_1 \cdot e^{-2t} + c_2 \cdot e^{-2t}(1 - 2 \cdot t^2) =$$

$$\left(-2 \cdot t \cdot c_1 + c_2(1 - 2 \cdot t^2)\right) \cdot e^{-2t} =$$

$$\left(-2 \cdot c_1 + c_2(1 - 2 \cdot t)\right) \cdot t \cdot e^{-2t}$$

We set the initial conditions:

$$x(0) = 1, \dot{x}(0) = 0$$

From the first condition we get:

$$x(t = 0) = 1 \rightarrow c_1 = 1$$

From the second condition we get:

$$\dot{x}(t = 0) = 0 \rightarrow c_2 = 0$$

$$-2 \cdot c_1 + c_2(1 - 2 \cdot t) = 0$$

$$c_2(1 - 2 \cdot t) = 2 \cdot c_1$$

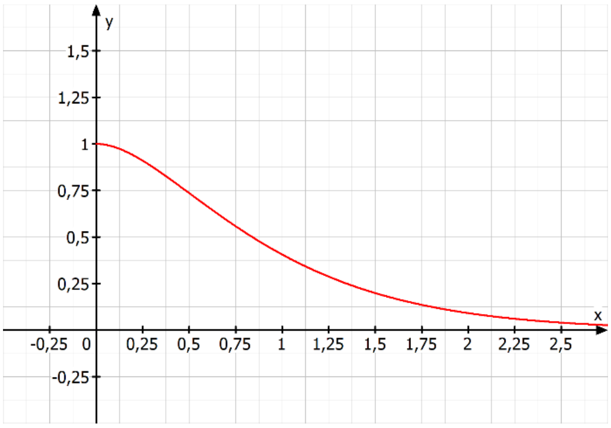
$$c_2 = 2 \cdot c_1$$

$$c_2 = 2$$

Result:

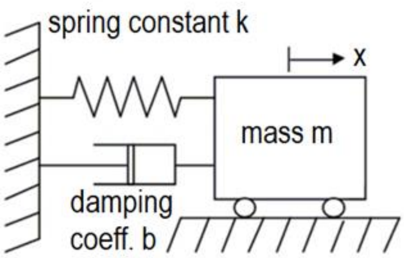
$$x(t) = e^{-2t} + 2 \cdot t \cdot e^{-2t}$$

We plot:



This is a critically damped case.

Some details.



The mass has value $1\ kg$, the force $F = 16\ {}^{Newton}/_{meter}$

Time is measured in seconds, position $x(t)$ is measured in meter.

We assume frictionless wheels, b is a real **positive** parameter.

The position is determined by the differential equation:

$$\ddot{x} + b \cdot \dot{x} + 16 \cdot x = 0$$

We check units:

The original equation:

$$m\ddot{x} + b \cdot \dot{x} + 16 \cdot x = 0$$

$$\left[\frac{kg \cdot m}{s^2} + b\frac{m}{2} + \frac{kg \cdot m \cdot m}{s^2 \cdot m}\right] = \left[\frac{kg \cdot m}{s^2} + b\frac{m}{s} + \frac{kg \cdot m}{s^2}\right]$$

The damping factor b must have the units $\left[\frac{kg}{s}\right]$

The characteristic polynomial:

$$p(r) = r^2 + b \cdot r + 16$$

The roots of the characteristic polynomial:

$$r_{12} = \frac{-b \pm \sqrt{b^2 - 64}}{2}$$

We get two distinct real roots if:

$$b^2 - 64 > 0$$

$$b > 8$$

In this case we have:

$$r_1 = \frac{-b - \sqrt{b^2 - 64}}{2}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 64}}{2}$$

We get the solutions:

$$x_1(t) = c_1 e^{\frac{-b-\sqrt{b^2-64}}{2}t}$$

$$x_2(t) = c_2 e^{\frac{-b+\sqrt{b^2-64}}{2}t}$$

The combined solution:

$$x(t) = c_1 e^{\frac{-b-\sqrt{b^2-64}}{2}t} + c_2 e^{\frac{-b+\sqrt{b^2-64}}{2}t}$$

$$\dot{x}(t) = c_1 \frac{-b - \sqrt{b^2 - 64}}{2} e^{\frac{-b-\sqrt{b^2-64}}{2}t} + c_2 \frac{-b + \sqrt{b^2 - 64}}{2} e^{\frac{-b+\sqrt{b^2-64}}{2}t}$$

$$\ddot{x}(t) = c_1 \left(\frac{-b - \sqrt{b^2 - 64}}{2}\right)^2 e^{\frac{-b-\sqrt{b^2-64}}{2}t} + c_2 \left(\frac{-b + \sqrt{b^2 - 64}}{2}\right)^2 e^{\frac{-b+\sqrt{b^2-64}}{2}t}$$

We set boundary conditions:

$$x(t = 0) = 1$$

$$\dot{x}(t = 0) = -\frac{b}{2}$$

First boundary condition $x(t = 0) = 1$:

$$c_1 + c_2 = 1 \rightarrow c_2 = 1 - c_1$$

Second boundary condition $\dot{x}(t = 0) = -\frac{b}{2}$:

$$c_1 \frac{-b - \sqrt{b^2 - 64}}{2} + (1 - c_1) \frac{-b + \sqrt{b^2 - 64}}{2} = -\frac{b}{2}$$

$$c_1 \frac{-b-\sqrt{b^2-64}}{2} + \frac{-b+\sqrt{b^2-64}}{2} - c_1 \frac{-b+\sqrt{b^2-64}}{2} = -\frac{b}{2}$$

$$c_1\left(-b-\sqrt{b^2-64}\right) - b + \sqrt{b^2-64} - c_1\left(-b+\sqrt{b^2-64}\right) = -b$$

$$-bc_1 - \sqrt{b^2-64}c_1 - b + \sqrt{b^2-64} + bc_1 - \sqrt{b^2-64}c_1 = -b$$

$$-2\sqrt{b^2-64}c_1 + \sqrt{b^2-64} = 0$$

$$c_1 = \frac{-\sqrt{b^2-64}}{-2\sqrt{b^2-64}} = \frac{1}{2}$$

$$c_2 = \frac{1}{2}$$

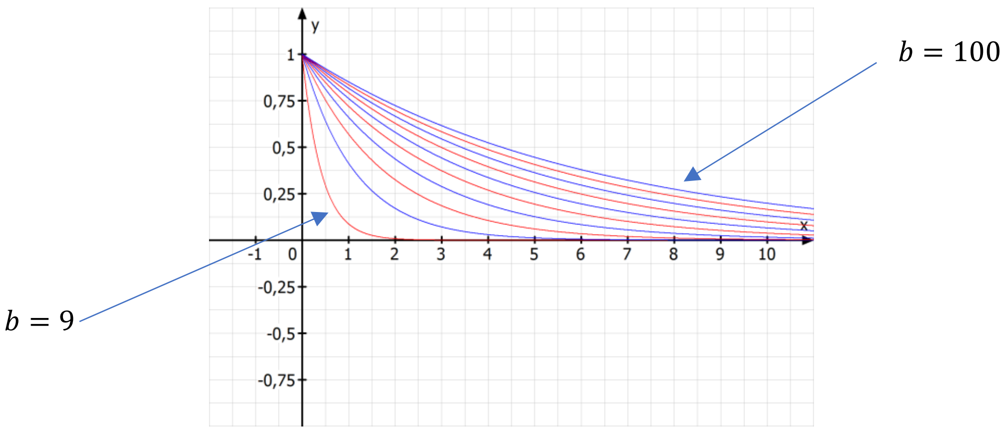
We get the solution:

$$x(t) = \frac{1}{2}e^{\frac{-b-\sqrt{b^2-64}}{2}t} + \frac{1}{2}e^{\frac{-b+\sqrt{b^2-64}}{2}t} =$$

$$\frac{1}{2}\left(e^{-\frac{b}{2}t}e^{-\frac{\sqrt{b^2-64}}{2}t} + e^{-\frac{b}{2}t}e^{\frac{\sqrt{b^2-64}}{2}t}\right) =$$

$$\frac{1}{2}e^{-\frac{b}{2}t}\left(e^{-\frac{\sqrt{b^2-64}}{2}t} + e^{\frac{\sqrt{b^2-64}}{2}t}\right)$$

We plot:



The higher the damping, the slower the mass moves to the origin.

In case of two real roots, how often can the mass pass the equilibrium?

We calculate $x(t) = 0$:

$$0 = c_1 e^{\frac{-b-\sqrt{b^2-64}}{2}t} + c_2 e^{\frac{-b+\sqrt{b^2-64}}{2}t}$$

$$0 = c_1 e^{-\frac{b}{2}t}e^{-\frac{\sqrt{b^2-64}}{2}t} + c_2 e^{-\frac{b}{2}t}e^{\frac{\sqrt{b^2-64}}{2}t}$$

$$0 = c_1 e^{-\frac{\sqrt{b^2-64}}{2}t} + c_2 e^{\frac{\sqrt{b^2-64}}{2}t}$$

$$0 = e^{-\frac{\sqrt{b^2-64}}{2}t} + \frac{c_2}{c_1}e^{\frac{\sqrt{b^2-64}}{2}t}$$

$$0 = 1 + \frac{c_2}{c_1}\frac{e^{\frac{\sqrt{b^2-64}}{2}t}}{e^{-\frac{\sqrt{b^2-64}}{2}t}} = 1 + \frac{c_2}{c_1}e^{\sqrt{b^2-64}t}$$

$$\frac{c_2}{c_1}e^{\sqrt{b^2-64}t} = -1$$

$$e^{\sqrt{b^2-64}t} = -\frac{c_1}{c_2}$$

If $e^{\sqrt{b^2-64}t} = -\frac{c_1}{c_2}$ and $-\frac{c_1}{c_2} > 0$ we get a one-time crossing of the zero point.

If $-\frac{c_1}{c_2} < 0$ we get the aperiodic limit without crossing the zero point.

The characteristic polynomial:

$$p(r) = r^2 + b \cdot r + 16$$

The roots of the characteristic polynomial:

$$r_{12} = \frac{-b \pm \sqrt{b^2 - 64}}{2}$$

We get two distinct complex roots:

$$b^2 - 64 < 0$$

$$b < 8$$

In this case we have:

$$r_1 = -\frac{b + i\sqrt{64 - b^2}}{2}$$

$$r_2 = -\frac{b - i\sqrt{64 - b^2}}{2}$$

We get the solutions:

$$x_1(t) = c_1 e^{-\frac{b}{2}t} e^{-i\frac{\sqrt{64-b^2}}{2}t}$$

$$x_2(t) = c_2 e^{-\frac{b}{2}t} e^{i\frac{\sqrt{64-b^2}}{2}t}$$

The combined solution:

$$x(t) = e^{-\frac{b}{2}t}\left(c_1 e^{-i\frac{\sqrt{64-b^2}}{2}t} + c_2 e^{i\frac{\sqrt{64-b^2}}{2}t}\right)$$

We use: $x(t=0) = 1$:

$$x(t=0) = c_1 + c_2 = 1 \rightarrow c_2 = 1 - c_1$$

We have:

$$x(t) = e^{-\frac{b}{2}t}\left(c_1 e^{-i\frac{\sqrt{64-b^2}}{2}t} + e^{i\frac{\sqrt{64-b^2}}{2}t} - c_1 e^{i\frac{\sqrt{64-b^2}}{2}t}\right) =$$

$$e^{-\frac{b}{2}t}e^{i\frac{\sqrt{64-b^2}}{2}t}+c_1e^{-\frac{b}{2}t}\left(e^{-i\frac{\sqrt{64-b^2}}{2}t}-e^{i\frac{\sqrt{64-b^2}}{2}t}\right)$$

Note:

$$e^{-i\frac{\sqrt{64-b^2}}{2}t}-e^{i\frac{\sqrt{64-b^2}}{2}t}=-2\cdot i\cdot sin\left(\frac{\sqrt{64-b^2}}{2}t\right)$$

We have:

$$x(t)=e^{-\frac{b-i\sqrt{64-b^2}}{2}t}-c_1\cdot 2\cdot i\cdot sin\left(\frac{\sqrt{64-b^2}}{2}t\right)$$

$$\dot{x}(t)=-\frac{b-i\sqrt{64-b^2}}{2}e^{-\frac{b-i\sqrt{64-b^2}}{2}t}-c_1\cdot 2\cdot i\cdot cos\left(\frac{\sqrt{64-b^2}}{2}t\right)\frac{\sqrt{64-b^2}}{2}$$

We use: $\dot{x}(t=0)=-\frac{b}{2}$:

$$\dot{x}(t=0)=-\frac{b-i\sqrt{64-b^2}}{2}-c_1\cdot 2\cdot i\cdot\frac{\sqrt{64-b^2}}{2}=-\frac{b}{2}$$

$$-\frac{b}{2}+\frac{i\sqrt{64-b^2}}{2}-c_1\cdot 2\cdot i\cdot\frac{\sqrt{64-b^2}}{2}+\frac{b}{2}=0$$

$$\frac{i\sqrt{64-b^2}}{2}-c_1\cdot 2\cdot\frac{i\sqrt{64-b^2}}{2}=0$$

$$\frac{1}{2}-c_1=0$$

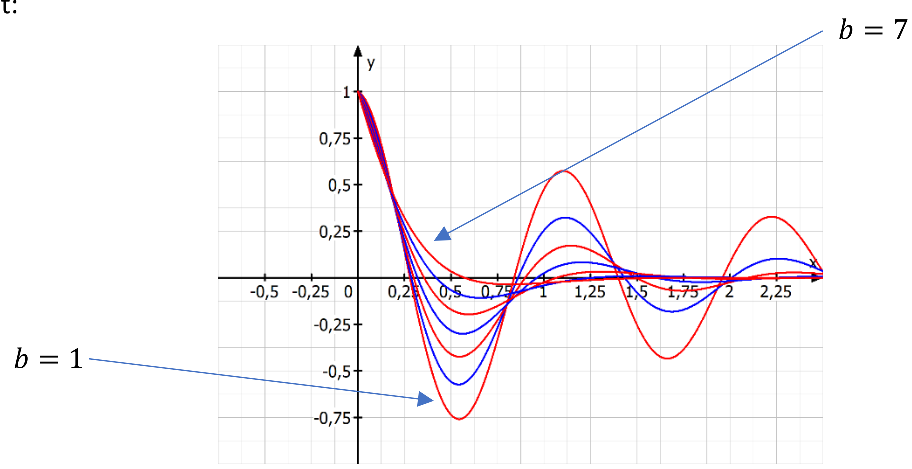
$$c_1=\frac{1}{2}\rightarrow c_2=\frac{1}{2}$$

We get the solution:

$$x(t)=\frac{1}{2}e^{-\frac{b}{2}t}\left(e^{-i\frac{\sqrt{64-b^2}}{2}t}+e^{i\frac{\sqrt{64-b^2}}{2}t}\right)=$$

$$e^{-\frac{b}{2}t}\cdot cos\left(\frac{\sqrt{64-b^2}}{2}t\right)$$

We plot:



The higher the damping, the smaller the amplitude of the oscillation.

We calculate the period of the oscillation:

$$\frac{\sqrt{64-b^2}}{2}t_1 = \frac{\pi}{2} \rightarrow t_1 = \frac{\pi}{\sqrt{64-b^2}}$$

$$\frac{\sqrt{64-b^2}}{2}t_2 = \frac{3\pi}{2} \rightarrow t_2 = \frac{3\pi}{\sqrt{64-b^2}}$$

$$t_2 - t_1 = \frac{3\pi}{\sqrt{64-b^2}} - \frac{\pi}{\sqrt{64-b^2}} = \frac{2\pi}{\sqrt{64-b^2}}$$

Pseudoperiod

The value of $2(t_2 - t_1) = \frac{4\pi}{\sqrt{64-b^2}}$ is called the pseudoperiod of the oscillation.

The bigger b, the bigger $\frac{4\pi}{\sqrt{64-b^2}}$. The damping leads to an increase of the oscillation period, the oscillation becomes slower.

Numeric example 2:

$$0.5 \cdot \ddot{x} + 1.5 \cdot \dot{x} + \frac{5}{8}x = 0$$

$$\ddot{x} + 3 \cdot \dot{x} + \frac{10}{8}x = 0$$

We get the characteristic equation:

$$r^2 + 3r + \frac{10}{8} = 0$$

We get r_1 and r_2:

$$r_{1/2} = \frac{-3 \pm \sqrt{9-5}}{2} = \frac{-3 \pm 2}{2}$$

$$r_1 = -\frac{1}{2}, r_2 = -\frac{5}{2}$$

We get the solution:

$$x(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-\frac{5}{2}t}$$

$$\dot{x}(t) = -\frac{1}{2}c_1 e^{-\frac{1}{2}t} - \frac{5}{2}c_2 e^{-\frac{5}{2}t}$$

We set the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

From the first condition we get:

$$x(t=0) = x_0$$

$$x_0 = c_1 + c_2$$

$$c_1 = x_0 - c_2$$

From the second condition we get:

$$\dot{x}(t=0)=v_0$$

$$v_0=-\frac{1}{2}c_1-\frac{5}{2}c_2$$

$$v_0=-\frac{1}{2}(x_0-c_2)-\frac{5}{2}c_2$$

$$v_0=-\frac{1}{2}x_0+\frac{1}{2}c_2-\frac{5}{2}c_2=-\frac{1}{2}x_0-2c_2$$

$$-2c_2=v_0+\frac{1}{2}x_0$$

$$-c_2=\frac{v_0}{2}+\frac{1}{4}x_0$$

$$c_2=-\frac{1}{4}x_0-\frac{v_0}{2}$$

$$c_1=x_0-c_2$$

$$c_1=x_0+\frac{v_0}{2}+\frac{1}{4}x_0=\frac{5}{4}x_0+\frac{v_0}{2}$$

We get the solution with this initial condition:

$$x(t)=\left(\frac{5}{4}x_0+\frac{v_0}{2}\right)e^{-\frac{1}{2}t}-\left(\frac{1}{4}x_0+\frac{v_0}{2}\right)e^{-\frac{5}{2}t}$$

We plot:

This is the case the cart having the velocity +1 outwards at position 1.

We set the initial position of the cart:

$$x_0=0.25$$

$$x(t)=\left(\frac{5}{16}+\frac{v_0}{2}\right)e^{-\frac{1}{2}t}-\left(\frac{1}{16}+\frac{v_0}{2}\right)e^{-\frac{5}{2}t}$$

$$x(t)=e^{-\frac{1}{2}t}\left(\left(\frac{5}{16}+\frac{v_0}{2}\right)-\left(\frac{1}{16}+\frac{v_0}{2}\right)e^{-\frac{4}{2}t}\right)$$

$x(t)$ becomes negative if $\left(\frac{5}{16}+\frac{v_0}{2}\right)$ becomes zero. This happens if:

$$\frac{5}{16}+\frac{v_0}{2}<0$$

$$\frac{v_0}{2}<-\frac{5}{16}$$

$$v_0<-\frac{8}{8}$$

Numeric example 3:

$$\frac{1}{2}\cdot\ddot{x}+\frac{1}{4}\cdot\dot{x}+\frac{51}{25}x=0$$

$$\ddot{x}+\frac{1}{2}\cdot\dot{x}+\frac{102}{25}x=0$$

We get the characteristic equation:

$$r^2+\frac{1}{2}\cdot r+\frac{102}{25}=0$$

We get r_1 and r_2:

$$r_{\frac{1}{2}}=\frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{408}{25}}}{2}=\frac{-\frac{1}{2}\pm\sqrt{-\frac{1607}{100}}}{2}=$$

$$-\frac{1}{4}\pm\sqrt{-4.0175}$$

$$r_1=-\frac{1}{4}+i\sqrt{4.0175},r_2=-\frac{1}{4}-i\sqrt{4.0175}$$

For better readability we rename:

$$\sqrt{4.0175}=\alpha$$

$$r_1=-\frac{1}{4}+i\alpha,r_2=-\frac{1}{4}-i\alpha$$

We get the solution:

$$x(t)=c_1e^{\left(-\frac{1}{4}+i\alpha\right)t}+c_2e^{\left(-\frac{1}{4}-i\alpha\right)t}$$

$$x(t)=e^{-\frac{1}{4}t}\left(c_1e^{i\alpha t}+c_2e^{-i\alpha t}\right)$$

We calculate $\dot{x}(t)$ by using the first expression:

$$\dot{x}(t)=\left(-\frac{1}{4}+i\alpha\right)c_1e^{\left(-\frac{1}{4}+i\alpha\right)t}+\left(-\frac{1}{4}-i\alpha\right)c_2e^{\left(-\frac{1}{4}-i\alpha\right)t}$$

$$=\left(-\frac{1}{4}+i\alpha\right)c_1e^{-\frac{1}{4}t}e^{i\alpha t}+\left(-\frac{1}{4}-i\alpha\right)c_2e^{-\frac{1}{4}t}e^{-i\alpha t}$$

$$=e^{-\frac{1}{4}t}\left(\left(-\frac{1}{4}+i\alpha\right)c_1e^{i\alpha t}-\left(\frac{1}{4}+i\alpha\right)c_2e^{-i\alpha t}\right)$$

We set the initial conditions:

$$x(0)=0$$

$$\dot{x}(0) = 1$$

At time $t = 0$ the cart is in position zero moving with velocity 1 outwards.

From the first condition we get:

$$x(t = 0) = 0$$

$$c_1 + c_2 = 0$$

$$c_2 = -c_1$$

From the second condition we get:

$$\dot{x}(t) = e^{-\frac{1}{4}t}\left(\left(-\frac{1}{4} + i\alpha\right)c_1 e^{i\alpha t} - \left(\frac{1}{4} + i\alpha\right)c_2 e^{-i\alpha t}\right)$$

$$= e^{-\frac{1}{4}t}\left(\left(-\frac{1}{4} + i\alpha\right)c_1 e^{i\alpha t} + \left(\frac{1}{4} + i\alpha\right)c_1 e^{-i\alpha t}\right)$$

$$\dot{x}(t = 0) = 1$$

$$\left(\left(-\frac{1}{4} + i\alpha\right)c_1 + \left(\frac{1}{4} + i\alpha\right)c_1\right) = 1$$

$$-\frac{1}{4}c_1 + i\alpha c_1 + \frac{1}{4}c_1 + i\alpha c_1 = 1$$

$$2i\alpha c_1 = 1$$

$$c_1 = \frac{1}{2i\alpha} = -\frac{i}{2\alpha}$$

$$c_2 = \frac{i}{2\alpha}$$

We get the solution with this initial condition:

$$x(t) = e^{-\frac{1}{4}t}\left(c_1 e^{i\alpha t} + c_2 e^{-i\alpha t}\right)$$

$$= e^{-\frac{1}{4}t}\left(-\frac{i}{2\alpha}e^{i\alpha t} + \frac{i}{2\alpha}e^{-i\alpha t}\right)$$

$$= -e^{-\frac{1}{4}t} \cdot \frac{i}{2\alpha} \cdot \left(e^{i\alpha t} - e^{-i\alpha t}\right)$$

We note:

$$e^{i\alpha t} - e^{-i\alpha t} = 2 \cdot i \cdot Im\left(e^{i\alpha t}\right)$$

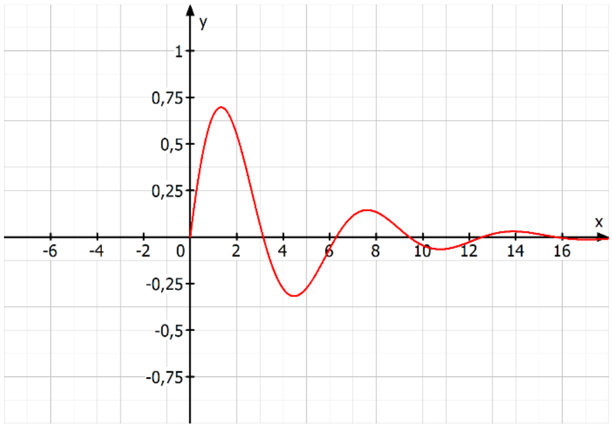
We get:

$$x(t) = e^{-\frac{1}{4}t} \cdot \frac{Im\left(e^{i\alpha t}\right)}{\alpha}$$

We replace α:

$$x(t) = e^{-\frac{1}{4}t} \cdot \frac{\sin\left(\sqrt{4.0175} \cdot t\right)}{\sqrt{4.0175}}$$

We plot:



This is the cart having the velocity +1 outwards at position 0.

Representing a sinusoidal function in rectangular form and in polar form.

Real part:

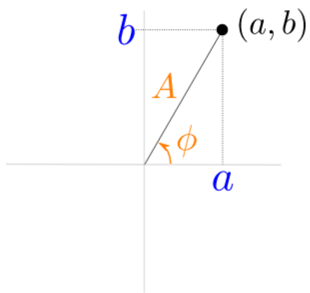
$$\frac{1}{1+\left(\frac{\omega}{k}\right)^2}\left(cos(\omega t)+\frac{\omega}{k}\cdot sin(\omega t)\right)$$

rectangular form

$$a\cdot cos(\theta)+b\cdot sin(\theta)=C\cdot cos(\theta-\varphi)$$

polar form

The transformation:



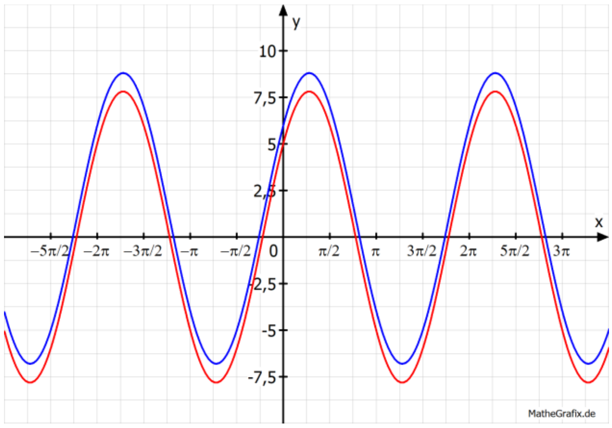
$$a\cdot cos(\theta)+b\cdot sin(\theta)=A\cdot cos(\theta-\phi)=$$

$$A\cdot cos(\phi)\cdot cos(\theta)+A\cdot sin(\phi)\cdot sin(\theta)=$$

$$A\cdot\big(cos(\phi)\cdot cos(\theta)+sin(\phi)\cdot sin(\theta)\big)=$$

$$\sqrt{a^2+b^2}\cdot\big(cos(\phi)\cdot cos(\theta)+sin(\phi)\cdot sin(\theta)\big)$$

Example 1:



$$f(x) = 5 * cos(x) + 6 * sin(x) \rightarrow f(x) = 7{,}8 * cos(x - pi/3{,}57) + 1$$

$$\frac{1}{1+\left(\frac{\omega}{k}\right)^2}\left(cos(\omega t) + \frac{\omega}{k} \cdot sin(\omega t)\right) \rightarrow \frac{1}{\sqrt{1+\left(\frac{\omega}{k}\right)^2}} \cdot cos(\omega t - \phi)$$

Example 2:

$$-1 * cos(5 * x) - \sqrt{3} * sin(5 * x)$$

Check:

$$a \cdot cos(\theta) + b \cdot sin(\theta) = C \cdot cos(\theta - \phi)$$

Left side:

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$sin(\theta) = -i \cdot \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$a \cdot \frac{e^{i\theta} + e^{-i\theta}}{2} - b \cdot i \cdot \frac{e^{i\theta} - e^{-i\theta}}{2} =$$

$$\frac{a}{2} \cdot e^{i\theta} + \frac{a}{2} \cdot e^{-i\theta} - \frac{b \cdot i}{2} \cdot e^{i\theta} + \frac{b \cdot i}{2} \cdot e^{-i\theta} =$$

$$(a - b \cdot i) \cdot \frac{e^{i\theta}}{2} + (a + b \cdot i) \cdot \frac{e^{-i\theta}}{2} =$$

$$(a - b \cdot i) \cdot \frac{e^{i\theta}}{2} + (a + b \cdot i) \cdot \frac{e^{-i\theta}}{2} =$$

$$c \cdot e^{-i\phi} \cdot \frac{e^{i\theta}}{2} + c \cdot e^{i\phi} \cdot \frac{e^{-i\theta}}{2} =$$

$$c \cdot \frac{e^{i(\theta-\phi)}}{2} + c \cdot \frac{e^{-i(\theta-\phi)}}{2} =$$

$$c \cdot \left(\frac{e^{i(\theta-\phi)}}{2} + \frac{e^{-i(\theta-\phi)}}{2}\right) =$$

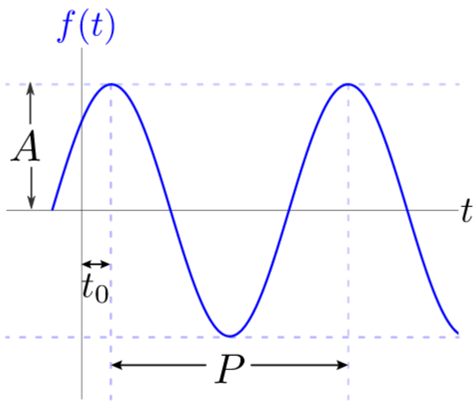
$$C \cdot cos(\theta - \phi)$$

We get the value of C:

$$|C| = \sqrt{a^2 + b^2}$$

$$\phi = arctan -\frac{b}{a}$$

Describing a wave



Graph of $f(t) = A\cos(\omega t - \phi)$

A: Amplitude

P: time for one complete period

t_0: time lag compared to the original cos-function.

These are related to ω and ϕ:

$$P = \frac{2\pi}{\omega}$$

$$t_0 = \frac{\phi}{\omega}$$

We remember:

$$\omega = 2\pi v = \frac{2\pi}{P}$$

Numeric example 4:

$$\ddot{x} + 0 \cdot \dot{x} + \pi^2 x = 0$$

We get the characteristic equation:

$$r^2 + 0 \cdot r + \pi^2 = 0$$

We get r_1 and r_2:

$$r_{\frac{1}{2}} = \frac{\pm\sqrt{-4\pi^2}}{2} = \pm i\pi$$

We get the solution:

$$x(t) = c_1 e^{i\pi t} + c_2 e^{-i\pi t}$$

We calculate $\dot{x}(t)$:

$$\dot{x}(t) = i\pi c_1 e^{i\pi t} - i\pi c_2 e^{-i\pi t} =$$

$$i\pi\left(c_1 e^{i\pi t} - c_2 e^{-i\pi t}\right)$$

We set the initial conditions:

$$x(0) = -1$$

$$\dot{x}(0) = \pi$$

From the first condition we get:

$$x(t = 0) = -1$$

$$c_1 + c_2 = -1$$

$$c_2 = -1 - c_1$$

From the second condition we get:

$$\dot{x}(t = 0) = i\pi\left(c_1 e^{i\pi t} - c_2 e^{-i\pi t}\right) = \pi$$

$$i\pi(c_1 - c_2) = \pi$$

$$i(c_1 - c_2) = 1$$

$$-(c_1 - c_2) = i$$

$$-c_1 + c_2 = -i$$

$$c_2 = c_1 - i$$

$$-1 - c_1 = c_1 - i$$

$$-1 + i = 2c_1$$

$$c_1 = -\frac{1}{2} + \frac{i}{2}$$

$$c_2 = -\frac{1}{2} - \frac{i}{2}$$

We get the solution with this initial condition:

$$x(t) = \left(-\frac{1}{2} + \frac{i}{2}\right) e^{i\pi t} + \left(-\frac{1}{2} - \frac{i}{2}\right) e^{-i\pi t} =$$

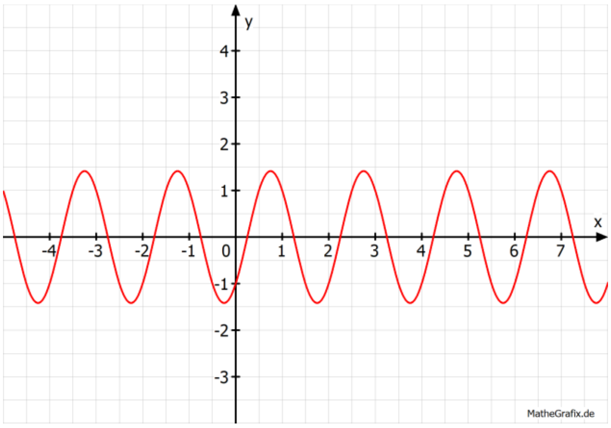
$$-\frac{1}{2} e^{i\pi t} + \frac{i}{2} e^{i\pi t} - \frac{1}{2} e^{-i\pi t} - \frac{i}{2} e^{-i\pi t} =$$

$$-\frac{1}{2} e^{i\pi t} - \frac{1}{2} e^{-i\pi t} + \frac{i}{2} e^{i\pi t} - \frac{i}{2} e^{-i\pi t} =$$

$$-\frac{1}{2}\left(e^{i\pi t} + e^{-i\pi t}\right) + \frac{i}{2}\left(e^{i\pi t} - e^{-i\pi t}\right) =$$

$$-cos(\pi t) + sin(\pi t)$$

We plot:



Numeric example 5:

$$\ddot{x} + \frac{1}{2} \cdot \dot{x} + \frac{17}{16} x = 0$$

We get the characteristic equation:

$$r^2 + \frac{1}{2} \cdot r + \frac{17}{16} = 0$$

We get r_1 and r_2:

$$r_{\frac{1}{2}} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{17}{4}}}{2} = \frac{-\frac{1}{2} \pm \sqrt{-4}}{2} =$$

$$-\frac{1}{4} \pm i$$

$$r_1 = -\frac{1}{4} + i, r_2 = -\frac{1}{4} - i$$

We get the solution:

$$x(t) = c_1 e^{\left(-\frac{1}{4}+i\right)t} + c_2 e^{\left(-\frac{1}{4}-i\right)t}$$

$$x(t) = e^{-\frac{1}{4}t}\left(c_1 e^{it} + c_2 e^{-it}\right)$$

We calculate $\dot{x}(t)$ by using the first expression:

$$\dot{x}(t) = \left(-\frac{1}{4} + i\right) c_1 e^{\left(-\frac{1}{4}+i\right)t} + \left(-\frac{1}{4} - i\right) c_2 e^{\left(-\frac{1}{4}-i\right)t}$$

$$= \left(-\frac{1}{4} + i\right) c_1 e^{-\frac{1}{4}t} e^{it} + \left(-\frac{1}{4} - i\right) c_2 e^{-\frac{1}{4}t} e^{-it}$$

$$= e^{-\frac{1}{4}t}\left(\left(-\frac{1}{4} + i\right) c_1 e^{it} - \left(\frac{1}{4} + i\right) c_2 e^{-it}\right)$$

We set the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0.75$$

At time $t = 0$ the door is in position 1 swinging with velocity 1 outwards.

From the first condition we get:

$$x(t = 0) = 1$$

$$c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

From the second condition we get:

$$\dot{x}(t = 0) = \left(\left(-\frac{1}{4} + i\right)c_1 - \left(\frac{1}{4} + i\right)c_2\right) = 0.75$$

$$\dot{x}(t = 0) = \left(\left(-\frac{1}{4} + i\right)c_1 - \left(\frac{1}{4} + i\right)(1 - c_1)\right) = 0.75$$

$$\left(\left(-\frac{1}{4} + i\right)c_1 - \left(\frac{1}{4} + i\right)(1 - c_1)\right) - 0.75 = 0$$

$$\left(\left(-\frac{1}{4} + i\right)c_1 - \left(\frac{1}{4} - \frac{1}{4}c_1 + i - ic_1\right)\right) - 0.75 = 0$$

$$-\frac{1}{4}c_1 + ic_1 - \frac{1}{4} + \frac{1}{4}c_1 - i + ic_1 - 0.75 = 0$$

$$ic_1 - 1 - i + ic_1 = 0$$

$$2ic_1 - 1 - i = 0$$

$$c_1 = \frac{1 + i}{2i}$$

$$c_1 = \frac{1 - i}{2}$$

$$c_2 = 1 - \frac{1 - i}{2} = \frac{1 + i}{2}$$

We get the solution with this initial condition:

$$x(t) = e^{-\frac{1}{4}t}\left(c_1 e^{it} + c_2 e^{-it}\right)$$

$$= e^{-\frac{1}{4}t}\left(\left(\frac{1 - i}{2}\right)e^{it} + \left(\frac{1 + i}{2}\right)e^{-it}\right)$$

$$= \frac{1}{2}e^{-\frac{1}{4}t}\left((1 - i)e^{it} + (1 + i)e^{-it}\right)$$

$$= \frac{1}{2}e^{-\frac{1}{4}t}\left(e^{it} - ie^{it} + e^{-it} + ie^{-it}\right)$$

$$= \frac{1}{2}e^{-\frac{1}{4}t}\left(e^{it} + e^{-it} + ie^{-it} - ie^{it}\right)$$

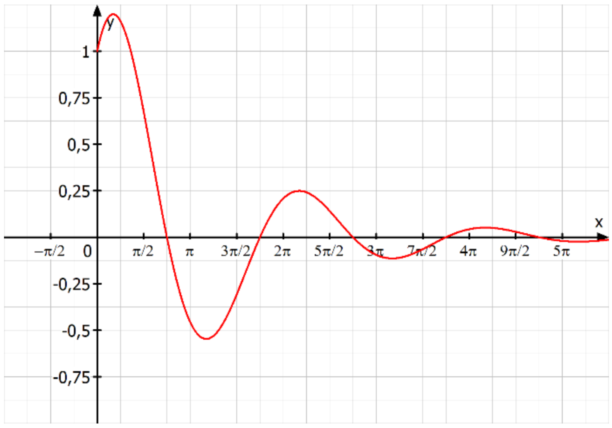
$$= \frac{1}{2}e^{-\frac{1}{4}t}\left(e^{it} + e^{-it} - i\left(ie^{it} - e^{-it}\right)\right)$$

We get:

$$x(t) = \frac{1}{2} e^{-\frac{1}{4}t}\big(2cos(t) - i \cdot 2 \cdot isin(t)\big) =$$

$$e^{-\frac{1}{4}t}\big(cos(t) + sin(t)\big)$$

We plot:



This is the cart having the velocity +0.75 outwards at position 1.

We bring into the phase form:

$$e^{-\frac{1}{4}t}\big(cos(t) + sin(t)\big)$$

We use:

$$a \cdot cos(t) + b \cdot sin(t) = C \cdot cos(t - \phi)$$

We get the value of C:

$$|C| = \sqrt{a^2 + b^2}$$

$$|C| = \sqrt{2}$$

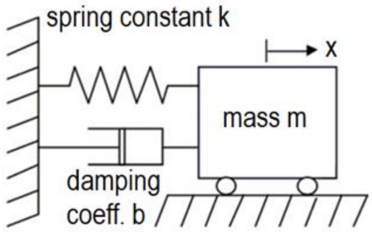
$$\phi = arctan - \frac{b}{a}$$

$$\phi = -\frac{\pi}{4}$$

We get the phase form:

$$e^{-\frac{1}{4}t}\big(cos(t) + sin(t)\big) = \sqrt{2} \cdot cos\left(t + \frac{\pi}{4}\right)$$

The general case



The position is determined by the differential equation:

$$m\ddot{x} + b \cdot \dot{x} + k \cdot x = 0$$

The characteristic polynomial:

$$p(r) = m \cdot r^2 + b \cdot r + k$$

The roots of the characteristic polynomial:

$$r_{12} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

We write:

$$\omega_n = \sqrt{\frac{k}{m}}$$

Note: ω_n is the frequency of the undamped oscillator.

We get:

$$r_{12} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - {\omega_n}^2}$$

Depending on the value of $\left(\frac{b}{2m}\right)^2 - {\omega_n}^2$ we distinguish three cases.

Case 1, underdamped:

$$\left(\frac{b}{2m}\right)^2 - {\omega_n}^2 \leftrightarrow b^2 - 4mk < 0$$

$$r_{12} = -\frac{b}{2m} \pm i\sqrt{{\omega_n}^2 - \left(\frac{b}{2m}\right)^2}$$

We note that $\sqrt{{\omega_n}^2 - \left(\frac{b}{2m}\right)^2}$ gives the frequency of the damped oscillator that is smaller than the frequency of the undamped oscillator by $\left(\frac{b}{2m}\right)^2$.

Case 2, overdamped:

$$\left(\frac{b}{2m}\right)^2 - {\omega_n}^2 > 0 \leftrightarrow b^2 - 4mk > 0$$

$$r_{12} = -\frac{b}{2m} \pm i\sqrt{{\omega_n}^2 - \left(\frac{b}{2m}\right)^2}$$

Case 3, limit:

$$\left(\frac{b}{2m}\right)^2 - {\omega_n}^2 = 0 \leftrightarrow b^2 - 4mk = 0$$

$$r_1 = -\frac{b}{2m}$$

Numeric example 6:

$$\ddot{x} + \dot{x} + 3x = 0$$

We get the characteristic equation:

$$r^2 + r + 3 = 0$$

We get r_1 and r_2:

$$r_{1/2} = \frac{-1 \pm \sqrt{1-12}}{2} = -\frac{1}{2} \pm i \cdot \frac{\sqrt{11}}{2}$$

We get the solution:

$$x(t) = c_1 e^{\left(-\frac{1}{2} + i\cdot\frac{\sqrt{11}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2} - i\cdot\frac{\sqrt{11}}{2}\right)t}$$

$$x(t) = e^{-\frac{1}{2}t}\left(c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} + c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)$$

$$\dot{x}(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} + c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\right) + e^{-\frac{1}{2}t}\left(i\cdot\frac{\sqrt{11}}{2}c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} - i\cdot\frac{\sqrt{11}}{2}c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\right) =$$

$$-\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} + c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t} + i\cdot\sqrt{11}c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} - i\cdot\sqrt{11}c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\right) =$$

$$-\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} + i\cdot\sqrt{11}c_1 e^{i\cdot\frac{\sqrt{11}}{2}t} + c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t} - i\cdot\sqrt{11}c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\right) =$$

$$-\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 e^{i\cdot\frac{\sqrt{11}}{2}t}\left(1 + i\cdot\sqrt{11}\right) + c_2 e^{-i\cdot\frac{\sqrt{11}}{2}t}\left(1 - i\cdot\sqrt{11}\right)\right)$$

We set the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

From the first condition we get:

$$c_2 = 1 - c_1$$

From the second condition we get:

$$\dot{x}(t=0)=0$$

$$-\frac{1}{2}\left(c_1\left(1+i\cdot\sqrt{11}\right)+c_2\left(1-i\cdot\sqrt{11}\right)\right)=0$$

$$c_1\left(1+i\cdot\sqrt{11}\right)+c_2\left(1-i\cdot\sqrt{11}\right)=0$$

$$c_1\left(1+i\cdot\sqrt{11}\right)+(1-c_1)\left(1-i\cdot\sqrt{11}\right)=0$$

$$c_1\left(1+i\cdot\sqrt{11}\right)+\left(1-i\cdot\sqrt{11}\right)+c_1\left(-1+i\cdot\sqrt{11}\right)=0$$

$$c_1\left(1+i\cdot\sqrt{11}-1+i\cdot\sqrt{11}\right)+\left(1-i\cdot\sqrt{11}\right)=0$$

$$c_1\left(2i\cdot\sqrt{11}\right)+1-i\cdot\sqrt{11}=0$$

$$c_1\left(2i\cdot\sqrt{11}\right)=i\sqrt{11}-1$$

$$c_1=\frac{i\sqrt{11}-1}{2i\cdot\sqrt{11}}=\frac{\sqrt{11}}{2\cdot\sqrt{11}}-\frac{1}{2i\cdot\sqrt{11}}=\frac{1}{2}+\frac{i}{2\sqrt{11}}$$

$$c_2=1-\left(\frac{1}{2}+\frac{i}{2\sqrt{11}}\right)=\frac{1}{2}-\frac{i}{2\sqrt{11}}$$

Result:

$$x(t)=e^{-\frac{1}{2}t}\left(\left(\frac{1}{2}+\frac{i}{2\sqrt{11}}\right)e^{i\cdot\frac{\sqrt{11}}{2}t}+\left(\frac{1}{2}-\frac{i}{2\sqrt{11}}\right)e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)$$

Note: This is of kind $z_1z_2+\overline{z_1z_2}$ and a real number.

We expand:

$$x(t)=e^{-\frac{1}{2}t}\left(\left(\frac{1}{2}+\frac{i}{2\sqrt{11}}\right)e^{i\cdot\frac{\sqrt{11}}{2}t}+\left(\frac{1}{2}-\frac{i}{2\sqrt{11}}\right)e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)=$$

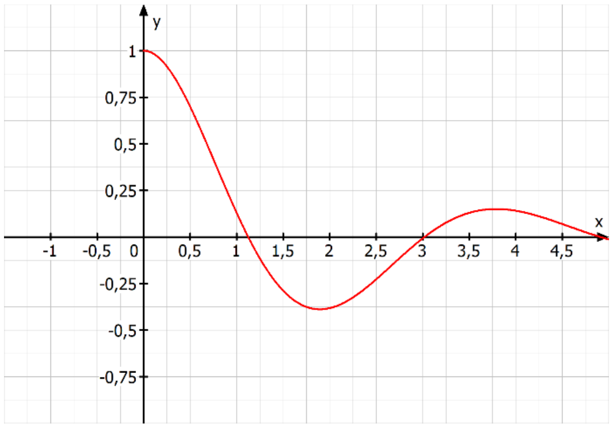
$$e^{-\frac{1}{2}t}\left(\frac{1}{2}e^{i\cdot\frac{\sqrt{11}}{2}t}+\frac{i}{2\sqrt{11}}e^{i\cdot\frac{\sqrt{11}}{2}t}+\frac{1}{2}e^{-i\cdot\frac{\sqrt{11}}{2}t}-\frac{i}{2\sqrt{11}}e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)=$$

$$e^{-\frac{1}{2}t}\left(\frac{1}{2}e^{i\cdot\frac{\sqrt{11}}{2}t}+\frac{1}{2}e^{-i\cdot\frac{\sqrt{11}}{2}t}+\frac{i}{2\sqrt{11}}e^{i\cdot\frac{\sqrt{11}}{2}t}-\frac{i}{2\sqrt{11}}e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)=$$

$$e^{-\frac{1}{2}t}\left(\frac{1}{2}\left(e^{i\cdot\frac{\sqrt{11}}{2}t}+e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)+\frac{i}{2\sqrt{11}}\left(e^{i\cdot\frac{\sqrt{11}}{2}t}-e^{-i\cdot\frac{\sqrt{11}}{2}t}\right)\right)=$$

$$e^{-\frac{1}{2}t}\left(cos\left(\frac{\sqrt{11}}{2}t\right)+\frac{1}{\sqrt{11}}sin\left(\frac{\sqrt{11}}{2}t\right)\right)$$

We plot:



This is a damped “oscillation”.

Higher order linear ode

Some theoretical background

The span of a set of functions $f_1, f_2, \dots\ f_n$ is the set of all linear combinations of $f_1, f_2, \dots\ f_n$:

$$span(f_1, f_2, \dots\ f_n) \coloneqq \sum c_i f_i \ \ c_i \in \mathbb{C}$$

In the set of functions, we have the zero-function $f(x) = 0$.

This is similar to a vector space. A vector space of dimension 3 is the set of all triples (vectors):

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Like the functions above vectors can be added and multiplied by numbers.

In a vector space we have linearly dependent and linearly independent vectors (we need at least dimension 2).

For a set of two vectors $\vec{v}_1, \vec{v}_2$ we have only a simple kind of linear dependent vectors, that are vectors that are a multiple of each other.

For a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ we write:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0 \Leftrightarrow linear\ dependent$$

provided that not all $c_1, c_2, \dots, c_n$ are zero.

This holds for functions too:

For a set of functions $f_1, f_2, \dots, f_n$:

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \Leftrightarrow linear\ dependent$$

provided that not all $c_1, c_2, \dots, c_n$ are zero.

In a vector space we call a set linear independent vectors a basis if the span of these vectors builds the whole vector space and the vectors are linearly independent.

A vector space of dimension 3 has (one of many possible) basis:

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Any set of three linearly independent vectors form a basis:

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Every list of linearly independent functions forms a vector space of functions.

The dimension of the solution space is given by the order of the differential equation.

The dimension of the space of the solutions to a homogeneous ode with constant coefficients of grade 1 is one-dimensional.

The dimension of the space of the solutions to a homogeneous ode with constant coefficients of grade 2 is two-dimensional.

The dimension of the space of the solutions to a homogeneous ode with constant coefficients of grade n is n-dimensional.

We want to solve a homogeneous ode with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

We write the characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

We calculate the roots and factor:

$$P(r) = a_n (r - r_1)(r - r_2) \dots (r - r_n)$$

If the roots $r_1, r_2, \dots, r_n$ are distinct, then the functions $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ form a basis for the vector space of solutions:

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

Note: complex roots always appear as pairs of conjugate complex numbers. If some of the roots are complex, the coefficients c_i also are complex.

If the roots $r_1, r_2, \dots, r_n$ are not distinct, then the functions $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ cannot form a basis for the vector space of solutions.

If a particular root has a multiplicity, is repeated m-times, then you can make these solutions linearly independent by multiplying with powers of t:

$$e^{rt}, e^{rt}, \dots, e^{rt} \ \rightarrow \ t^0 e^{rt}, t^1 e^{rt}, \dots, t^{m-1} e^{rt}$$

Numeric example 7:

Find the general solution:

$$y^{(6)} + 6y^{(5)} + 9y^{(4)} = 0$$

The characteristic polynomial:

$$r^6 + 6r^5 + 9r^4 = 0$$

We get the roots:

$$0,0,0,0,-3,-3$$

We get the basis:

$$e^{0t}, te^{0t}, t^2e^{0t}, t^3e^{0t}, e^{-3t}, te^{-3t}$$

We write short:

$$1, t, t^2, t^3, e^{-3t}, te^{-3t}$$

We get the general solution:

$$c_1 + c_2t + c_3t^2 + c_4t^3 + c_5e^{-3t} + c_6te^{-3t}$$

Numeric example 8:

Find the (simplest) homogeneous linear ode with constant coefficients having as one of its solutions:

$$(5t + 7)e^{-t} - 9e^{2t}$$

The function given is a linear combination of:

$$e^{-t}, te^{-t}, e^{2t}$$

The roots of the characteristic polynomial include $-1, -1, 2$.

The simplest characteristic polynomial:

$$(r + 1)(r + 1)(r - 2) = r^3 - 3r - 2$$

The corresponding ode:

$$y^{(3)} - 3y^{(1)} - 2y = 0$$

Numeric example 9:

Find a basis for the general solution:

$$y^{(3)} + 3y^{(2)} + 9y^{(1)} - 13y = 0$$

The characteristic polynomial:

$$r^3 + 3r^2 + 9r - 13 = 0$$

We guess: $r = 1$ is a solution.

We factor out:

$$(r - 1)(r^2 + 4r + 13)$$

We calculate the roots:

$$r_1 = 1$$

$$r^2 + 4r + 13 = 0$$

$$r_{23} = -2 \pm 3i$$

We get the basis:

$$e^t, e^{(-2+3i)t}, e^{(-2-3i)t}$$

We get the general solution:

$$c_1e^t + c_2e^{(-2+3i)t} + c_3e^{(-2-3i)t}$$

We look for a pure real basis:

$$e^{(-2+3i)t} = e^{-2t}e^{3it} = e^{-2t}\big(cos(3t) + i \cdot sin(3t)\big)$$

We get:

$$e^t, e^{-2t}cos(3t), e^{-2t}sin(3t)$$

Note: The "i" goes into the complex coefficient c_n.

Numeric example 10:

Suppose the roots —with multiplicity— of the characteristic polynomial of a certain homogeneous constant coefficient linear equation:

$$3,4,4,4,5 \pm 2i, 5 \mp 2i$$

We get the roots:

$$0,0,0,0,-3,-3$$

We get the basis:

$$e^{3t}, e^{4t}, te^{4t}, t^2e^{4t}, e^{(5+2i)t}, te^{(5+2i)t}, e^{(5-2i)t}, te^{(5-2i)t}$$

We get the general solution:

$$c_1e^{3t} + c_2e^{4t} + c_3te^{4t} + c_4t^2e^{4t} + c_5e^{(5+2i)t} + c_6te^{(5+2i)t} + c_7e^{(5-2i)t} + c_8te^{(5-2i)t}$$

We look for a pure real basis:

$$e^{(5+2i)t} = e^{5t}e^{2it} = e^{5t}\big(cos(2t) + i \cdot sin(2t)\big)$$

We get:

$$e^{3t}, e^{4t}, te^{4t}, t^2e^{4t}, e^{5t}cos(2t), e^{5t}sin(2t), te^{5t}cos(2t), te^{5t}sin(2t)$$

We get the general real solution:

$$c_1e^{3t} + c_2e^{4t} + c_3te^{4t} + c_4t^2e^{4t} + c_5e^{5t}cos(2t) + c_6e^{5t}sin(2t) + c_7te^{5t}cos(2t) + c_8te^{5t}sin(2t)$$

Numeric example 11:

$$x(t)^{(3)} - x(t)^{(1)} = 0$$

The characteristic polynomial:

$$r(r-1)(r+1) = 0$$

The roots:

$$r_1 = 0, r_2 = 1, r_3 = -1$$

We get the basis:

$$e^0, e^t, e^{-t}$$

We get the general solution:

$$x(t) = c_1e^t + c_2e^{-t} + c_3$$

$$\dot{x}(t) = c_1e^t - c_2e^{-t}$$

$$\ddot{x}(t) = c_1 e^t + c_2 e^{-t}$$

Initial conditions:

$$x(0) = 0, \dot{x}(0) = 1, \ddot{x}(0) = 1$$

The first condition:

$$c_1 + c_2 + c_3 = 0$$

The second condition:

$$c_1 - c_2 = 1 \rightarrow c_2 = c_1 - 1$$

The third condition:

$$c_1 + c_2 = 1 \rightarrow c_2 = -(c_1 - 1)$$

We get:

$$c_2 = 0, c_1 = 1, c_3 = -1$$

We check:

$$x(t) = e^t - 1$$

$$\dot{x}(t) = e^t$$

$$\ddot{x}(t) = e^t$$

We check the initial conditions:

$$x(0) = 0, \dot{x}(0) = 1, \ddot{x}(0) = 1$$

$$x(0) = 0 \; ok$$

$$\dot{x}(0) = 1 \; ok$$

$$\ddot{x}(0) = 1 \; ok$$

Numeric example 12:

Find the (simplest) homogeneous linear ode with constant coefficients having as one of its solutions:

$$(1 + 4t - t^2)e^{-2t} + 4e^{-t}cos(3t)$$

Th function given is a linear combination of:

$$e^{-2t}, te^{-2t}, t^2e^{-2t}, e^{-t}cos(3t)$$

$$e^{-t}cos(3t) = e^{-t}\left(\frac{e^{i3t} + e^{-i3t}}{2}\right) = \frac{1}{2}\left(e^{(-1+i3)t} + e^{(-1-i3)t}\right)$$

The roots of the characteristic polynomial include $-2, -2, -2, (-1 + i3), (-1 - i3)$.

The simplest characteristic polynomial:

$$(r + 2)(r + 2)(r + 2)\big(r - (-1 + i3)\big)\big(r - (-1 - i3)\big) =;$$

First part:

$$(r^3 + 6r^2 + 12r + 8)$$

Second part:

$$\big(r-(-1+i3)\big)\big(r-(-1-i3)\big)=$$

$$r^2-r(-1-i3)-r(-1+i3)+(-1+i3)(-1-i3)=$$

$$r^2+r+i3r+r-i3r+10=$$

$$r^2+2r+10$$

We assemble:

$$(r^3+6r^2+12r+8)(r^2+2r+10)=$$

$$r^5+8r^4+34r^3+92r^2+136r+80$$

The corresponding ode:

$$y^{(5)}+8y^{(4)}+34y^{(3)}+92y^{(2)}+136y^{(1)}+80y=0$$

Numeric example 13:

Find the complex number c such that $Re\big(ce^{i\omega t}\big)=3cos(\omega t)-sin(\omega t)$.

$$(x+iy)e^{i\omega t}=xe^{i\omega t}+iye^{i\omega t}$$

$$Re\big(xe^{i\omega t}\big)=3cos(\omega t)\rightarrow x=3$$

$$Re\big(iye^{i\omega t}\big)=-sin(\omega t)\rightarrow y=1$$

We can write this as:

$$\sqrt{10}e^{iarctan(\frac{1}{3})}$$

Example 8

$$\ddot{x}+0\cdot\dot{x}+4x=0$$

We get the characteristic equation:

$$r^2=-4$$

We get r_1 and r_2:

$$r_{\frac{1}{2}}=\pm i2$$

We get the solution:

$$x(t)=c_1e^{i2t}+c_2e^{-i2t}$$

We calculate $\dot{x}(t)$:

$$\dot{x}(t)=i2c_1e^{i2t}-i2c_2e^{-i2t}=$$

$$i2\big(c_1e^{i2t}-c_2e^{-i2t}\big)$$

We set the initial conditions:

$$x(0)=1$$

$$\dot{x}(0)=-2$$

From the first condition we get:

$$x(t=0)=1$$

$$c_1+c_2=1$$

$$c_2=1-c_1$$

From the second condition we get:

$$\dot{x}(t=0)=i2\left(c_1e^{i2t}-c_2e^{-i2t}\right)=-2$$

$$i2(c_1-c_2)=-2$$

$$i(c_1-c_2)=-1$$

$$-(c_1-c_2)=-i$$

$$-c_1+c_2=i$$

$$c_2=c_1+i$$

$$1-c_1=c_1+i$$

$$1-i=2c_1$$

$$c_1=\frac{1}{2}-\frac{i}{2}$$

$$c_2=\frac{1}{2}+\frac{i}{2}$$

We get the solution with this initial condition:

$$x(t)=\left(\frac{1}{2}-\frac{i}{2}\right)e^{i2t}+\left(\frac{1}{2}+\frac{i}{2}\right)e^{-i2t}=$$

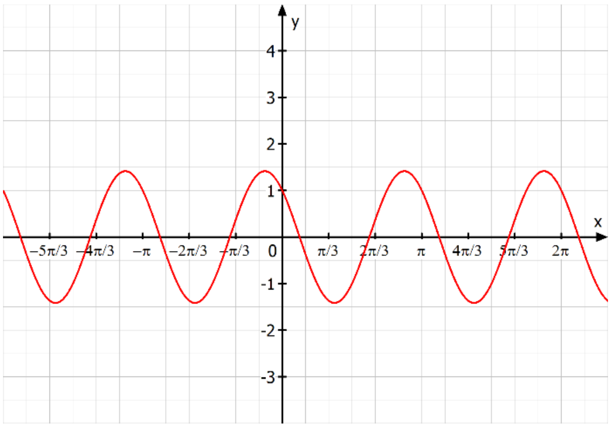
$$\frac{1}{2}e^{i2t}-\frac{i}{2}e^{i2t}+\frac{1}{2}e^{-i2t}+\frac{i}{2}e^{-i2t}=$$

$$\frac{1}{2}e^{i2t}+\frac{1}{2}e^{-i2t}-\frac{i}{2}e^{i2t}+\frac{i}{2}e^{-i2t}=$$

$$\frac{1}{2}\left(e^{i2t}+e^{-i2t}\right)-\frac{i}{2}\left(e^{i2t}-e^{-i2t}\right)=$$

$$cos(2t)-sin(2t)$$

We plot:



Numeric example 14:

$$\ddot{x} + \pi^2 x = 0$$

We get the characteristic equation:

$$r^2 = -\pi^2$$

We get r_1 and r_2:

$$r_{\frac{1}{2}} = \pm i\pi$$

We get the solution:

$$x(t) = c_1 e^{i\pi t} + c_2 e^{-i\pi t}$$

We calculate $\dot{x}(t)$:

$$\dot{x}(t) = i\pi c_1 e^{i\pi t} - i\pi c_2 e^{-i\pi t} =$$

$$i\pi\left(c_1 e^{i\pi t} - c_2 e^{-i\pi t}\right)$$

We set the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = \pi\sqrt{3}$$

From the first condition we get:

$$x(t = 0) = 1$$

$$c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

From the second condition we get:

$$\dot{x}(t = 0) = i\pi\left(c_1 e^{i\pi t} - c_2 e^{-i\pi t}\right) = \pi\sqrt{3}$$

$$i\pi(c_1 - c_2) = \pi\sqrt{3}$$

$$i(c_1 - c_2) = \frac{\pi\sqrt{3}}{\pi} = \sqrt{3}$$

$$-(c_1 - c_2) = i\sqrt{3}$$

$$-c_1 + c_2 = i\sqrt{3}$$

$$c_2 = c_1 + i\sqrt{3}$$

$$1 - c_1 = c_1 + i\sqrt{3}$$

$$1 - i\sqrt{3} = 2c_1$$

$$c_1 = \frac{1}{2} - i\sqrt{3}$$

$$c_2 = \frac{1}{2} + i\sqrt{3}$$

We get the solution with this initial condition:

$$x(t) = \left(\frac{1}{2} - i\sqrt{3}\right)e^{i\pi t} + \left(\frac{1}{2} + i\sqrt{3}\right)e^{-i\pi t} =$$

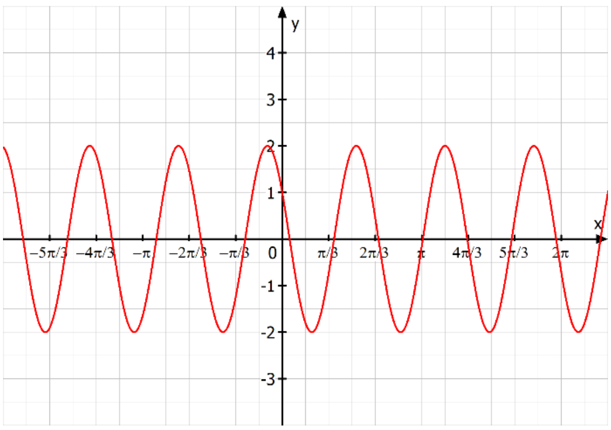
$$\frac{1}{2}e^{i\pi t} - i\sqrt{3}e^{i\pi t} + \frac{1}{2}e^{-i\pi t} + i\sqrt{3}e^{-i\pi t} =$$

$$\frac{1}{2}e^{i\pi t} + \frac{1}{2}e^{-i\pi t} - i\sqrt{3}e^{i\pi t} + i\sqrt{3}e^{-i\pi t} =$$

$$\frac{1}{2}\left(e^{i\pi t} + e^{-i\pi t}\right) - i\sqrt{3}\left(e^{i\pi t} - e^{-i\pi t}\right) =$$

$$cos(\pi t) - \sqrt{3}sin(\pi t)$$

We plot:



Numeric example 15:

Find the (simplest) homogeneous linear ode with constant coefficients having as one of its solutions:

$$y(t) = 4te^{-2t} + e^{-5t}sin(t)$$

We write exponential style:

$$\tilde{y}(t) = 4te^{-2t} + e^{-5t}e^{it} =$$

$$4te^{-2t} + e^{(i-5)t}$$

We see that $4te^{-2t}$ is a double root coefficient because otherwise we would have $4e^{-2t}$ as a single root.

We get the characteristic polynomial:

$$\big(r-(-2)\big)^2\big(r-(i-5)\big)\big(r-(-i-5)\big)=$$

$$(r+2)^2(r+5-i)(r+5+i)=$$

$$(r^2+4r+4)(r^2+5r+ri+5r+25+5i-ri-5i+1)=$$

$$(r^2+4r+4)(r^2+10r+26)=$$

$$r^4+14r^3+70r^2+144r+104$$

We get the differential equation:

$$y^{(4)}+14y^{(3)}+70y^{(2)}+144y^{(1)}+104y^{(0)}=0$$

We check by derivating:

$$y^{(0)}=4te^{-2t}+e^{(i-5)t}$$

$$y^{(1)}=4e^{-2t}-8te^{-2t}+(i-5)e^{(i-5)t}$$

$$y^{(2)}=-16e^{-2t}+16te^{-2t}+(i-5)^2e^{(i-5)t}$$

$$y^{(3)}=48e^{-2t}-32te^{-2t}+(i-5)^3e^{(i-5)t}$$

$$y^{(4)}=-128e^{-2t}+64te^{-2t}+(i-5)^4e^{(i-5)t}$$

We insert into the differential equation:

$$y^{(4)}+14y^{(3)}+70y^{(2)}+144y^{(1)}+104y^{(0)}=0$$

$$-128e^{-2t}+64te^{-2t}+(i-5)^4e^{(i-5)t}+14\big(48e^{-2t}-32te^{-2t}+(i-5)^3e^{(i-5)t}\big)$$
$$+70\big(-16e^{-2t}+16te^{-2t}+(i-5)^2e^{(i-5)t}\big)$$
$$+144\big(4e^{-2t}-8te^{-2t}+(i-5)e^{(i-5)t}\big)+104\big(4te^{-2t}+e^{(i-5)t}\big)=$$

$$-128e^{-2t}+64te^{-2t}+(i-5)^4e^{(i-5)t}+672e^{-2t}-448te^{-2t}+14(i-5)^3e^{(i-5)t}-1120e^{-2t}$$
$$+1120te^{-2t}+70(i-5)^2e^{(i-5)t}+576e^{-2t}-1152te^{-2t}+144(i-5)e^{(i-5)t}$$
$$+416te^{-2t}+104e^{(i-5)t}=$$

$$-128e^{-2t}+672e^{-2t}-1120e^{-2t}+576e^{-2t}+416te^{-2t}+64te^{-2t}-448te^{-2t}+1120te^{-2t}$$
$$-1152te^{-2t}+(i-5)^4e^{(i-5)t}+14(i-5)^3e^{(i-5)t}+70(i-5)^2e^{(i-5)t}$$
$$+144(i-5)e^{(i-5)t}+104e^{(i-5)t}=$$

$$(-128+672-1120+576)e^{-2t}+(416+64-448+1120-1152)te^{-2t}$$
$$+((i-5)^4+14(i-5)^3+70(i-5)^2+144(i-5)+104)e^{(i-5)t}=$$

$$(0)e^{-2t}+(0)te^{-2t}+0e^{(i-5)t}=0$$

Numeric example 16:

$$\ddot{x}-t^4\dot{x}=0$$

We get the characteristic equation:

$$r^2-t^4r=0$$

$$r_{1/2}=\frac{t^4\pm\sqrt{t^8}}{2}=\frac{t^4\pm t^4}{2}$$

We have:

$$r_1 = t^4$$

$$r_2 = 0$$

We get the first solution:

$$e^{t^4}$$

We get the second solution:

$$1$$

The complete solution:

$$x(t) = c_1 \cdot e^{t^4} + c_2$$

$$\dot{x}(t) = t^4 \cdot c_1 \cdot e^{t^4}$$

$$\ddot{x}(t) = 4t^3 \cdot c_1 \cdot e^{t^4} + t^8 \cdot c_1 \cdot e^{t^4}$$

We set the initial conditions:

$$x(0) = 1, \dot{x}(0) = 0$$

From the first condition we get:

$$x(t = 0) = 1 \rightarrow c_2 = 1 - c_1$$

From the second condition we get:

$$\dot{x}(t = 0) = 0 \rightarrow c_1 = 0$$

$$c_2 = 1$$

Result:

$$x(t) = 1$$

Numeric example 17:

$$\ddot{x} + t^2 = 0$$

We get the characteristic equation:

$$r^2 + t^2 = 0$$

$$r_{1/2} = \frac{\pm\sqrt{-4t^2}}{2} = \frac{\pm i\sqrt{4t^2}}{2} = \pm it$$

We have:

$$r_1 = it$$

$$r_2 = -it$$

The complete solution:

$$x(t) = c_1 \cdot e^{it} + c_2 \cdot e^{-it}$$

$$\dot{x}(t) = t\left(c_1 \cdot e^{it} - c_2 \cdot e^{-it}\right)$$

We set the initial condition:

$$x(0) = 6$$

From the first condition we get:

$$x(t = 0) = c_1 + c_2 = 6$$

$$c_2 = 6 - c_1$$

Result:

$$x(t) = c_1 \cdot e^{it} + 6 \cdot e^{-it} - c_1 \cdot e^{-it} =$$

$$c_1 \cdot \left(e^{it} - e^{-it}\right) + 6 \cdot e^{-it}$$

Damping

We have the ode:

$$\frac{\ddot{x}}{2} + b\dot{x} + \frac{x}{2} = 0$$

$$\ddot{x} + 2b\dot{x} + x = 0$$

We get the characteristic equation:

$$r^2 + 2br + 1 = 0$$

$$r_{1/2} = \frac{-2b \pm \sqrt{4b^2 - 4}}{2} = -b \pm \sqrt{b^2 - 1}$$

We have:

$$r_1 = -b + \sqrt{b^2 - 1}$$

$$r_2 = -b - \sqrt{b^2 - 1}$$

Case 1: $b < 1$

$$r_1 = -b + i\sqrt{1 - b^2}$$

$$r_2 = -b - i\sqrt{1 - b^2}$$

Solution:

$$x(t) = e^{-bt} \cdot \left(c_1 \cdot e^{i\sqrt{1-b^2}t} + c_2 \cdot e^{-i\sqrt{1-b^2}t}\right)$$

$$\dot{x}(t) = -b \cdot e^{-bt} \cdot \left(c_1 \cdot e^{i\sqrt{1-b^2}t} + c_2 \cdot e^{-i\sqrt{1-b^2}t}\right) + e^{-bt} \cdot i \cdot \sqrt{1 - b^2}$$
$$\cdot \left(c_1 \cdot e^{i\sqrt{1-b^2}t} - c_2 \cdot e^{-i\sqrt{1-b^2}t}\right) =$$

$$e^{-bt} \cdot \left(-b \cdot \left(c_1 \cdot e^{i\sqrt{1-b^2}t} + c_2 \cdot e^{-i\sqrt{1-b^2}t}\right) + i \cdot \sqrt{1 - b^2} \cdot \left(c_1 \cdot e^{i\sqrt{1-b^2}t} - c_2 \cdot e^{-i\sqrt{1-b^2}t}\right)\right)$$

We set the initial conditions:

$$x(0) = 1, \dot{x}(0) = -1$$

From the first condition $x(0) = 1$:

$$x(t = 0) = c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

From the second condition $\dot{x}(0) = -1$:

$$\dot{x}(t = 0) = -b \cdot (c_1 + c_2) + i \cdot \sqrt{1 - b^2} \cdot (c_1 - c_2) = -1$$

$$1 - b + i \cdot \sqrt{1 - b^2} \cdot (2c_1 - 1) = 0$$

$$1 - b + 2 \cdot c_1 \cdot i \cdot \sqrt{1 - b^2} - i \cdot \sqrt{1 - b^2} = 0$$

$$2 \cdot c_1 \cdot i \cdot \sqrt{1 - b^2} = b - 1 + i \cdot \sqrt{1 - b^2}$$

$$c_1 = \frac{b - 1 + i \cdot \sqrt{1 - b^2}}{2 \cdot i \cdot \sqrt{1 - b^2}} =$$

$$\frac{b - 1}{2 \cdot i \cdot \sqrt{1 - b^2}} + \frac{i \cdot \sqrt{1 - b^2}}{2 \cdot i \cdot \sqrt{1 - b^2}} =$$

$$\frac{b - 1}{2 \cdot i \cdot \sqrt{1 - b^2}} + \frac{1}{2} =$$

$$-\frac{1 - b}{2 \cdot i \cdot \sqrt{1 - b^2}} + \frac{1}{2} =$$

$$-\frac{1 - b}{2 \cdot i \cdot \sqrt{1 - b} \cdot \sqrt{1 + b}} + \frac{1}{2}$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}}$$

$$c_2 = \frac{1}{2} + \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}}$$

We get the solution:

$$x(t) = e^{-bt} \cdot \left(\left(\frac{1}{2} - \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \right) \cdot e^{i\sqrt{1-b^2}t} + \left(\frac{1}{2} + \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \right) \cdot e^{-i\sqrt{1-b^2}t} \right) =$$

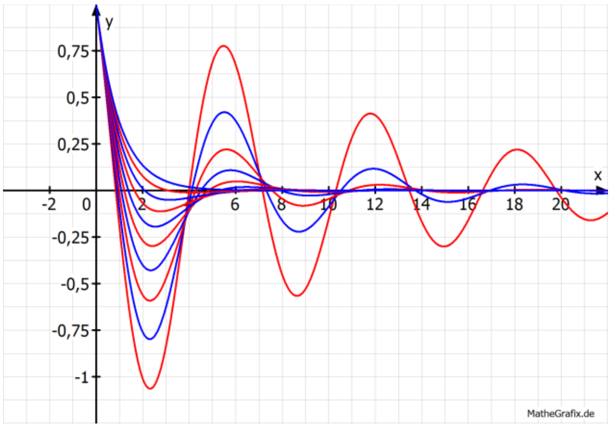
$$e^{-bt} \cdot \left(\frac{1}{2} \cdot e^{i\sqrt{1-b^2}t} - \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \cdot e^{i\sqrt{1-b^2}t} + \frac{1}{2} \cdot e^{-i\sqrt{1-b^2}t} + \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \cdot e^{-i\sqrt{1-b^2}t} \right) =$$

$$e^{-bt} \cdot \left(\frac{1}{2} \cdot e^{i\sqrt{1-b^2}t} + \frac{1}{2} \cdot e^{-i\sqrt{1-b^2}t} - \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \cdot e^{i\sqrt{1-b^2}t} + \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \cdot e^{-i\sqrt{1-b^2}t} \right) =$$

$$e^{-bt} \cdot \left(\frac{1}{2} \cdot \left(e^{i\sqrt{1-b^2}t} + e^{-i\sqrt{1-b^2}t} \right) - \frac{\sqrt{1 - b}}{2 \cdot i \cdot \sqrt{1 + b}} \cdot \left(e^{i\sqrt{1-b^2}t} - e^{-i\sqrt{1-b^2}t} \right) \right) =$$

$$e^{-bt} \cdot \left(cos\left(\sqrt{1 - b^2}t \right) - \sqrt{\frac{1 - b}{1 + b}} \cdot sin\left(\sqrt{1 - b^2}t \right) \right)$$

We plot:



Case 1: $b = 1$

$$r_{12} = -1$$

Solution:

$$x(t) = c_1 \cdot e^{-t} + c_2 \cdot t \cdot e^{-t}$$

$$\dot{x}(t) = -c_1 \cdot e^{-t} + c_2 \cdot e^{-t} - c_2 \cdot t \cdot e^{-t} =$$

$$e^{-t}(-c_1 + c_2 - c_2 \cdot t) =$$

$$e^{-t}\big(-c_1 + c_2(1 - t)\big)$$

We set the initial conditions:

$$x(0) = 1, \dot{x}(0) = -1$$

From the first condition $x(0) = 1$:

$$x(t = 0) = c_1 = 1$$

From the second condition $\dot{x}(0) = -1$:

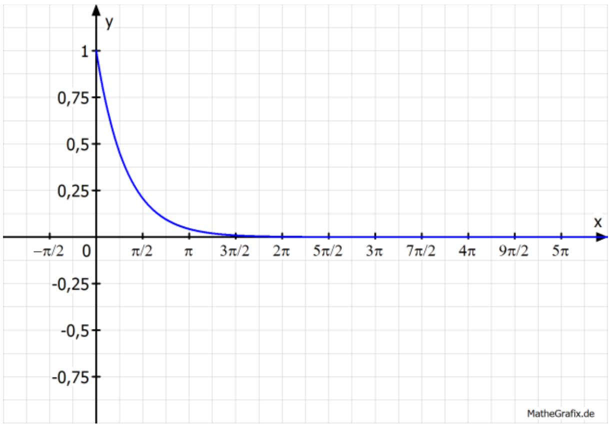
$$\dot{x}(t = 0) = c_2 - c_1 = -1$$

$$c_2 = c_1 - 1 = 0$$

We get the solution:

$$x(t) = e^{-t}$$

We plot:



Case 1: $b > 1$

$$r_1 = -b + \sqrt{b^2 - 1}$$

$$r_2 = -b - \sqrt{b^2 - 1}$$

Solution:

$$x(t) = e^{-bt} \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} + c_2 \cdot e^{-\sqrt{b^2-1}t} \right)$$

$$\dot{x}(t) = -b \cdot e^{-bt} \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} + c_2 \cdot e^{-\sqrt{b^2-1}t} \right) + e^{-bt} \cdot \sqrt{b^2-1} \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} - c_2 \cdot e^{-\sqrt{b^2-1}t} \right) =$$

$$e^{-bt} \cdot \left(-b \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} + c_2 \cdot e^{-\sqrt{b^2-1}t} \right) + \sqrt{b^2-1} \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} - c_2 \cdot e^{-\sqrt{b^2-1}t} \right) \right) =$$

$$e^{-bt} \cdot \left(-b \cdot c_1 \cdot e^{\sqrt{b^2-1}t} - b \cdot c_2 \cdot e^{-\sqrt{b^2-1}t} + \sqrt{b^2-1} \cdot c_1 \cdot e^{\sqrt{b^2-1}t} - \sqrt{b^2-1} \cdot c_2 \cdot e^{-\sqrt{b^2-1}t} \right) =$$

$$e^{-bt} \cdot \left(c_1 \cdot e^{\sqrt{b^2-1}t} \left(\sqrt{b^2-1} - b \right) - c_2 \cdot e^{-\sqrt{b^2-1}t} \left(\sqrt{b^2-1} + b \right) \right)$$

We set the initial conditions:

$$x(0) = 1, \dot{x}(0) = -1$$

From the first condition $x(0) = 1$:

$$x(t = 0) = c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

From the second condition $\dot{x}(0) = -1$:

$$\dot{x}(t = 0) = -1$$

$$\left(c_1 \left(\sqrt{b^2-1} - b \right) - c_2 \left(\sqrt{b^2-1} + b \right) \right) = -1$$

$$\left(c_1 \left(\sqrt{b^2-1} - b \right) - (1 - c_1) \left(\sqrt{b^2-1} + b \right) \right) = -1$$

$$\left(c_1 \left(\sqrt{b^2-1} - b \right) - \left(\sqrt{b^2-1} + b \right) + c_1 \cdot \left(\sqrt{b^2-1} + b \right) \right) = -1$$

$$\left(c_1 \left(\sqrt{b^2-1} - b + \sqrt{b^2-1} + b \right) - \left(\sqrt{b^2-1} + b \right) \right) = -1$$

$$c_1\left(2\sqrt{b^2-1}\right)-\left(\sqrt{b^2-1}+b\right)=-1$$

$$c_1\left(2\sqrt{b^2-1}\right)=\sqrt{b^2-1}+b-1$$

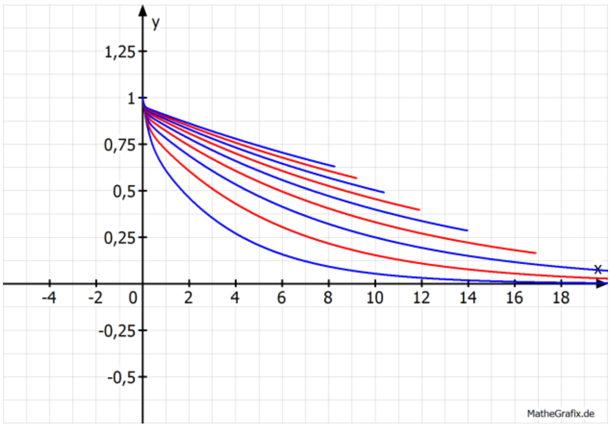
$$c_1=\frac{\sqrt{b^2-1}+b-1}{2\sqrt{b^2-1}}=\frac{1}{2}+\frac{b-1}{2\sqrt{b^2-1}}$$

$$c_2=\frac{1}{2}-\frac{b-1}{2\sqrt{b^2-1}}=$$

We get the solution:

$$x(t)=e^{-bt}\cdot\left(\left(\frac{1}{2}+\frac{b-1}{2\sqrt{b^2-1}}\right)\cdot e^{\sqrt{b^2-1}t}+\left(\frac{1}{2}-\frac{b-1}{2\sqrt{b^2-1}}\right)\cdot e^{-\sqrt{b^2-1}t}\right)$$

We plot:



The superposition principle

$$y^{(2)}+py^{(1)}+qy^{(0)}=0$$

We rewrite:

$$D^2y+pDy+qy=0$$

Note: D is the differentiation operator.

We "factor" out y:

$$(D^2+pD+q)y=0$$

Note: This is quantum mechanics style, D is an operator.

We call L a linear operator:

$$D^2+pD+q\rightarrow L$$

We get:

$$Ly=0$$

We use the picture:

$$u(x)\rightarrow[L]\rightarrow v(x)$$

L acts on functions and transform a function into a new function.

L is a linear operator:

$$L\big(u_1(x) + u_2(x)\big) = L\big(u_1(x)\big) + L\big(u_2(x)\big)$$

$$L\big(c \cdot u_1(x)\big) = c \cdot L\big(u_1(x)\big)$$

We proof the superposition principle.

We have:

$$L(y) = 0$$

We calculate:

$$L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2) =$$

$$c_1 L(y_1) + c_2 L(y_2) =$$

$$c_1 0 + c_2 0 = 0$$

Numeric example 18:

We have the differential equation:

$$2\ddot{y} + 3\dot{y} + 5y = 0$$

We know the characteristic polynomial:

$$2r^2 + 3r + 5 = 0$$

We rewrite this:

$$(2 \cdot D^2 + 3 \cdot D + 5)y = 0$$

A polynomial in D applied to y results in zero.

A polynomial differential operator:

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0$$

This is a linear time invariant operator:

If

$x(t)$ is a solution to $P(D)x = f(t)$

then

$y(t) = x(t - t_0)$ is a solution to $P(D)y = f(t - t_0)$.

Numeric example 19:

We have the differential equation:

$$\dot{y} = cos\left(t + \frac{\pi}{2}\right)$$

We solve by time invariance:

$$\dot{y} = cos(t) \rightarrow y(t) = sin(t)$$

$$\dot{y} = cos\left(t + \frac{\pi}{2}\right) \rightarrow y(t) = sin\left(t + \frac{\pi}{2}\right)$$

Note:

$$\cos\left(t+\frac{\pi}{2}\right) = -\sin(t)$$

$$\sin\left(t+\frac{\pi}{2}\right) = \cos(t)$$

$$\cos\left(t-\frac{\pi}{2}\right) = \sin(t)$$

$$\sin\left(t-\frac{\pi}{2}\right) = -\cos(t)$$

Numeric example 20:

We have the differential equation:

$$\dot{x} + x = \cos(t)$$

General solution:

$$x(t) = \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) + c_1 e^{-t}$$

We search for the solution for:

$$\dot{y} + y = \sin(t)$$

We solve by time invariance:

$$\sin(t) = \cos\left(t-\frac{\pi}{2}\right)$$

$$y(t) = x\left(t-\frac{\pi}{2}\right) =$$

$$\frac{1}{2}\cos\left(t-\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(t-\frac{\pi}{2}\right) + c_1 e^{-t-\frac{\pi}{2}} =$$

$$\frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) + c_2 e^{-t}$$

Note: $c_2 = c_1 e^{-\frac{\pi}{2}}$

Shortcut

We have:

$$(2D^2 + 3D + 5)e^{rt} =$$

$$2D^2 e^{rt} + 3De^{rt} + 5e^{rt} =$$

$$2rDe^{rt} + 3re^{rt} + 5e^{rt} =$$

$$2r^2 e^{rt} + 3re^{rt} + 5e^{rt} =$$

$$(2r^2 + 3r + 5)e^{rt}$$

We get the theorem:

For any polynomial P and any number r holds:

$$P(D)e^{rt} = P(r)e^{rt}$$

Particular solution

We want to get the solution to the inhomogeneous ode:

$$P(D)y = e^{rt}$$

We use:

$$P(D)e^{rt} = P(r)e^{rt}$$

We divide by $P(r)$:

$$P(D)\left(\frac{e^{rt}}{P(r)}\right) = e^{rt}$$

What we did was: We applied the differential equation onto the inhomogeneous part and got the characteristic polynomial in r. Then we divide the inhomogeneous part by this polynomial in r.

This is called the exponential response formula ERF.

For any polynomial P and any number r such that $P(r) \neq 0$ holds:

$$\frac{e^{rt}}{P(r)}$$

is a particular solution to $P(D)y = e^{rt}$.

Numeric example 21:

We have the ode:

$$\ddot{y} + 7\dot{y} + 12y = -5e^{2t}$$

We solve the homogeneous part:

$$\ddot{y} + 7\dot{y} + 12y = 0$$

We get the characteristic polynomial:

$$r^2 + 7r + 12 = 0$$

Solution for the characteristic polynomial:

$$r_{12} = \frac{-7 \pm \sqrt{49 - 48}}{2} = -\frac{7}{2} \pm \frac{1}{2}$$

Solution for the homogeneous ode:

$$y(t) = c_1 e^{-3t} + c_2 e^{-4t}$$

We assemble the particular solution and use $r = 2$:

$$\frac{-5e^{2t}}{4 + 14 + 12} = -\frac{e^{-2t}}{6}$$

The complete set of solutions:

$$-\frac{1}{6}e^{-2t} + c_1 e^{-3t} + c_2 e^{-4t}$$

Numeric example 22:

We have the ode:

$$\ddot{x} + x = e^{2it}$$

We rewrite in operator notation:

$$(D^2 + 1)x = e^{2it}$$

The corresponding characteristic polynomial (for the homogeneous ode):

$$r^2 + 1 = 0$$

We get the roots:

$$r_1 = i, r_2 = -i$$

The solution to the homogeneous ode:

$$x(t) = c_1 e^{it} + c_2 e^{-it}$$

One particular solution to the inhomogeneous ode:

$$x(t) = \frac{e^{rt}}{r^2 + 1} \to \frac{e^{2it}}{(2i)^2 + 1} = \frac{e^{2it}}{1 - 4} = -\frac{e^{2it}}{3}$$

The complete solution:

$$x(t) = -\frac{e^{2it}}{3} + c_1 e^{it} + c_2 e^{-it}$$

Numeric example 23:

We have the ode:

$$\ddot{x} + x = e^{-t}$$

We rewrite in operator notation:

$$(D^2 + 1)x = e^{-t}$$

The corresponding characteristic polynomial (for the homogeneous ode):

$$r^2 + 1 = 0$$

We get the roots:

$$r_1 = i, r_2 = -i$$

The solution to the homogeneous ode:

$$x(t) = c_1 e^{it} + c_2 e^{-it}$$

One particular solution to the inhomogeneous ode:

$$x(t) = \frac{e^{rt}}{r^2 + 1} \to \frac{e^{-t}}{(-1)^2 + 1} = \frac{e^{-t}}{2}$$

The complete solution:

$$x(t) = \frac{e^{-t}}{2} + c_1 e^{it} + c_2 e^{-it}$$

Numeric example 24:

We have the ode:

$$\ddot{x} + x = e^{-t} - 3e^{2it}$$

We rewrite in operator notation:

$$(D^2 + 1)x = e^{-t} - 3e^{2it}$$

The corresponding characteristic polynomial (for the homogeneous ode):

$$r^2 + 1 = 0$$

We get the roots:

$$r_1 = i, r_2 = -i$$

The solution to the homogeneous ode:

$$x(t) = c_1 e^{it} + c_2 e^{-it}$$

We split the inhomogeneous version into:

$$(I)\ \ddot{x} + x = e^{-t}$$

$$(II)\ \ddot{x} + x = -3e^{2it}$$

One particular solution to the inhomogeneous ode (I):

$$x(t) = \frac{e^{rt}}{r^2 + 1} \rightarrow \frac{e^{-t}}{(-1)^2 + 1} = \frac{e^{-t}}{2}$$

One particular solution to the inhomogeneous ode (II):

$$x(t) = \frac{-3e^{rt}}{r^2 + 1} \rightarrow \frac{-3e^{2it}}{(2i)^2 + 1} = \frac{-3e^{2it}}{1 - 4} = e^{2it}$$

The complete solution by superposition:

$$x(t) = c_1 \frac{e^{-t}}{2} + c_2 e^{2it} + c_3 e^{it} + c_4 e^{-it}$$

The ERF does not work if we have $r^2 + 1 = 0$. This is the case for e^{it}.

We generalize.

Suppose P is a polynomial and $P(r_0) = 0$. We check whether $P'(r_0) \neq 0$ for some value of r_0.

If this is the case ten we have a particular solution:

$$x_p = \frac{1}{P'(r_0)} t e^{r_0 t}$$

Numeric example 25:

We have the ode:

$$\ddot{x} - 4x = e^{-2t}$$

We solve the homogeneous part:

$$\ddot{x} - 4x = 0$$

We get the characteristic polynomial:

$$P(r) = r^2 - 4 = 0$$

Solution for the characteristic polynomial:

$$r_{12} = \pm 2$$

Solution for the homogeneous ode:

$$x(t) = c_1 e^{2t} + c_2 e^{-t}$$

We assemble the particular solution and use $r = 2$:

$$\frac{e^{-2t}}{0} = ?$$

We check:

$$P'(r) = 2r$$

$$P'(r = \pm 2) = \pm 4 \neq 0$$

We get the particular solution:

$$x_p = \frac{1}{4} t e^{2t}$$

The complete set of solutions:

$$\frac{1}{4} t e^{2t} + c_1 e^{-3t} + c_2 e^{-4t}$$

Generalization exponential response formula erf

If the characteristic polynomial becomes zero, search for the first derivative where it doesn't become zero. We assemble the solution via:

$$y_p = \frac{1}{P^{(m)}(r_0)} \cdot t^m \cdot e^{r_0 t}$$

Note the corresponding $P^{(m)}$ and t^m.

Numeric example 26:

We have the ode:

$$\ddot{x} - 4x = e^{-2t}$$

We solve the homogeneous part:

$$\ddot{x} - 4x = 0$$

We get the characteristic polynomial:

$$P(r) = r^2 - 4 = 0$$

Solution for the characteristic polynomial:

$$r_{12} = \pm 2$$

Solution for the homogeneous ode:

$$x(t) = c_1 e^{2t} + c_2 e^{-t}$$

We assemble the particular solution and use $r = 2$:

$$\frac{e^{-2t}}{0} = ?$$

We check:

$$P'(r) = 2r$$

$$P'(r = \pm 2) = \pm 4 \neq 0$$

We get the particular solution:

$$x_p = \frac{1}{4} t e^{2t}$$

The complete set of solutions:

$$\frac{1}{4} t e^{2t} + c_1 e^{-3t} + c_2 e^{-4t}$$

Proof of generalized exponential response formula erf

We want to solve:

$$P(D)y = e^{rt}$$

We know that:

$$P(D)e^{r_0 t} = P(r_0)e^{r_0 t}$$

Note: This says that we have one particular solution of the ode:

$$P(D)\left(\frac{e^{r_0 t}}{P(r_0)}\right) = e^{r_0 t}$$

We work with the case $P(r_0) = 0$ so we can't divide by $P(r_0)$.

We check the surrounding of r_0 and take a look at the derivatives:

$$P(D)e^{rt} = P(r)e^{rt}$$

$$\frac{\partial}{\partial r}\left(P(D)e^{rt}\right) = \frac{\partial}{\partial r}\left(P(r)e^{rt}\right)$$

The polynomial in D is independent of r:

$$P(D)\left(\frac{\partial}{\partial r}(e^{rt})\right) = \left(\frac{d}{dr}P(r)\right)e^{rt} + P(r)\frac{\partial}{\partial r}(e^{rt}) =$$

$$\left(\frac{d}{dr}P(r)\right)e^{rt} + P(r) \cdot t \cdot e^{rt}$$

We derivate repeatedly until we get one derivative $\neq 0$:

$$\left(\frac{d}{dr}P(r)\right)e^{rt} + P(r)\cdot t\cdot e^{rt}\ may\ be\ zero$$

Note: $P(r = r_0) = 0$.

$$\left(\frac{d^2}{dr^2}P(r)\right)e^{rt} + P(r)\cdot t^2\cdot e^{rt}\ may\ be\ zero$$

$$\dots$$

$$\left(\frac{d^m}{dr^m}P(r)\right)e^{rt} + P(r)\cdot t^m\cdot e^{rt}\ may\ be\ not\ zero$$

Because of:

$$P(r = r_0)\cdot t^m\cdot e^{rt} = 0$$

We get for derivative m:

$$\left(\frac{d^m}{dr^m}P(r)\right)e^{rt} \neq 0$$

We combine:

$$\frac{\partial^m}{\partial r^m}(P(D)e^{rt}) = \left(\frac{d^m}{dr^m}P(r)\right)e^{rt} + P(r)\cdot t^m\cdot e^{rt} \rightarrow$$

$$\frac{\partial^m}{\partial r^m}(P(D)e^{rt}) = \left(\frac{d^m}{dr^m}P(r)\right)e^{rt} \rightarrow$$

$$P(D)\cdot t^m\cdot e^{rt} = \left(\frac{d^m}{dr^m}P(r)\right)e^{rt} \rightarrow$$

$$P(D)\cdot\left(\frac{t^m\cdot e^{r_0 t}}{\frac{d^m}{dr^m}P(r_0)}\right) = e^{r_0 t}$$

Numeric example 27:

We have the ode:

$$\ddot{x} + x = e^{it}$$

Initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

We rewrite in operator notation:

$$(D^2 + 1)x = e^{it}$$

The corresponding characteristic polynomial (for the homogeneous ode):

$$r^2 + 1 = 0$$

We get the roots:

$$r_1 = i, r_2 = -i$$

The solution to the homogeneous ode:

$$x(t) = c_1 e^{it} + c_2 e^{-it}$$

One particular solution to the inhomogeneous ode:

$$x(t) = \frac{e^{rt}}{r^2 + 1} \rightarrow \frac{e^{it}}{(i)^2 + 1} = \frac{e^{it}}{0} \nexists$$

We derive:

$$\frac{d}{dr} P(r) = 2r \rightarrow \frac{d}{dr} P(r = i) = 2i$$

We get the particular solution:

$$\frac{t^m \cdot e^{r_0 t}}{\frac{d^m}{dr^m} P(r_0)} \rightarrow \frac{t \cdot e^{it}}{2i} = -\frac{1}{2} \cdot i \cdot t \cdot e^{it}$$

We check the particular solution:

$$\ddot{x} + x = e^{it}$$

$$x_p(t) = -\frac{1}{2} \cdot i \cdot t \cdot e^{it}$$

$$\dot{x}_p(t) = -\frac{1}{2} \cdot i \cdot e^{it} - \frac{1}{2} \cdot i^2 \cdot t \cdot e^{it} =$$

$$= -\frac{1}{2} \cdot i \cdot e^{it} + \frac{1}{2} \cdot t \cdot e^{it}$$

$$\ddot{x}_p(t) = -\frac{1}{2} \cdot i^2 \cdot e^{it} + \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot i \cdot t \cdot e^{it} =$$

$$= \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot i \cdot t \cdot e^{it} =$$

$$= e^{it} + \frac{1}{2} \cdot i \cdot t \cdot e^{it}$$

$$\ddot{x} + x = e^{it} \rightarrow$$

$$e^{it} + \frac{1}{2} \cdot i \cdot t \cdot e^{it} - \frac{1}{2} \cdot i \cdot t \cdot e^{it} =$$

$$e^{it} + \frac{1}{2} \cdot i \cdot t \cdot e^{it} - \frac{1}{2} \cdot i \cdot t \cdot e^{it} = e^{it}$$

The complete solution:

$$x(t) = -\frac{1}{2} \cdot i \cdot t \cdot e^{it} + c_1 e^{it} + c_2 e^{-it}$$

$$\dot{x}(t) = -\frac{1}{2} \cdot i \cdot e^{it} + \frac{1}{2} \cdot t \cdot e^{it} + c_1 \cdot ie^{it} - c_2 \cdot ie^{-it}$$

We apply the initial conditions:

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

$$x(0) = c_1 + c_2 = 1 \rightarrow c_2 = 1 - c_1$$

$$\dot{x}(0) = 0 \rightarrow$$

$$-\frac{1}{2} \cdot i \cdot e^{it} + \frac{1}{2} \cdot t \cdot e^{it} + c_1 \cdot ie^{it} - c_2 \cdot ie^{-it} = 0$$

$$-\frac{1}{2} \cdot i + c_1 \cdot i - c_2 \cdot i = 0$$

$$-\frac{1}{2} \cdot i + c_1 \cdot i - (1 - c_1) \cdot i = 0$$

$$-\frac{1}{2} \cdot i + c_1 \cdot i - i + c_1 \cdot i = 0$$

$$-\frac{3}{2} \cdot i + 2 \cdot c_1 \cdot i = 0$$

$$2 \cdot c_1 \cdot i = \frac{3}{2} \cdot i$$

$$c_1 = \frac{3}{4} \rightarrow c_2 = \frac{1}{4}$$

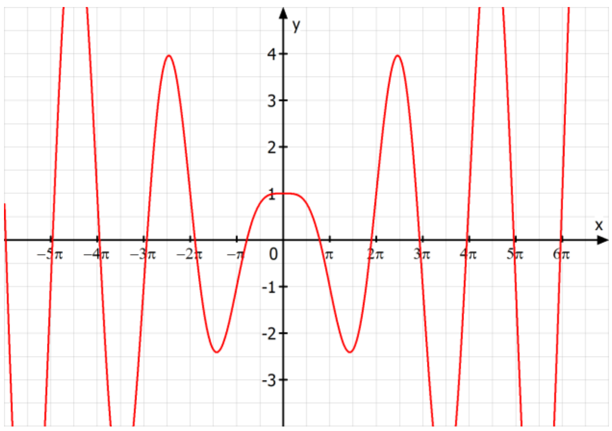
Solution:

$$x(t) = -\frac{1}{2} \cdot i \cdot t \cdot e^{it} + \frac{3}{4} e^{it} + \frac{1}{4} e^{-it}$$

We plot the real part:

$$Re\big(x(t)\big) = \frac{1}{2} \cdot t \cdot sin(t) + \frac{3}{4} \cdot cos(t) + \frac{1}{4} \cdot cos(t) =$$

$$\frac{1}{2} \cdot t \cdot sin(t) + cos(t)$$



Triple example 28:

We want to solve:

$$a)\ \dot{x} + kx = 1$$

$$b)\ \dot{x} + kx = e^{-5t}$$

$$c)\ \dot{x} + kx = 4 + 7e^{-5t}$$

We begin with a):

We calculate the homogeneous solution.

$$\dot{x} + kx = 0$$

Characteristic polynomial:

$$r + k = 0 \rightarrow r = -k$$

Homogeneous solution:

$$x_h(t) = c_1 e^{-kt}$$

The inhomogeneous ode:

$$\dot{x} + kx = 1$$

We guess:

$$x_p(t) = \frac{1}{k}$$

We try:

$$\dot{x} + kx = 1 \rightarrow 0 + \frac{k}{k} = 1 \surd$$

The complete solution:

$$x(t) = c_1 e^{-kt} + \frac{1}{k}$$

Note: This is not the only possibility. We rewrite the ode:

$$\dot{x} + kx = 1 \rightarrow \dot{x} + kx = e^{0t}$$

We can now use ERF and get the particular solution:

$$x_p(t) = \frac{e^{0t}}{P(0)} = \frac{e^{0t}}{k}$$

This is the same result we got with our trial.

We work with b):

$$\dot{x} + kx = e^{-5t}$$

We already know the homogeneous solution:

$$x(t) = c_1 e^{-kt}$$

We use ERF and get the particular solution:

$$x_p(t) = \frac{e^{-5t}}{P(-5)} = \frac{e^{-5t}}{k-5}$$

We get the particular solution:

$$x_p = \frac{e^{-5t}}{k-5}$$

$$\dot{x}_p = -5\frac{e^{-5t}}{k-5}$$

The inhomogeneous ode:

$$\dot{x} + kx = e^{-5t}$$

We check:

$$-5\frac{e^{-5t}}{k-5} + k\frac{e^{-5t}}{k-5} = e^{-5t}$$

$$(k-5)\frac{e^{-5t}}{k-5} = e^{-5t} \surd$$

The complete solution:

$$x(t) = c_1 e^{-kt} + \frac{e^{-5t}}{k-5}$$

Note: This solution is valid for $k \neq 5$.

In case $k = 5$ we use variation of parameters.

We set $k = 5$:

$$\dot{x} + 5x = e^{-5t}$$

We know the homogeneous solution:

$$x_h(t) = c_1 e^{-5t}$$

We replace c_1 by $u(t)$:

$$x_p(t) = u(t)e^{-5t}$$

We insert this into the ode:

$$\frac{d}{dt}\left(u(t)e^{-5t}\right) + 5u(t)e^{-5t} = e^{-5t}$$

$$\frac{\dot{d}}{dt}\big(u(t)\big)e^{-5t} - 5u(t)e^{-5t} + 5u(t)e^{-5t} = e^{-5t}$$

$$\frac{\dot{d}}{dt}\big(u(t)\big)e^{-5t} = e^{-5t}$$

$$\frac{\dot{d}}{dt}\big(u(t)\big) = 1$$

$$u(t) = t + c_2$$

We get the complete solution for the case $k = 5$:

$$x(t) = (t + c_2)e^{-5t}$$

We note that this solution contains

We work with $c)$:

$$\dot{x} + kx = 4 + 7e^{-5t}$$

We see that this contains the odes $a)$ and $b)$ so we can get a particular solution to this ode by linear superposition:

$$x_p(t) = 4 \cdot a) + 7 \cdot b) =$$

Case $k \neq 5$	Case $k = 5$
$4\left(c_1 e^{-kt} + \frac{1}{k}\right) + 7\left(c_2 e^{-kt} + \frac{e^{-5t}}{k-5}\right)$	$4\left(c_1 e^{-kt} + \frac{1}{k}\right) + 7(t + c_2)e^{-5t}$

For the general solution we need to add the homogeneous solution and get:

Case $k \neq 5$	Case $k = 5$
$4\left(c_1 e^{-kt} + \frac{1}{k}\right) + 7\left(c_2 e^{-kt} + \frac{e^{-5t}}{k-5}\right) + c_3 e^{-kt}$	$4\left(c_1 e^{-kt} + \frac{1}{k}\right) + 7(t + c_2)e^{-5t} + c_3 e^{-kt}$

Numeric example 29:

We use the homogeneous ode:

$$y^{(3)} - 10y^{(2)} + 31y^{(1)} - 30y^{(0)} = 0$$

We write in operator style:

$$P(D)y = 0$$

We get the characteristic polynomial:

$$P(r) = r^3 - 10r^2 + 31r^1 - 30r^0 = 0$$

We get the roots:

$$(r-2)(r-3)(r-5)$$

We get three solutions:

$$e^{2t}, e^{3t}, e^{5t}$$

We know:

$$e^{2t}: P(D)e^{2t} = P(2)e^{2t} = 0e^{2t} = 0$$

$$e^{3t}: P(D)e^{3t} = P(3)e^{3t} = 0e^{3t} = 0$$

$$e^{5t}: P(D)e^{5t} = P(5)e^{5t} = 0e^{5t} = 0$$

In order to know that the three solutions form a complete basis we must check whether they are linearly independent.

We assume:

$$ae^{2t} + b\, e^{3t} = e^{5t}$$

We apply $(D-2)(D-3)$ to both sides of the equation:

$$(D-2)(D-3)(ae^{2t}+b\,e^{3t})=(D-2)(D-3)e^{5t}$$

Left side, using the property of the polynomials above:

$$(D-2)(D-3)ae^{2t}+(D-2)(D-3)b\,e^{3t}=$$

$$(D-3)(D-2)ae^{2t}+(D-2)(D-3)b\,e^{3t}=$$

$$(2-3)(2-2)ae^{2t}+(2-2)(2-3)b\,e^{3t}=$$

$$0+0=0$$

Right side:

$$(D-2)(D-3)e^{5t}=(5-2)(5-3)e^{5t}\neq 0$$

Numeric example 30:

We use the homogeneous ode:

$$D^3y=0$$

We get the characteristic polynomial:

$$P(r)=r^3=0$$

We get the triple root:

$$r=0$$

We integrate three times:

$$D^3y=0\rightarrow$$

$$D^2y=c_1e^{0t}$$

$$D^1y=tc_2e^{0t}$$

$$D^0y=t^2\frac{c'_3}{2}\rightarrow t^2c_3e^{0t}$$

We get complete solution:

$$y(t)=c_1e^{0t}+tc_2e^{0t}+t^2c_3e^{0t}$$

Numeric example 31:

We use the homogeneous ode:

$$(D-5)^3y=0$$

We get the characteristic polynomial:

$$P(r)=(r-5)^3=0$$

We get the triple root:

$$r=5$$

We integrate three times:

$$(D-5)^3y=0\rightarrow$$

$$(D-5)^2 y = c_1 e^{5t}$$

$$(D-5)^1 y = c_2 t e^{5t}$$

$$(D-5)^0 y = t^2 \frac{c'_3}{2} e^{5t} \rightarrow c_3 t^2 e^{5t}$$

We get complete solution:

$$y(t) = c_1 e^{5t} + c_2 t e^{5t} + c_3 t^2 e^{5t}$$

Numeric example 32:

We use the ode:

$$2\ddot{x} + \dot{x} + x = 1 + 2e^t$$

Note: We can write the right side as:

$$2\ddot{x} + \dot{x} + x = e^{0t} + 2e^t$$

We work with the homogeneous ode:

$$2\ddot{x} + \dot{x} + x = 0$$

We get the characteristic polynomial:

$$P(r) = 2r^2 + r + 1 = 0$$

$$r_{12} = \frac{-1 \pm \sqrt{1-8}}{4} = -\frac{1}{4} \pm i\sqrt{\frac{7}{16}}$$

We get the roots:

$$r_1 = -\frac{1}{4} + i\sqrt{\frac{7}{16}}$$

$$r_2 = -\frac{1}{4} - i\sqrt{\frac{7}{16}}$$

We get solution of the homogeneous ode:

$$x_h(t) = c_1 e^{\left(-\frac{1}{4}+i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-\frac{1}{4}-i\sqrt{\frac{7}{16}}\right)t}$$

We assemble the particular solution and use $r = 1$:

$$x_p(t) = 1 + \frac{2e^t}{2+1+1} = 1 + \frac{2e^t}{4}$$

Note: We can write this as:

$$x_p(t) = \frac{e^{0t}}{0+0+1} + \frac{2e^t}{2+1+1} = 1 + \frac{2e^t}{4}$$

We check:

$$\dot{x}_p(t) = D\left(1 + \frac{e^t}{2}\right) = \frac{e^t}{2}$$

$$\ddot{x}_p(t) = D\left(\frac{e^t}{2}\right) = \frac{e^t}{2}$$

$$2\ddot{x} + \dot{x} + x = 1 + 2e^t \rightarrow$$

$$\frac{2e^t}{2} + \frac{e^t}{2} + 1 + \frac{e^t}{2} =$$

$$2e^t + 1 \surd$$

The complete set of solutions:

$$c_1 e^{\left(-\frac{1}{4}+i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-\frac{1}{4}-i\sqrt{\frac{7}{16}}\right)t} + 1 + \frac{e^t}{2}$$

We want to write the left part in sin/cos style.

$$c_1 e^{\left(-\frac{1}{4}+i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-\frac{1}{4}-i\sqrt{\frac{7}{16}}\right)t} =$$

$$c_1 e^{\left(-\frac{1}{4}\right)t} e^{\left(i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-\frac{1}{4}\right)t} e^{\left(-i\sqrt{\frac{7}{16}}\right)t} =$$

$$e^{\left(-\frac{1}{4}\right)t}\left(c_1 e^{\left(i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-i\sqrt{\frac{7}{16}}\right)t}\right) =$$

$$e^{\left(-\frac{1}{4}\right)t}\left(c_1 cos\left(\sqrt{\frac{7}{16}}t\right) + ic_1 sin\left(\sqrt{\frac{7}{16}}t\right) + c_2 cos\left(\sqrt{\frac{7}{16}}t\right) - ic_2 sin\left(\sqrt{\frac{7}{16}}t\right)\right) =$$

$$e^{\left(-\frac{1}{4}\right)t}\left(cos\left(\sqrt{\frac{7}{16}}t\right)(c_1 + c_2) + isin\left(\sqrt{\frac{7}{16}}t\right)(c_1 - c_2)\right)$$

Depending on the initial conditions this will give something like:

$$Ae^{\left(-\frac{1}{4}\right)t}\left(cos\left(\sqrt{\frac{7}{16}}t - \phi\right)\right)$$

Numeric example 33:

We use the ode:

$$\ddot{x} + 8\dot{x} + 15x = e^{-5t}$$

We work with the homogeneous ode:

$$\ddot{x} + 8\dot{x} + 15x = 0$$

We get the characteristic polynomial:

$$P(r) = r^2 + 8r + 15 = 0$$

$$r_{12} = \frac{-8 \pm \sqrt{64 - 60}}{2} = -4 \pm 1$$

We get the roots:

$$r_1 = -3$$

$$r_2 = -5$$

We get solution of the homogeneous ode:

$$x_h(t) = c_1 e^{-3t} + c_2 e^{-5t}$$

We assemble the particular solution and use $r = -5$:

$$x_p(t) = \frac{e^{-5t}}{25 - 40 + 15} = \frac{e^{-5t}}{0}$$

This does not work.

We have $P(r_0) = 0$. We check whether $P'(r_0) \neq 0$ for some value of r_0.

If this is the case, then we have a particular solution:

$$P(r) = r^2 + 8r + 15$$

$$P'(r) = 2r + 8$$

$$P'(r = -5) = -10 + 8 = -2 \neq 0$$

If the characteristic polynomial becomes zero, search for the first derivative where it doesn't become zero. Assemble the solution via:

$$x_p = -\frac{1}{2} \cdot t \cdot e^{-5t}$$

We calculate:

$$\dot{x}_p(t) = D\left(-\frac{1}{2} \cdot t \cdot e^{-5t}\right) = -\frac{1}{2} \cdot e^{-5t} + \frac{5}{2} \cdot t \cdot e^{-5t}$$

$$\ddot{x}_p(t) = D\left(-\frac{1}{2} \cdot e^{-5t} + \frac{5}{2} \cdot t \cdot e^{-5t}\right) = \frac{5}{2} \cdot e^{-5t} + \frac{5}{2} \cdot e^{-5t} - \frac{25}{2} \cdot t \cdot e^{-5t} =$$

$$= 5 \cdot e^{-5t} - \frac{25}{2} \cdot t \cdot e^{-5t}$$

We check:

$$\ddot{x} + 8\dot{x} + 15x = e^{-5t} \rightarrow$$

$$5 \cdot e^{-5t} - \frac{25}{2} \cdot t \cdot e^{-5t} + 8\left(-\frac{1}{2} \cdot e^{-5t} + \frac{5}{2} \cdot t \cdot e^{-5t}\right) + 15\left(-\frac{1}{2} \cdot t \cdot e^{-5t}\right) =$$

$$5 \cdot e^{-5t} - \frac{25}{2} \cdot t \cdot e^{-5t} - 4 \cdot e^{-5t} + 20 \cdot t \cdot e^{-5t} - \frac{15}{2} \cdot t \cdot e^{-5t} =$$

$$5 \cdot e^{-5t} - 4 \cdot e^{-5t} = e^{-5t}$$

The complete set of solutions:

$$x(t) = c_1 e^{-3t} + c_2 e^{-5t} - \frac{1}{2} \cdot t \cdot e^{-5t}$$

Comparison of exponential response formula and variation of parameters

ERF	Variation of parameters
n-th order DEs	1^{st} order DEs
Constant coefficients	Any linear DE
Exponential input	Any input

Operator notations

We have:

$$(D^3 - D)(x + y) =$$

$$D^3(x + y) - D(x + y) =$$

$$D^3x + D^3y - Dx - Dy =$$

$$D^3x - Dx + D^3y - Dy$$

$$(D^3 - D)x + (D^3 - D)y$$

Numeric example 34:

We use the ode:

$$x^{(3)} - x^{(1)} = e^{2t}$$

We write in operator style:

$$(D^3 - D)x = e^{2t}$$

We work with the homogeneous ode:

$$x^{(3)} - x^{(1)} = 0$$

We get the characteristic polynomial:

$$P(r) = r^3 - r = r(r - 1)(r + 1) = 0$$

We get three solutions:

$$e^{0t}, e^t, e^{-t}$$

$$x_h(t) = c_1 + c_2e^t + c_3e^{-t}$$

We assemble the particular solution and use $r = 2$:

$$x_p(t) = \frac{e^{2t}}{8 - 2} = \frac{e^{2t}}{6}$$

We calculate:

$$x^{(1)}{}_p(t) = D\left(\frac{1}{6}e^{2t}\right) = \frac{1}{3}e^{2t}$$

$$x^{(2)}{}_p(t) = D\left(\frac{1}{3}e^{2t}\right) = \frac{2}{3}e^{2t}$$

$$x^{(3)}{}_p(t) = D\left(\frac{2}{3}e^{2t}\right) = \frac{4}{3}e^{2t}$$

We check:

$$x^{(3)} - x^{(1)} = e^{2t} \rightarrow$$

$$\frac{4}{3}e^{2t} - \frac{1}{3}e^{2t} = e^{2t}\ \surd$$

The complete set of solutions:

$$x = c_1 + c_2 e^t + c_3 e^{-t} + e^{2t}$$

Numeric example 35:

We use the ode:

$$x^{(3)} - x^{(1)} = e^{-t}$$

We write in operator style:

$$(D^3 - D)x = e^{-t}$$

We work with the homogeneous ode and get the solution:

$$x_h(t) = c_1 + c_2 e^t + c_3 e^{-t}$$

We assemble the particular solution and use $r = -1$:

$$x_p(t) = \frac{e^{-t}}{(-1) - (-1)} = \frac{e^{-t}}{0}$$

This doesn't work.

We use the generalized ERF:

$$x_p(t) = \frac{te^{-t}}{3(-1)^2 - 1} = \frac{1}{2}te^{-t}$$

We calculate:

$$x^{(1)}{}_p(t) = D\left(\frac{1}{2}te^{-t}\right) = \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t}$$

$$x^{(2)}{}_p(t) = D\left(\frac{1}{2}e^{-t} - \frac{1}{2}te^{-t}\right) = -e^{-t} + \frac{1}{2}te^{-t}$$

$$x^{(3)}{}_p(t) = D\left(-e^{-t} + \frac{1}{2}te^{-t}\right) = \frac{3}{2}e^{-t} - \frac{1}{2}te^{-t}$$

We check:

$$x^{(3)} - x^{(1)} = e^{-t} \rightarrow$$

$$\frac{3}{2}e^{-t} - \frac{1}{2}te^{-t} - \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} = e^{-t}$$

The complete set of solutions:

$$x = c_1 + c_2 e^t + c_3 e^{-t} + \frac{1}{2}te^{-t}$$

Numeric example 36:

We use the ode:

$$x^{(3)} - x^{(1)} = 1$$

We write in operator style:

$$(D^3 - D)x = e^{-t}$$

We work with the homogeneous ode and get the solution:

$$x_h(t) = c_1 + c_2 e^t + c_3 e^{-t}$$

We guess a particular solution:

$$x_p(t) = -t$$

We calculate:

$$x^{(1)}{}_p(t) = -1$$

$$x^{(2)}{}_p(t) = 0$$

$$x^{(3)}{}_p(t) = 0$$

We check:

$$x^{(3)} - x^{(1)} = e^{-t} \rightarrow$$

$$0 - (-1) = 1$$

The complete set of solutions:

$$x = c_1 + c_2 e^t + c_3 e^{-t} - t$$

Exponential response formula

We use the ode:

$$\ddot{x} + 9x = 9 + e^{3t}$$

Note: We can write the right side as:

$$(D^2 + 9)x = 9e^{0t} + e^{3t}$$

We work with the homogeneous ode:

$$\ddot{x} + 9x = 0$$

We get the characteristic polynomial:

$$P(r) = r^2 + 9 = 0$$

$$r_{12} = \pm i\sqrt{3}$$

We get the roots:

$$r_1 = +i\sqrt{3}$$

$$r_2 = -i\sqrt{3}$$

We get solution of the homogeneous ode:

$$x_h(t) = c_1 e^{+i\sqrt{3}t} + c_2 e^{-i\sqrt{3}t}$$

We assemble the particular solution and use $r = 1$:

$$x_p(t) = \frac{9e^{0t}}{0+9} + \frac{e^{3t}}{9+9} = 1 + \frac{e^{3t}}{18}$$

We check:

$$\dot{x}_p(t) = D\left(1 + \frac{e^{3t}}{18}\right) = \frac{e^{3t}}{6}$$

$$\ddot{x}_p(t) = D\left(\frac{e^{3t}}{6}\right) = \frac{e^{3t}}{2}$$

$$\ddot{x} + 9x = 9 + e^{3t} \rightarrow$$

$$\frac{e^{3t}}{2} + 9\left(1 + \frac{e^{3t}}{18}\right) = \frac{e^{3t}}{2} + 9 + \frac{e^{3t}}{2} =$$

$$9 + e^{3t} \surd$$

The complete set of solutions:

$$c_1 e^{\left(-\frac{1}{4}+i\sqrt{\frac{7}{16}}\right)t} + c_2 e^{\left(-\frac{1}{4}-i\sqrt{\frac{7}{16}}\right)t} + 1 + \frac{e^{3t}}{18}$$

Higher order inhomogeneous differential equation

We use the homogeneous ode:

$$(D^4 - 1)x = e^{-2t}$$

We work with the homogeneous ode:

$$(D^4 - 1)x = 0$$

We get the characteristic polynomial:

$$P(r) = r^4 - 1 = 0$$

$$r^4 = 1$$

We get the roots:

$$1, -1, i, -i$$

Solution of the homogeneous ode:

$$x_h(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{it} + c_4 e^{-it}$$

We use ERF:

$$x_p(t) = \frac{e^{-2t}}{15}$$

We check:

$$x^{(1)}{}_p(t) = -\frac{2e^{-2t}}{15}$$

$$x^{(2)}{}_p(t) = \frac{4e^{-2t}}{15}$$

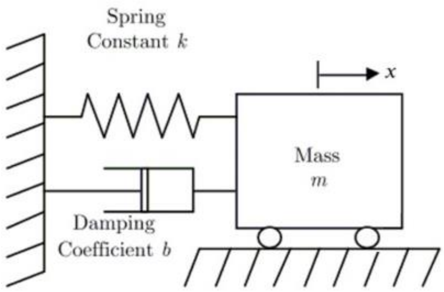
$$x^{(3)}{}_p(t) = -\frac{8e^{-2t}}{15}$$

$$x^{(4)}{}_p(t) = \frac{16e^{-2t}}{15}$$

$$x^{(4)}{}_p(t) - x_p(t) = e^{2t} \rightarrow$$

$$\frac{16e^{-2t}}{15} - \frac{e^{-2t}}{15} = e^{-2t} \surd$$

Review damped oscillators



The position is determined by the differential equation:

$$m\ddot{x} + b \cdot \dot{x} + k \cdot x = 0$$

We write this in operator style:

$$mD^2 + bD + kI$$

The characteristic polynomial:

$$p(r) = m \cdot r^2 + b \cdot r + k$$

The roots of the characteristic polynomial:

$$r_{12} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

We remember:

$$\omega_n = \sqrt{\frac{k}{m}}$$

Note: ω_n is the frequency of the undamped oscillator.

The exponential decay in case of damping:

$$-\frac{b}{2m}$$

The frequency of the damped oscillator that is smaller than the frequency of the undamped oscillator by $\left(\frac{b}{2m}\right)^2$:

$$\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

Advantages of complex replacement

$z := x_1 + ix_2$ is a solution to the complex replacement ode:

$$P(D)z = e^{i\omega t}$$

We can rewrite this as:

$$P(D)(x_1 + ix_2) = cos(\omega t) + isin(\omega t)$$

We get two solutions:

$$P(D)(x_1) = cos(\omega t)$$

$$P(D)(x_2) = sin(\omega t)$$

real input signal	complex replacement
$cos(\omega t)$	$e^{i\omega t}$
$Acos(\omega t - \phi)$	$Ae^{i(\omega t-\phi)}$
$e^{at}cos(\omega t)$	$e^{(a+i\omega)t}$
$acos(\omega t) + bsin(\omega t)$	$(a - ib)e^{i\omega t}$

Examples using complex replacement

Numeric example 37:

We have the ode:

$$\ddot{x} + 4x = cos(2t)$$

We replace:

$$\ddot{z} + 4z = e^{2it}$$

We get the characteristic polynomial:

$$P(r) = r^2 + 4 = 0$$

$$r_{12} = \pm 2i$$

We find that $P(r = 2i) = 0$

We derivate:

$$P'(r) = 2r$$

$$P'(r = 2i) \neq 0$$

We get:

$$z_p = \frac{te^{2it}}{4i} = -\frac{1}{4} \cdot i \cdot te^{2it} =$$

$$-\frac{1}{4}\cdot e^{\frac{1}{2}i\pi}\cdot te^{2it}=-\frac{1}{4}\cdot t\cdot e^{i\left(\frac{1}{2}\pi+2t\right)}$$

We have the solutions:

$$z_p=-\frac{1}{4}\cdot t\cdot cos\left(\frac{1}{2}\pi+2t\right)+i\cdot\frac{1}{4}\cdot t\cdot sin\left(\frac{1}{2}\pi+2t\right)$$

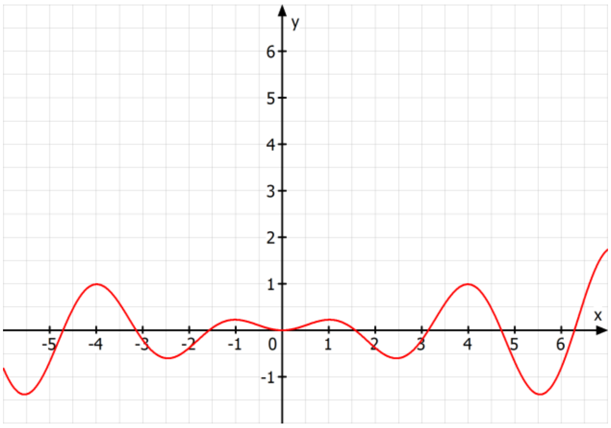
The real part of it:

$$x_p=-\frac{1}{4}\cdot t\cdot cos\left(\frac{1}{2}\pi+2t\right)$$

We remove the phase shift and get:

$$x_p=\frac{1}{4}\cdot t\cdot sin\left(\frac{1}{2}\pi+2t\right)$$

We plot:



Numeric example 38:

We have the ode:

$$\ddot{x}+\dot{x}+2x=cos(2t)$$

We replace:

$$\ddot{z}+\dot{z}+2z=e^{2it}$$

We get the characteristic polynomial:

$$P(r)=r^2+r+2=0$$

$$r_{12}=\frac{-1\pm\sqrt{1-8}}{2}=-\frac{1}{2}\pm i\sqrt{\frac{7}{4}}$$

We use erf to find a particular solution in z:

$$z_p=\frac{e^{i\omega t}}{P(i\omega)}\to\frac{e^{i2t}}{-2+2i}$$

We decompose:

$$z_p=\frac{e^{i2t}}{-2+2i}=\frac{(-2-2i)e^{i2t}}{(-2+2i)(-2-2i)}=$$

$$= \frac{(-2-2i)e^{i2t}}{8} = -\frac{(1+i)e^{i2t}}{4} =$$

$$= -\frac{e^{i2t}}{4} - \frac{ie^{i2t}}{4} = -\frac{e^{i2t}}{4} - \frac{e^{i\left(\frac{\pi}{2}+2t\right)}}{4} =$$

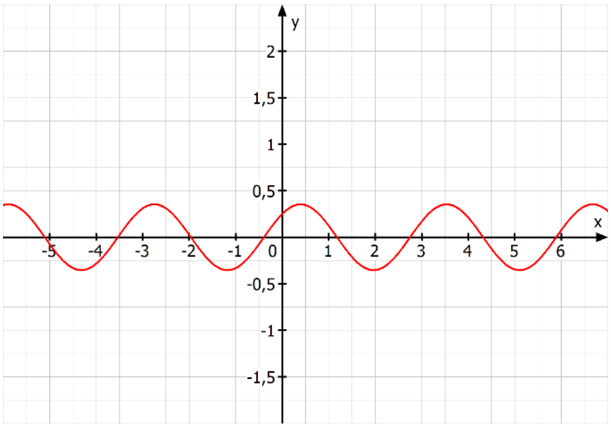
$$-\frac{1}{4}cos(2t) - i\frac{1}{4}sin(2t) - \frac{1}{4}cos\left(\frac{\pi}{2}+2t\right) - i\frac{1}{4}sin\left(\frac{\pi}{2}+2t\right)$$

We take the real part of it:

$$x_p = -\frac{1}{4}\left(cos(2t) + cos\left(\frac{\pi}{2}+2t\right)\right) =$$

$$= -\frac{1}{4}\left(cos(2t) - sin(2t)\right)$$

We plot:



Damped sinusoidal inputs

We can use complex replacement to solve any ode of the form:

$$P(D)x = e^{at}cos(\omega t - \phi)$$

Numeric example 39:

We use the ode:

$$\ddot{x} + 2x = e^{-t}cos(3t - \phi)$$

We replace the right-hand side with a complex exponential:

$$e^{-t}cos(3t-\phi) \rightarrow e^{-t}e^{i(3t-\phi)} = e^{i3t-i\phi-t} =$$

$$e^{-i\phi}e^{t(-1+i3)}$$

We complexify:

$$\ddot{z} + 2z = e^{-i\phi}e^{t(-1+i3)}$$

We get the characteristic polynomial:

$$P(r) = r^2 + 2 = 0$$

$$r_{12} = \pm i\sqrt{2}$$

We use erf:

$$P(r = -1 + i3) = (-1 + i3)^2 + 2 =$$

$$1 - 2i3 - 9 + 2 = -6 - i6$$

We get a particular complex solution:

$$z_p = -\frac{e^{-i\phi}e^{t(-1+i3)}}{6 + i6} = \frac{e^{-i\phi}e^{-t+i3t}}{6 + i6} =$$

$$\frac{e^{-i\phi-t+i3t}}{6 + i6} = \frac{e^{-t}e^{i(3t-\phi)}}{6 + i6}$$

We convert $6 + i6$ into polar form:

$$6 + i6 \rightarrow \sqrt{72}e^{i\frac{\pi}{4}}$$

We get:

$$z_p = \frac{e^{-t}e^{i(3t-\phi)}}{\sqrt{72}e^{i\frac{\pi}{4}}} = \frac{1}{\sqrt{72}}e^{-t}e^{i\left(3t-\phi-\frac{\pi}{4}\right)}$$

A real particular solution:

$$x_p = \frac{1}{\sqrt{72}}e^{-t}cos\left(3t - \phi - \frac{\pi}{4}\right)$$

Numeric example 40:

We use the ode:

$$\ddot{y} - \dot{y} + 2y = 10e^{-t}sin(t)$$

We replace the right-hand side with a complex exponential:

$$10e^{-t}sin(t) \rightarrow 10e^{-t}e^{it}$$

We note that $10e^{-t}sin(t)$ is the imaginary part of $10e^{-t}e^{it}$.

We complexify:

$$\ddot{z} - \dot{z} + 2z = 10e^{-t}e^{it} = 10e^{t(i-1)}$$

We get the characteristic polynomial:

$$P(r) = r^2 - r + 2 = 0$$

$$r_{12} = \frac{1 \pm i\sqrt{7}}{2} = \frac{1}{2} \pm i\sqrt{\frac{7}{4}}$$

We use erf:

$$P\left(r = \frac{1}{2} + i\sqrt{\frac{7}{4}}\right) = \left(\frac{1}{2} + i\sqrt{\frac{7}{4}}\right)^2 - \frac{1}{2} - i\sqrt{\frac{7}{4}} + 2 =$$

$$\frac{1}{4}+i\sqrt{\frac{7}{4}}-\frac{7}{4}-\frac{1}{2}-i\sqrt{\frac{7}{4}}+2=$$

$$\frac{1}{4}-\frac{7}{4}-\frac{2}{4}+\frac{8}{4}=0$$

We derivate:

$$P'(r)=2r-1$$

$$P'\left(r=\frac{1}{2}+i\sqrt{\frac{7}{4}}\right)=2\left(\frac{1}{2}+i\sqrt{\frac{7}{4}}\right)-1=$$

$$1+i\sqrt{7}=2\sqrt{2}e^{i\cdot tan^{-1}(\sqrt{7})}$$

We call $tan^{-1}(\sqrt{7}) \coloneqq \varphi$

We get the solution of the homogeneous erf:

$$z=c_1te^{i\varphi}+c_1te^{-i\varphi}$$

We get a particular complex solution:

$$z_p=\frac{10e^{t(i-1)}}{P(r=i-1)}=\frac{10e^{t(i-1)}}{(i-1)^2-(i-1)+2}=$$

$$\frac{10e^{t(i-1)}}{-1-2i+1-i+1+2}=$$

$$\frac{10e^{t(i-1)}}{3-3i}$$

We convert $3-3i$ into polar form:

$$3-3i\rightarrow\sqrt{18}e^{-i\frac{\pi}{4}}$$

We get:

$$z_p=\frac{10e^{t(i-1)}}{3-3i}=\frac{10}{\sqrt{18}}e^{it-t+i\frac{\pi}{4}}=$$

$$\frac{10}{\sqrt{18}}e^{-t}e^{i\left(t+\frac{\pi}{4}\right)}$$

A real particular solution:

$$x_p=\frac{10}{\sqrt{18}}e^{-t}cos\left(t+\frac{\pi}{4}\right)$$

An imaginary particular solution:

$$x_p=\frac{10}{\sqrt{18}}e^{-t}sin\left(t+\frac{\pi}{4}\right)$$

We plot the imaginary particular solution