This paper shows how Prof. Zwiebach introduced the Dirac delta function by using Fourier transform:

https://courses.learn.mit.edu/learn/course/course-v1:MITxT+8.04x+1T2026/home

Hope I can help you learning quantum mechanics.

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \phi(k) \cdot e^{ikx} dk$$

$\Psi(x)$ is a superposition of plane waves $\phi(k) \cdot e^{ikx}$ where $\phi(k)$ is the weight of every phase e^{ikx}.

Fourier's theorem states that we can do the reverse thing:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \Psi(x) \cdot e^{-ikx} dx$$

$\Psi(x)$ and $\phi(k)$ give you the same information.

What we can do now is to insert the second equation into the first one:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \phi(k) \cdot e^{ikx} dk \rightarrow$$

$$\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \Psi(\xi) \cdot e^{-ik\xi} d\xi \right] \cdot e^{ikx} dk =$$

$$\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \Psi(\xi) \cdot e^{-ik\xi} d\xi \right] \cdot e^{ikx} dk$$

Note: In the inner integral we replaced x by ξ because x in $\phi(k)$ is independent of ξ in $\Psi(\xi)$.

We should get back $\Psi(x)$.

We note that $\Psi(\xi)$ is independent of k, so we could rewrite the integral:

$$\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \Psi(\xi) d\xi \cdot \int_{-\infty}^{\infty} e^{ik(x-\xi)} dk$$

This must be $\Psi(x)$:

$$\Psi(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \Psi(\xi) d\xi \cdot \int_{-\infty}^{\infty} e^{ik(x-\xi)} dk$$

$e^{ik(x-\xi)}$ is a phase. The integration of a phase over the interval from $-\infty$ to ∞ cancels out and gives zero with one exception: if $x = \xi$ the exponent becomes zero and the integral blows up to infinity. This is the property of the Dirac delta function:

$$\frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{ik(x-\xi)} dk$$

The Dirac delta function $\delta(x-\xi)$, extracting the value of $\Psi(x)$:

$$\Psi(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \Psi(\xi) \delta(x-\xi) d\xi$$