From classical physics to quantum mechanics via the Ehrenfest theorem.

Related information you may find at:

https://farside.ph.utexas.edu/teaching/qmech/Quantum/node36.html

Hope I can help you with learning quantum mechanics.

Contents

Prerequisite

The commutator of two operators (matrices) A, B:

$$[A, B] \coloneqq AB - BA$$

The position operator X multiplies any functions applied to by x:

$$Xf(x) = x \cdot f(x)$$

The differentiation operator D derives any functions with respect to x:

$$Df(x) = \frac{d}{dx} f(x)$$

The commutator of D and X:

$$[X, D]f(x) =$$

$$XDf(x) - DXf(x) =$$

$$x \frac{d}{dx} f(x) - \frac{d}{dx}\big(xf(x)\big) =$$

$$x \frac{d}{dx} f(x) - f(x) - x \frac{d}{dx} f(x) = -f(x)$$

In other words:

$$[X, D] = -1$$

Position and momentum classical

A particle moving along the x-axis has position x and momentum p.

The particle is moving on the x-axis in a potential $V(x)$.

We use the potential:

$$V(x) = a \cdot x^2$$

Note: This is the potential of the harmonic oscillator.

The Hamiltonian $h(x, p)$ describes the total energy of the particle, the sum of kinetic energy and potential energy:

$$h(x, p) = \frac{p^2}{2m} + V(x)$$

The speed of the particle:

$$\frac{dx}{dt} = \frac{\partial h}{\partial p} = \frac{p}{m}$$

The force acting on the particle:

$$\frac{dp}{dt} = -\frac{\partial h}{\partial x} = -\frac{dV(x)}{dx}$$

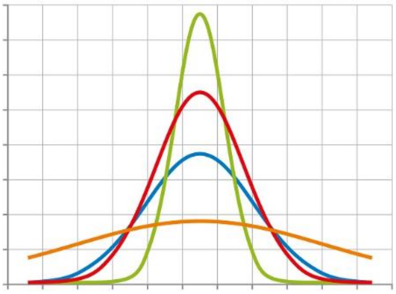
Note: x and p are observables that can be measured (exactly).

Position and momentum quantum

X and P are operators. The state of the particle becomes a vector $|\psi\rangle$ in Hilbert space of square integrable functions $L^2(\mathbb{R},\mathbb{C})$. The function $\psi(x)$ is named wave function.

The operator X corresponds to the observable x, the operator P to the observable p.

The set of eigenfunctions of the operator X are elements $\{|x\rangle, x \in \mathbb{R}\}$. The corresponding eigenvalues are the position of the particle, x.



$|x\rangle$, the position of the particle, is not part of the Hilbert space of square integrable functions $L^2(\mathbb{R},\mathbb{C})$. This causes that the position of the particle cannot be measured exactly. The position after measurement (after the application of the operator) must be a more or less sharp localized square integrable function – a wave function.

Ehrenfest

We transform the Classical Hamiltonian into the Quantum one:

$$h(x,p) = \frac{p^2}{2m} + V(x) \rightarrow H = \frac{P^2}{2m} + V(X)$$

The Ehrenfest Theorem states:

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle [A,H]\rangle$$

Note: $\langle A\rangle$ denotes the expectation value (average).

Note: The operator A has no explicit time dependency.

Note: If A commutes with H, the expectation value of A does not change with time.

From the Ehrenfest Theorem we get the Ehrenfest equations:

$$\frac{d\langle X\rangle}{dt} = \langle\frac{\partial H}{\partial P}\rangle = \frac{\langle P\rangle}{m}$$

$$\frac{d\langle P\rangle}{dt} = -\langle\frac{\partial H}{\partial X}\rangle = -\frac{d}{dx}\langle V(X)\rangle$$

We combine the Ehrenfest Theorem and the Ehrenfest equations:

$$\frac{d\langle X\rangle}{dt} = \frac{\langle P\rangle}{m}, \frac{d\langle X\rangle}{dt} = \frac{1}{i\hbar}\langle [X,H]\rangle \;\rightarrow\; \frac{\langle P\rangle}{m} = \frac{1}{i\hbar}\langle [X,H]\rangle$$

$$\frac{d\langle P\rangle}{dt} = -\frac{d}{dx}\langle V(X)\rangle, \frac{d\langle P\rangle}{dt} = \frac{1}{i\hbar}\langle [P,H]\rangle \;\rightarrow\; -\frac{d}{dx}\langle V(X)\rangle = \frac{1}{i\hbar}\langle [P,H]\rangle$$

We take the first equation:

$$\frac{\langle P\rangle}{m} = \frac{1}{i\hbar}\langle [X,H]\rangle$$

We check the commutator.

$$[X,H] = XH - HX =$$

$$X\left(\frac{P^2}{2m}+V(X)\right)-\left(\frac{P^2}{2m}+V(X)\right)X=$$

$$\frac{XP^2}{2m}+XV(X)-\frac{P^2X}{2m}-V(X)X=$$

$$\frac{XP^2}{2m}-\frac{P^2X}{2m}+\ XV(X)-V(X)X$$

We examine the second part $XV(X)-V(X)X$:

$$XV(X)=xV(X)=xax^2=ax^3$$

$$V(X)X=ax^2x=ax^3$$

$$XV(X)-V(X)X=0$$

We get:

$$[X,H]=\frac{1}{2m}[X,P^2]$$

We disassemble this:

$$\frac{1}{2m}[X,P^2]=\frac{1}{2m}[X,P]P+P[X,P]$$

We go back to the first equation:

$$\frac{\langle P\rangle}{m}=\frac{1}{i\hbar}\langle[X,H]\rangle\rightarrow\frac{\langle P\rangle}{m}=\frac{1}{i\hbar}\frac{1}{2m}\langle[X,P]P+P[X,P]\rangle$$

This equation is valid if:

$$[X,P]=i\hbar\cdot id$$

Note: we work with operators (matrices), id is the identity matrix.

From the prerequisite we have:

$$[X,D]=-1$$

If we assign: $P\coloneqq -i\hbar D$ we get:

$$\frac{\langle P\rangle}{m}=\frac{1}{i\hbar}\langle[X,H]\rangle$$

We check whether the second equation holds with this P:

$$-\frac{d}{dx}\langle V(X)\rangle=\frac{1}{i\hbar}\langle[P,H]\rangle$$

We check the commutator:

$$[P,H]=PH-HP=$$

$$P\left(\frac{P^2}{2m}+V(X)\right)-\left(\frac{P^2}{2m}+V(X)\right)P=$$

$$P\frac{P^2}{2m}+PV(X)-\frac{P^2}{2m}P-V(X)P=$$

$$P\frac{P^2}{2m} - \frac{P^2}{2m}P + PV(X) - V(X)P =;$$

Note: $PP^2 = P^2P$

$$PV(X) - V(X)P =$$

$$-i\hbar DV(X) - V(X)(-i\hbar D) =;$$

Note: we need a function $f(x)$ the operators are applied to:

$$-i\hbar\left(\frac{d}{dx}(ax^2) - (ax^2)\left(\frac{d}{dx}\right)\right)f(x) =$$

$$-i\hbar\left(\frac{d}{dx}\big((ax^2)f(x)\big) - (ax^2)\left(\frac{d}{dx}f(x)\right)\right) =$$

$$-i\hbar\left(2axf(x) + ax^2\frac{d}{dx}f(x) - ax^2\frac{d}{dx}f(x)\right) =$$

$$-i\hbar 2axf(x)$$

In other words:

$$[P,H] = -i\hbar 2ax = -i\hbar\frac{d}{dx}\langle V(X)\rangle$$

We check the second equation:

$$-\frac{d}{dx}\langle V(X)\rangle = \frac{1}{i\hbar}\langle [P,H]\rangle \rightarrow$$

$$-\frac{d}{dx}\langle V(X)\rangle = \frac{1}{i\hbar}\langle -i\hbar\frac{d}{dx}\langle V(X)\rangle\rangle \rightarrow$$

$$\frac{d}{dx}\langle V(X)\rangle = \frac{d}{dx}\langle\langle V(X)\rangle\rangle$$

Note: the average of an average remains the same.

Note: This will work with every potential that can be expressed in polynomial functions of x.

Result: If we set $P \coloneqq -i\hbar D$ the Ehrenfest equations for our scenario are valid. The momentum operator, the equivalence to the classical momentum becomes the differentiation operator, multiplied by $-i\hbar$.

We can treat results of classical processes as averages of quantum processes and try to find the corresponding quantum operators. This may work but must not work in every case.