This short paper presents a translation of the article of P. Ehrenfest (from 1927).

In the appendix you find the original German text and two comments marked as (a) and (b).

Hope I can help you learning quantum mechanics.

Remarks on the approximate validity of classical mechanics within quantum mechanics. By P. Ehrenfest in Leiden, Holland. Received September 5, 1927

The following relationship can be derived from Schrödinger's equation using a short elementary calculation without neglect:

(a)
$$m\frac{d^2}{dt^2}\iiint d\tau\cdot\Psi\Psi^*\cdot x = \iiint d\tau\cdot\Psi\Psi^*\left(-\frac{\partial V}{\partial x}\right)$$

For a small and remaining small wave packet (m of the order of 1 gram) the acceleration of its position coordinates matches the local force $-\frac{\partial V}{\partial x}$ according to Newton's equation of motion.

It is desirable to be able to answer the following question as elementary as possible: What perspective does Newton's basic equations of classical mechanics have from the view of quantum mechanics?

A number of recent publications[1] have essentially clarified that and to what extent classical mechanics remain correct for macroscopic processes to a high degree of approximation.

But it is permissible to briefly point out a particular elementary relation that follows exactly from Schrödinger's equation without any neglect, perhaps it makes understanding the connection between wave mechanics and classical mechanics a little easier.

It is sufficient to present the formulas for the case of a single degree of freedom, i.e. for the following form of the Schrödinger equation:

$$-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\Psi = \mathrm{ih}\frac{\partial\Psi}{\partial t} \quad (1)$$

$$-\frac{h^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2}+V(x)\Psi^* = -\mathrm{ih}\frac{\partial\Psi^*}{\partial t} \quad (1^*)$$

Then define:

$$\int_{-\infty}^{\infty} dx\, x\,\Psi\Psi^* \equiv Q(t) \quad (2)$$

$$ih\int_{-\infty}^{\infty} dx\,\Psi\,\frac{\partial\Psi^*}{\partial x} \equiv P(t) \quad (3)$$ [2]

(b) Now by using (1) and (2) calculate $\frac{dQ}{dt}$ and $\frac{dP}{dt}$.

Substitution and partial integration results immediately (and without neglect):

$$\frac{dQ}{dt} = \frac{1}{m}P \quad (4)$$

$$m\frac{d^2Q}{dt^2} = \frac{dP}{dt} = \int dx\Psi\Psi^*\left(-\frac{\partial V}{\partial x}\right) \quad (5)$$

[1] Louis de Broglie, Thèse 1924; Journ. de phys. et le Rad. (6) **7**, 1,32,1926; C. R. **180**, 498, 1925; **183**,272,1926. – L. Brillouin, Journ. de phys. et le Rad. **7**, 353,1926. – E Schrödinger, Naturwiss. **14**, 664, 1926. – P. Debye, Phys. ZS. **28**, 170, 1927. – W. Heisenberg, ZS. f. Phys. **43**, 172, 1927. – E. H. Kennard, ZS. f. Phys. **44**, 326, 1927.

[2] Expansion of Ψ according to the eigenfunctions $\Psi = \sum c_n e^{\frac{iE_n}{h}t}\varphi_n(x)$ provides the relationship to the matrices $q_{nm} = e^{\frac{i}{h}(E_n-E_m)t}\int dx\cdot x\,\varphi_n\varphi_m$ and p_{nm}

Equation (5) obviously says: Every time the width of the (probability) wave packet $\Psi\Psi^*$ is quite small (in relation to macroscopic distances), the acceleration (of the center of gravity Q) of the wave packet matches the “dominant force at the location of the wave packet” $\left(-\frac{\partial V}{\partial x}\right)$ in the sense of Newton's equations.

Remarks: The gradual divergence of a wave packet has been discussed by Heisenberg, loc. cit. in detail. His calculation for the force-free movement of a mass point in one-dimensional space can perhaps be made a little more familiar with the help of an obvious connection to well-known calculations about heat conduction. For $V(x) = 0$, the Schrödinger equation has the structure of the heat conduction equation:

$$\frac{\partial \Psi}{\partial t} = a^2 \frac{\partial^2 \Psi}{\partial x^2} \tag{6}$$

$$a^2 = i\frac{h}{2m} \tag{7}$$

Substitute into the general solution (cf. e. g. Riemann-Weber, Bd. II):

$$\Psi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} d\xi \; e^{-\frac{(x-\xi)^2}{4a^2 t}} \Psi(0,\xi) \tag{8}$$

the following for the initial state:

$$\Psi(0,\xi) = Ce^{-\frac{\xi^2}{2\omega^2}+i\mu\xi} \tag{9}$$

Then:

$$(\Psi\Psi^*)_{t=0} = C^2 \cdot e^{-\frac{\xi^2}{\omega^2}} \tag{10}$$

(with μ being an arbitrary real constant), just like Heisenberg one finds the position and distribution of the “wave packet”:

$$\Psi\Psi^* = c(t) \cdot e^{-\frac{\left(x-\frac{h\mu}{m}t\right)^2}{\Omega^2}} \tag{11}$$

with

$$\Omega^2 = \omega^2 + \frac{h^2 t^2}{m^2 \omega^2} \tag{12}$$

a displacement of the wave packet with the speed $\frac{h\mu}{m}$ and an increasing dissolution over time. A doubling of the initial width ($\Omega^2 = 4\omega^2$) occurs after the following time:

$$T = \sqrt{3}\frac{m\omega^2}{h} \left(h = \frac{6.6 \times 10^{-27}}{2\pi}\right) \tag{13}$$

For $m = 1\ g, \omega = 10^{-3}\ cm$ we get $T = 10^{21}\ sec$; however, for $= 1.7 \times 10^{-24}\ g, \omega = 10^{-8}\ cm$ we get $T = 10^{-13}\ sec$.

Remark (a)

$\iiint d\tau \cdot \Psi\Psi^* \cdot x$ is the expectation value of the position in R^3, written as $\langle x \rangle$.

$m\frac{d^2}{dt^2}\iiint d\tau \cdot \Psi\Psi^* \cdot x$ corresponds to $m\langle \ddot{x} \rangle$.

$\iiint d\tau \cdot \Psi\Psi^* \left(-\frac{\partial V}{\partial x}\right)$ is the expectation value of the force, written as $-\langle V'(x) \rangle$ or maybe as $\langle F(x) \rangle$.

We interpret

$$m\frac{d^2}{dt^2}\iiint d\tau \cdot \Psi\Psi^* \cdot x = \iiint d\tau \cdot \Psi\Psi^* \left(-\frac{\partial V}{\partial x}\right)$$

as Newton's equation:

$$m\langle \ddot{x} \rangle = -\langle V'(x) \rangle$$

First time derivative of $\langle x \rangle$.	Second time derivative of $\langle x \rangle$.
We calculate: $$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\langle [H,x] \rangle + \langle \frac{\partial x}{\partial t} \rangle$$	We calculate: $$\frac{d\langle p_x \rangle}{dt} = \frac{i}{\hbar}\langle [H,p_x] \rangle + \langle \frac{\partial p}{\partial t} \rangle$$
The position operator $\langle x \rangle$ has no explicit time dependency: $$\langle \frac{\partial x}{\partial t} \rangle = 0$$	The momentum operator $\langle p \rangle$ has no explicit time dependency: $$\langle \frac{\partial p}{\partial t} \rangle = 0$$
We get: $$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\langle [H,x] \rangle$$ Note: $[H,x]$ is the commutator relation:	We get: $$\frac{d\langle p \rangle}{dt} = \frac{i}{\hbar}\langle [H,p] \rangle$$ Note: $[H,p]$ is the commutator relation:
We write the Hamiltonian: $$H = \frac{p^2}{2m} + V(x)$$	We write the Hamiltonian: $$H = \frac{p^2}{2m} + V(x)$$
We get the commutator: $$[H,x] = \left[\left(\frac{p^2}{2m} + V(x)\right), x\right]$$ $$= \frac{1}{2m}\left[\left(p_x^2 + p_y^2 + p_z^2\right), x\right] =$$ $$= \frac{1}{2m}[p_x^2, x]$$ $$= \frac{1}{2m}(p_x[p_x,x] + [p_x,x]p_x)$$ $$= \frac{1}{2m}\left(-i\hbar p_x + (-i\hbar p_x)\right)$$ $$= -\frac{i\hbar}{m}p_x$$ Note: Any function of x commutes with x. Note: p_y and p_z commute with x.	We get the commutator: $$\langle [H,p_x] \rangle = \langle \left[\left(\frac{p^2}{2m} + V(x)\right), p_x\right] \rangle$$ $$= \langle [V(x), p_x] \rangle = \frac{i}{\hbar}\langle \frac{\partial V}{\partial x} \rangle$$ Note: Any function of p commutes with p. Note: p_x, p_y and p_z commute with p_x.
We get: $$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\left(-\frac{i\hbar}{m}\langle p_x \rangle\right) = \frac{1}{m}\langle p_x \rangle$$	We get: $$\frac{d\langle p_x \rangle}{dt} = i\hbar\frac{i}{\hbar}\langle \frac{\partial V}{\partial x} \rangle = -\langle \frac{\partial V}{\partial x} \rangle$$

There is a difference in this derivation of Newton's law and the classic Newton:

classic	quantum
$m\langle\ddot{x}\rangle = -V'\langle x\rangle$	$m\langle\ddot{x}\rangle = -\langle V'(x)\rangle$

If the potential is quadratic, $V(x)\sim x^2$, then $V'(x)\sim x$ and $\langle V'(x)\rangle = V'\langle x\rangle$.

In all other cases we have:

$$\langle V'(x)\rangle \approx V'\langle x\rangle$$

The bigger the uncertainty in x, the bigger the difference between $\langle V'(x)\rangle$ and $V'\langle x\rangle$.

Remark (b)

We calculate $\dot{Q} = \frac{dQ}{dt}$:

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\int_{-\infty}^{\infty} dx\, x\, \Psi\Psi^*\right) = \int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx + \int_{-\infty}^{\infty} x\, \frac{d\Psi}{dt}\Psi^* dx + \int_{-\infty}^{\infty} x\, \Psi\frac{d\Psi^*}{dt} dx =$$

$$\int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx + \int_{-\infty}^{\infty} x\left(\frac{d\Psi}{dt}\Psi^* + \Psi\frac{d\Psi^*}{dt}\right) dx =$$

Note: We use partial integration for the second integral.

$$\int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx + x\int_{-\infty}^{\infty}\left(\frac{d\Psi}{dt}\Psi^* + \Psi\frac{d\Psi^*}{dt}\right) dx - \int_{-\infty}^{\infty}\left(\frac{d\Psi}{dt}\Psi^* + \Psi\frac{d\Psi^*}{dt}\right) dx =$$

$$\int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx + x\int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx + x\int_{-\infty}^{\infty} \Psi\frac{d\Psi^*}{dt} dx - \int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx - \int_{-\infty}^{\infty} \Psi\frac{d\Psi^*}{dt} dx =$$

Note: We use $\int_{-\infty}^{\infty} f\cdot g' = -\int_{-\infty}^{\infty} f'\cdot g$

$$\int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx + x\int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx - x\int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx - \int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx + \int_{-\infty}^{\infty} \frac{d\Psi}{dt}\Psi^* dx =$$

$$\int_{-\infty}^{\infty} \frac{dx}{dt}\, \Psi\Psi^* dx = \langle\dot{x}\rangle$$

We calculate $\dot{P} = \frac{dP}{dt}$:

$$\frac{dP}{dt} = ih\frac{d}{dt}\left(\int_{-\infty}^{\infty} \Psi\, \frac{\partial\Psi^*}{\partial x} dx\right) =$$

$$ih\left(\int_{-\infty}^{\infty}\left(\frac{d}{dt}\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x} dx + \int_{-\infty}^{\infty} \Psi\cdot\frac{d}{dt}\frac{\partial}{\partial x}\Psi^* dx\right) =$$

$$ih\left(\int_{-\infty}^{\infty}\left(\frac{d}{dt}\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x} dx + \int_{-\infty}^{\infty} \Psi\cdot\frac{\partial}{\partial x}\frac{d}{dt}\Psi^* dx\right) =$$

$$ih\left(\int_{-\infty}^{\infty}\left(\frac{-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\cdot\Psi}{ih}\right)\cdot\frac{\partial\Psi^*}{\partial x} dx + \int_{-\infty}^{\infty} \Psi\cdot\frac{\partial}{\partial x}\frac{\frac{h^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} - V(x)\cdot\Psi^*}{ih} dx\right) =$$

$$\int_{-\infty}^{\infty}\left(-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\cdot\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x}dx+\int_{-\infty}^{\infty}\Psi\cdot\frac{h^2}{2m}\cdot\frac{\partial}{\partial x}\frac{\partial^2\Psi^*}{\partial x^2}-\Psi\cdot\frac{\partial}{\partial x}(V(x)\cdot\Psi^*)dx=$$

$$\int_{-\infty}^{\infty}\left(-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\cdot\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x}dx+\int_{-\infty}^{\infty}\Psi\cdot\frac{h^2}{2m}\cdot\frac{\partial^3\Psi^*}{\partial x^3}-\Psi\cdot\frac{\partial}{\partial x}(V(x)\cdot\Psi^*)dx=$$

$$\int_{-\infty}^{\infty}\left(-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\cdot\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x}dx+\int_{-\infty}^{\infty}\Psi\cdot\frac{h^2}{2m}\cdot\frac{\partial^3\Psi^*}{\partial x^3}-\Psi\cdot\left(\frac{\partial V(x)}{\partial x}\cdot\Psi^*+V(x)\cdot\frac{\partial\Psi^*}{\partial x}\right)dx=$$

$$\int_{-\infty}^{\infty}\left(-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\cdot\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x}dx+\int_{-\infty}^{\infty}\Psi\cdot\frac{h^2}{2m}\frac{\partial^3\Psi^*}{\partial x^3}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*-V(x)\cdot\Psi\cdot\frac{\partial\Psi^*}{\partial x}dx=$$

$$\int_{-\infty}^{\infty}\left(-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\cdot\Psi\right)\cdot\frac{\partial\Psi^*}{\partial x}+\Psi\cdot\frac{h^2}{2m}\frac{\partial^3\Psi^*}{\partial x^3}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*-V(x)\cdot\Psi\cdot\frac{\partial\Psi^*}{\partial x}dx=$$

$$\int_{-\infty}^{\infty}-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\cdot\frac{\partial\Psi^*}{\partial x}+V(x)\cdot\Psi\cdot\frac{\partial\Psi^*}{\partial x}+\Psi\cdot\frac{h^2}{2m}\frac{\partial^3\Psi^*}{\partial x^3}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*-V(x)\cdot\Psi\cdot\frac{\partial\Psi^*}{\partial x}dx=$$

$$\int_{-\infty}^{\infty}-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\cdot\frac{\partial\Psi^*}{\partial x}+\Psi\cdot\frac{h^2}{2m}\frac{\partial^3\Psi^*}{\partial x^3}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*dx=$$

$$\int_{-\infty}^{\infty}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*dx$$

Note: Integration by parts gives:

$$\int_{-\infty}^{\infty}-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\cdot\frac{\partial\Psi^*}{\partial x}+\Psi\cdot\frac{h^2}{2m}\frac{\partial^3\Psi^*}{\partial x^3}dx=$$

$$-\frac{h^2}{2m}\left(\int_{-\infty}^{\infty}\frac{\partial^2\Psi}{\partial x^2}\cdot\frac{\partial\Psi^*}{\partial x}dx-\int_{-\infty}^{\infty}\Psi\cdot\frac{\partial^3\Psi^*}{\partial x^3}dx\right)=$$

$$-\frac{h^2}{2m}\left(-\int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial x}\cdot\frac{\partial^2\Psi^*}{\partial x^2}dx-\int_{-\infty}^{\infty}\Psi\cdot\frac{\partial^3\Psi^*}{\partial x^3}dx\right)=$$

$$-\frac{h^2}{2m}\left(\int_{-\infty}^{\infty}\Psi\cdot\frac{\partial^3\Psi^*}{\partial x^3}dx-\int_{-\infty}^{\infty}\Psi\cdot\frac{\partial^3\Psi^*}{\partial x^3}dx\right)=0$$

Conclusion:

$$\frac{dQ}{dt}=\int_{-\infty}^{\infty}\frac{dx}{dt}\;\Psi\Psi^*dx=\langle\dot{x}\rangle$$

$$\frac{dP}{dt}=\int_{-\infty}^{\infty}-\frac{\partial V(x)}{\partial x}\cdot\Psi\cdot\Psi^*dx=\langle F\rangle$$

The change in momentum results from the expectation value of the force.

Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik.

Von **P. Ehrenfest** in Leiden, Holland.

(Eingegangen am 5. September 1927.)

Aus der Schrödingerschen Gleichung läßt sich durch eine kurze elementare Rechnung ohne Vernachlässigung die Beziehung

$$m \frac{d^2}{dt^2} \iiint d\tau \,.\, \Psi\Psi^* \,.\, x = \iiint d\tau \,.\, \Psi\Psi^* \left(-\frac{\partial V}{\partial x}\right)$$

ableiten, die für ein kleines und klein bleibendes Wellenpaket (m von der Ordnung 1 g) besagt, daß die Beschleunigung seiner Lagekoordinaten im Sinne der Newtonschen Bewegungsgleichungen zur örtlichen Kraft $-\frac{\partial V}{\partial x}$ paßt.

Es ist wünschenswert, die folgende Frage möglichst elementar beantworten zu können: Welcher Rückblick ergibt sich vom Standpunkt der Quantenmechanik auf die Newtonschen Grundgleichungen der klassischen Mechanik? Durch eine ganze Reihe neuerer Publikationen ist im wesentlichen völlig aufgeklärt, daß und inwieweit die klassische Mechanik für makroskopische Vorgänge in hoher Näherung richtig bleibt[1]. Aber es sei erlaubt, kurz auf eine besonders elementare Relation hinzuweisen, die ohne jede Vernachlässigung exakt aus der Schrödingerschen Gleichung folgt, weil sie den Zusammenhang zwischen Wellenmechanik und klassischer Mechanik vielleicht doch noch etwas bequemer überblickbar macht.

Es genügt, die Formeln für den Fall eines einzigen Freiheitsgrades, also für die folgende Form der Schrödingergleichung

$$-\frac{h^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\,\Psi = i\,h\,\frac{\partial \Psi}{\partial t}, \tag{1}$$

$$-\frac{h^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + V(x)\,\Psi^* = -i\,h\,\frac{\partial \Psi^*}{\partial t} \tag{1*}$$

[1] Louis de Broglie, Thèse 1924; Journ. de phys. et le Rad. (6) **7**, 1, 32, 1926; C. R. **180**, 498, 1925; **183**, 272, 1926. — L. Brillouin, Journ. de phys. et le Rad. **7**, 353, 1926. — E. Schrödinger, Naturwiss. **14**, 664, 1926. — P. Debye, Phys. ZS. **28**, 170, 1927. — W. Heisenberg, ZS. f. Phys. **43**, 172, 1927. — E. H. Kennard, ZS. f Phys. **44**, 326, 1927.

darzustellen. — Nenne dann

$$\int_{-\infty}^{+\infty} dx\, x\, \Psi \Psi^* \equiv Q(t), \tag{2}$$

$$i h \int_{-\infty}^{+\infty} dx\, \Psi \frac{\partial \Psi^*}{\partial x} \equiv P(t)\,^1 \tag{3}$$

und berechne $\frac{dQ}{dt}$ und $\frac{dP}{dt}$ unter Benutzung von (1) und (2). Durch Einsetzen und partielle Integrationen ergibt sich sofort (und ohne Vernachlässigungen):

$$\frac{dQ}{dt} = \frac{1}{m} P, \tag{4}$$

$$m \frac{d^2 Q}{dt^2} = \frac{dP}{dt} = \int dx\, \Psi \Psi^* \left(-\frac{\partial V}{\partial x}\right). \tag{5}$$

Gleichung (5) besagt aber offenbar: Jedesmal, wenn die Breite des (Wahrscheinlichkeits-)Wellenpaketes $\Psi \Psi^*$ (im Verhältnis zu makroskopischen Distanzen) ziemlich klein ist, paßt die Beschleunigung (des Schwerpunktes Q) des Wellenpaketes im Sinne der Newtonschen Gleichungen zu der „am Orte des Wellenpaketes herrschenden" Kraft $\left(-\frac{\partial V}{\partial x}\right)$.

Bemerkungen: Das allmähliche Auseinanderfließen eines Wellenpaketes ist durch Heisenberg, l. c., ausführlich diskutiert worden. Seine Rechnung für das Beispiel der kraftfreien Bewegung eines materiellen Punktes im eindimensionalen Raume kann sich mit Hilfe einer naheliegenden Beziehung zu bekannten Rechnungen über Wärmeleitung vielleicht noch etwas vertrauter machen: Die Schrödingergleichung hat für $V(x) = 0$ die Struktur der Wärmeleitungsgleichung:

$$\frac{\partial \Psi}{\partial t} = a^2 \frac{\partial^2 \Psi}{\partial x^2} \tag{6}$$

mit

$$a^2 = i \frac{h}{2m}. \tag{7}$$

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1 Entwicklung von Ψ nach den Eigenfunktionen: $\Psi = \Sigma c_n e^{\frac{i E_n}{h} t} \varphi_n(x)$ liefert die Beziehung zu den Matrizen $q_{nm} = e^{\frac{i}{h}(E_n - E_m)t} \int dx \,.\, x \varphi_n \varphi_m$ und p_{nm}.

Setze in ihre allgemeine Lösung (siehe z. B. Riemann-Weber, Bd. II)

$$\Psi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} d\xi \, e^{-\frac{(x-\xi)^2}{4a^2 t}} \Psi(0,\xi) \tag{8}$$

für den Beginnzustand ein:

$$\Psi(0,\xi) = C e^{-\frac{\xi^2}{2\omega^2} + i\mu\xi}, \tag{9}$$

also

$$(\Psi\Psi^*)_{t=0} = C^2 \cdot e^{-\frac{\xi^2}{\omega^2}} \tag{10}$$

(μ eine willkürliche reelle Konstante), so findet man ganz wie bei Heisenberg für die spätere Lage und Verteilung des „Wellenpaketes“:

$$\Psi\Psi^* = c(t) \cdot e^{-\frac{(x - \frac{h\mu}{m} t)^2}{\Omega^2}}, \tag{11}$$

wo

$$\Omega^2 = \omega^2 + \frac{h^2 t^2}{m^2 \omega^2}, \tag{12}$$

also eine Verschiebung des Wellenpaketes mit der Geschwindigkeit $h\mu/m$ und ein mit der Zeit zunehmendes Zerfließen. Eine Verdopplung der anfänglichen Breite (d. h. $\Omega^2 = 4\omega^2$) tritt also nach der Zeit

$$T = \sqrt{3}\,\frac{m\omega^2}{h} \qquad \left(h = \frac{6{,}6 \cdot 10^{-27}}{2\pi}\right) \tag{13}$$

ein. Für $m = 1$ g, $\omega = 10^{-3}$ cm ist $T = 10^{21}$ sec; hingegen für $m = 1{,}7 \cdot 10^{-24}$ g und $\omega = 10^{-8}$ cm ist $T = 10^{-13}$ sec!