This paper deals with expressions like:

$$e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3}$$

Note: σ_i are the Pauli matrices.

The solution can be expressed either with trigonometric functions or exponentials.

We will work through this problem with exponential and trigonometric approach. A third and a fourth approach are shown in the end.

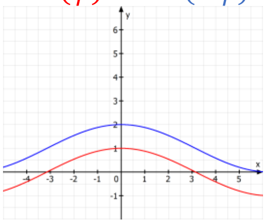
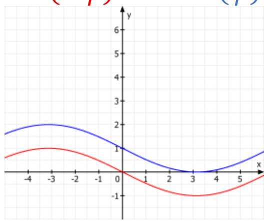
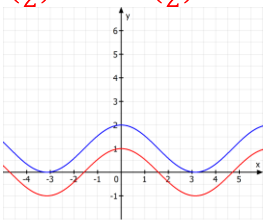
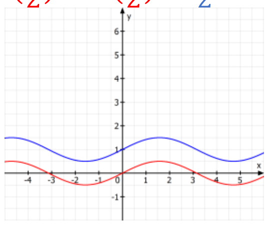
More information you find at

https://www.math.utah.edu/~gustafso/2250matrixexponential.pdf

Hope I can help you with learning quantum mechanics.

Prerequisite

The following trigonometric identities hold:

$\cos(\varphi) = \cos(-\varphi)$ 	$\sin(-\varphi) = -\sin(\varphi)$ 
$\cos^2\left(\frac{\varphi}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right) = \cos(\varphi)$ 	$\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right) = \frac{1}{2}\sin(\varphi)$ 

Note: blue functions shifted upwards.

The Pauli-matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: The identity matrix could be regarded as a Pauli matrix: $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note: Pauli-matrices are Hermitian.

For Pauli-matrices holds:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = id$$

Note: id is the identity matrix

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \; for \; i \neq j$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$\sigma_i \sigma_j = \delta_{ij} id + i\epsilon_{ijk}\sigma_k$$

ϵ_{ijk} is the Levi-Civita symbol that controls the $\mp$-sign:

$$\epsilon_{ijk} = \begin{cases} +1 \; if \; ijk \; is \; an \; even \; permutation \; of \; 123 \\ -1 \; if \; ijk \; is \; an \; odd \; permutation \; of \; 123 \\ 0 \qquad if \; two \; indices \; are \; the \; same \end{cases}$$

Note: Even permutations of $\{123\}$ are $\{123\}, \{231\}, \{312\}$. Odd permutations are $\{132\}, \{321\}, \{213\}$

For a quadratic matrix A holds:

$$\frac{d}{d\varphi} e^{A\varphi} = A \cdot e^{A\varphi}$$

End prerequisite

We treat the expression $e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3}$ as function of φ:

$$f(\varphi) = e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3}$$

Note: $f(0) = \sigma_1$

We derivate the function twice to get a differential equation.

First derivative:

$$\frac{d}{d\varphi}f(\varphi) = e^{i\frac{1}{2}\varphi\sigma_3}\left(i\frac{1}{2}\sigma_3\right)\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} + e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3}\left(-i\frac{1}{2}\sigma_3\right) =$$

$$\frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_3\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} - \frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1\sigma_3 e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$\frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3} - \frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}(-i\sigma_2) e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$\frac{i}{2}\left(e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3} + e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3}\right) =$$

$$-e^{i\frac{1}{2}\varphi\sigma_3}\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3}$$

Note: $f'(0) = -\sigma_2$

Second derivative:

$$\frac{d^2}{d\varphi^2}f(\varphi) = \frac{d}{d\varphi}\left(\frac{d}{d\varphi}f(\varphi)\right) = -e^{i\frac{1}{2}\varphi\sigma_3}\left(i\frac{1}{2}\sigma_3\right)\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3} - e^{i\frac{1}{2}\varphi\sigma_3}\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3}\left(-i\frac{1}{2}\sigma_3\right) =$$

$$-\frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_3\sigma_2 e^{-i\frac{1}{2}\varphi\sigma_3} + \frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_2\sigma_3 e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$\frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} + \frac{i}{2}e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$-\frac{1}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} - \frac{1}{2}e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$-e^{i\frac{1}{2}\varphi\sigma_3}i\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3}$$

Note: $f''(0) = -\sigma_1$

This resembles the derivation chain of sin and cos.

We get the differential equation:

$$f''(\varphi) = -f(\varphi)$$

Solution with exponential functions	Solution with trigonometric functions
$f(\varphi) = Ae^{i\varphi} + Be^{-i\varphi}$ $f'(\varphi) = iAe^{i\varphi} - iBe^{-i\varphi}$ $f''(\varphi) = -Ae^{i\varphi} - Be^{-i\varphi}$	$f(\varphi) = \mathcal{A}sin(\varphi) + \mathcal{B}cos(\varphi)$ $f'(\varphi) = \mathcal{A}cos(\varphi) - \mathcal{B}sin(\varphi)$ $f''(\varphi) = -\mathcal{A}sin(\varphi) - \mathcal{B}cos(\varphi)$
We use: $f(0) = \sigma_1$ $\sigma_1 = Ae^{i0} + Be^{-i0}$ $\sigma_1 = A + B$ $f'(0) = -\sigma_2$ $\sigma_2 = iB - iA$ We get for A and B: $A = \sigma_1 - B$ $B = \frac{\sigma_2 + iA}{i} = \frac{\sigma_2 + i(\sigma_1 - B)}{i}$ $B = \frac{\sigma_2}{i} + \sigma_1 - B$ $B = \frac{\sigma_2}{2i} + \frac{\sigma_1}{2}$ $A = \sigma_1 - \frac{\sigma_2}{2i} - \frac{\sigma_1}{2} = \frac{\sigma_1}{2} - \frac{\sigma_2}{2i}$	We use: $f(0) = \sigma_1 \rightarrow$ $\sigma_1 = \mathcal{B}$ $f'(0) = -\sigma_2 \rightarrow$ $-\sigma_2 = \mathcal{A}$
Solution: $f(\varphi) = \left(\frac{\sigma_1}{2} - \frac{\sigma_2}{2i}\right) e^{i\varphi} + \left(\frac{\sigma_2}{2i} + \frac{\sigma_1}{2}\right) e^{-i\varphi}$	Solution: $f(\varphi) = \sigma_1 cos(\varphi) - \sigma_2 sin(\varphi)$

We check whether the solutions are different and rewrite the exponential solution in the cartesian representation. For simplicity we use:

$$a := \left(\frac{\sigma_1}{2} - \frac{\sigma_2}{2i}\right)$$

$$b := \left(\frac{\sigma_2}{2i} + \frac{\sigma_1}{2}\right)$$

$$f(\varphi) = ae^{i\varphi} + be^{-i\varphi} =$$

$$a \cdot cos(\varphi) + i \cdot a \cdot sin(\varphi) + b \cdot cos(\varphi) - i \cdot b \cdot sin(\varphi) =$$

$$(a + b)cos(\varphi) + i(a - b)sin(\varphi) =;$$

We replace a and b by their originals:

$$\left(\frac{\sigma_1}{2} - \frac{\sigma_2}{2i} + \frac{\sigma_2}{2i} + \frac{\sigma_1}{2}\right) cos(\varphi) + i\left(\frac{\sigma_1}{2} - \frac{\sigma_2}{2i} - \frac{\sigma_2}{2i} - \frac{\sigma_1}{2}\right) sin(\varphi) =$$

$$\sigma_1 cos(\varphi) - \sigma_2 sin(\varphi)$$

The Ansatz with exponential functions gives the same result – the trigonometric way is faster in this case.

Third approach: For involutory matrices P like the Pauli matrices holds:

$$e^{iP\varphi} = cos(\varphi) \cdot id + i \cdot P \cdot sin(\varphi)$$

Note: an involutory matrix is its own inverse.

Note: id is the identity matrix.

With this we can work through our problem:

$$e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} \rightarrow \left(cos\left(\frac{\varphi}{2}\right) \cdot id + i \cdot \sigma_3 \cdot sin\left(\frac{\varphi}{2}\right)\right)\sigma_1\left(cos\left(\frac{\varphi}{2}\right) \cdot id - i \cdot \sigma_3 \cdot sin\left(\frac{\varphi}{2}\right)\right)$$

After some lengthy calculation and use of the trigonometric identities of the prerequisite we will arrive at the known result:

$$e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} = \sigma_1 cos(\varphi) - \sigma_2 sin(\varphi)$$

Fourth approach: For diagonal matrices holds:

$$e^{\begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n \end{pmatrix}} = \begin{pmatrix} e^a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^n \end{pmatrix}$$

σ_3 is a diagonal matrix:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In this case we get:

$$e^{i\frac{1}{2}\varphi\sigma_3}\sigma_1 e^{-i\frac{1}{2}\varphi\sigma_3} =$$

$$e^{i\frac{1}{2}\varphi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}e^{-i\frac{1}{2}\varphi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} =$$

$$\begin{pmatrix} e^{i\frac{1}{2}\varphi} & 0 \\ 0 & e^{-i\frac{1}{2}\varphi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{1}{2}\varphi} & 0 \\ 0 & e^{i\frac{1}{2}\varphi} \end{pmatrix} =$$

$$\ldots =$$

$$\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & cos(\varphi) + i \cdot sin(\varphi) \\ cos(\varphi) - i \cdot sin(\varphi) & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & cos(\varphi) \\ cos(\varphi) & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \cdot sin(\varphi) \\ -i \cdot sin(\varphi) & 0 \end{pmatrix} =$$

$$cos(\varphi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - sin(\varphi)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =$$

$$cos(\varphi)\sigma_1 - sin(\varphi)\sigma_2$$