In working with spin operators, we often have the expression $e^{i\theta\sigma_n}$ with σ_n standing for the pauli matrices $\sigma_x, \sigma_y, \sigma_z$, especially when working with unitary time evolution. This short paper shows how to transform them from exponential form into cartesian format with sin/cos:

$$e^{i\theta\sigma_n} \rightarrow cos(\theta) \cdot id + i \cdot \sigma_n \cdot sin(\theta)$$

Note: id is the identity matrix.

Hope I can help you with learning quantum mechanics.

The Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: the identity matrix could be named the fourth Pauli matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note:

$$\sigma_1{}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = id$$

$$\sigma_2{}^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = id$$

$$\sigma_3{}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = id$$

When working with unitary time development operators, we have something like:

$$U \sim e^{i\theta\sigma_n}$$

The "official" definition of power series (Taylor) means development at a starting point x_0:

$$T_{f,x_0}(x) = \sum_{i=0}^{\infty} \frac{f^i(x_0)}{i!}(x - x_0)^i$$

Note: f^i is the i-th derivation of f with respect to x.

Written out we get:

$$T_{f,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

Note: not all function shows convergence, meaning they cannot be developed into power series.

For real numbers, the power series of an exponential around $x_0 = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We apply this procedure to the pauli matrices:

$$e^{i\theta\sigma_1} = id + i\theta\sigma_1 - \frac{\theta^2}{2!}id - i\frac{\theta^3}{3!}\sigma_1 + \frac{\theta^4}{4!}id + i\frac{\theta^5}{5!}\sigma_1 \ldots =;$$

We rearrange:

$$e^{i\theta\sigma_1} = id \cdot \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\sigma_1\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) =;$$

We recognize the power series expansion of sin and $cosin$:

$$e^{i\theta\sigma_1} = id \cdot cos(\theta) + i\sigma_1 \cdot sin(\theta) = \begin{pmatrix} cos(\theta) & -i \cdot sin(\theta) \\ -i \cdot sin(\theta) & cos(\theta) \end{pmatrix}$$

The same holds for the other Pauli matrices too.