This paper introduces basic ideas of the Fourier transform. It follows a discussion at:

https://math.stackexchange.com/questions/1002/fourier-transform-for-dummies

https://studyflix.de/mathematik/fourier-transformation-1471 (in German)

Hope I can help you with learning Quantum mechanics.

The Fourier transform constructs (replaces)a periodic function $f(t)$ with period T by help of sin and cos functions:

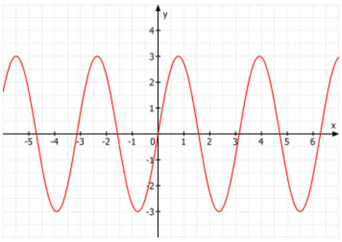
$$f(t) = \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t)\right), k \in \{0,1,2, \dots\}$$

A_k, B_k are amplitudes, intensities.

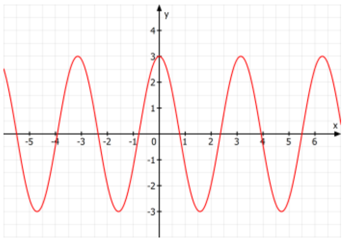
ω_k are frequencies:

$$\omega_k = \frac{2\pi}{T} k, k \in \{0,1,2, \dots\}$$

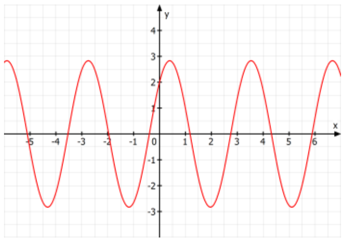
We distinguish between periodic functions:



odd



even



neither nor

Even or odd functions are easier to transform.

We want to calculate the amplitudes belonging to different numbers k.

We multiply $f(t)$ by $cos(\omega_{k'} \cdot t)$:

$$f(t) \cdot cos(\omega_{k'} \cdot t) = \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t)\right) \cdot cos(\omega_{k'} \cdot t), k \in \{0,1,2, \dots\}$$

We integrate:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos(\omega_{k'} \cdot t) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t)\right) \cdot cos(\omega_{k'} \cdot t)\, dt$$

We use orthogonality relations for sin and cos:

$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot cos(\omega_{k'} \cdot t) dt$	$=$	$\begin{cases} 0 \ for \quad k = k' \\ \frac{T}{2} \ for \ 0 \neq k = k' \\ T \ for \ 0 = k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} sin(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t) dt$	$=$	$\begin{cases} 0 \ for \quad k \neq k', k = 0 \ or \ k' = 0 \\ \frac{T}{2} \ for \quad 0 \neq k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t) dt$	$=$	0

Using the orthogonality relations, we see:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} A_k\cdot cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t) + B_k\cdot sin(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)\,dt$$

The sum vanishes because there is exactly one term that is not zero depending on k'.

The products of sin and cos are zero altogether.

We get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} A_k\cdot cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)dt$$

Result:

$k = k' \neq 0$	$k = k' = 0$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt = A_k\cdot\frac{T}{2}$	$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt = A_0\cdot T$

We get the amplitudes:

$A_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt$	$A_0 = \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$

Note: $k = k'$

The same way we calculate B_k using $sin(\omega_{k'}\cdot t)$.

We multiply $f(t)$ with $sin(\omega_{k'}\cdot t)$:

$$f(t)\cdot sin(\omega_{k'}\cdot t) = \sum_{k=0}^{\infty}\big(A_k\cdot cos(\omega_k\cdot t) + B_k\cdot sin(\omega_k\cdot t)\big)\cdot sin(\omega_{k'}\cdot t)\,, k\in\{0,1,2,\dots\}$$

We integrate:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{k=0}^{\infty}\big(A_k\cdot cos(\omega_k\cdot t) + B_k\cdot sin(\omega_k\cdot t)\big)\cdot sin(\omega_{k'}\cdot t)\,dt$$

We use orthogonality relations:

$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)dt$	=	$\begin{cases} 0 \;\; for \quad k = k' \\ \frac{T}{2} \;\; for\; 0 \neq k = k' \\ T \;\; for\; 0 = k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} sin(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)dt$	=	$\begin{cases} 0 \;\; for \quad k \neq k', k = 0\; or\; k' = 0 \\ \frac{T}{2} \;\; for \quad 0 \neq k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)dt$	=	0

Using the orthogonality relations, we see:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_{k'} \cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} A_k \cdot cos(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t) + B_k \cdot sin(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t)\, dt$$

The sum vanishes because there is exactly one term that is not zero depending on $k = k'$.

The products of sin and cos are zero altogether.

We get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_{k'} \cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} B_k \cdot sin(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t)dt$$

Result:

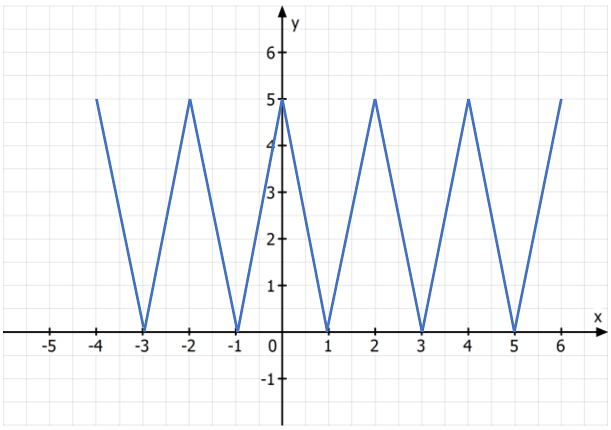
$k = k' \neq 0$	$k = k' = 0$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_k \cdot t)dt = B_k \cdot \frac{T}{2}$	0

We get the amplitudes:

$B_k = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_k \cdot t)dt$	$B_0 = 0$

Example 1

The even triangle function with period $T = 2$ and amplitude $\frac{5}{2}$.



Note: The amplitude is half of the total swing of the function.

We use the definition:

$$f(t) = \begin{cases} 5 \cdot \left(1 + \frac{2}{T} \cdot t\right) \; for -\frac{T}{2} \leq t \leq 0 \\ 5 \cdot \left(1 - \frac{2}{T} \cdot t\right) \quad for\; 0 \leq t \leq \frac{T}{2} \end{cases}$$

This is an even function.

We need calculate A_k only, because the product of an even and an odd function gives an odd function.

We use the triangle function with $T = 2$ to calculate A_k:

$$f(t) = \begin{cases} 5 \cdot (1+t)\ for -1 \le t \le 0 \\ 5 \cdot (1-t) \quad for\ 0 \le t \le 1 \end{cases}$$

We calculate A_k:

$$A_k = \frac{2}{2} \cdot \int_{-1}^{1} f(t) \cdot cos(\omega_k \cdot t) dt = \int_{-1}^{0} 5 \cdot (1+t) \cdot cos(\omega_k \cdot t) dt + \int_{0}^{1} 5 \cdot (1-t) \cdot cos(\omega_k \cdot t) dt =$$

$$5 \int_{-1}^{0} cos(\omega_k \cdot t) dt + 5 \int_{-1}^{0} t \cdot cos(\omega_k \cdot t) dt + 5 \int_{0}^{1} cos(\omega_k \cdot t) dt - 5 \int_{0}^{1} t \cdot cos(\omega_k \cdot t) dt =$$

$$5 \int_{-1}^{1} cos(\omega_k \cdot t) dt + 5 \int_{-1}^{0} t \cdot cos(\omega_k \cdot t) dt - 5 \int_{0}^{1} t \cdot cos(\omega_k \cdot t) dt$$

We calculate part two:

$$5 \int_{-1}^{0} t \cdot cos(\omega_k \cdot t) dt - 5 \int_{0}^{1} t \cdot cos(\omega_k \cdot t) dt$$

We take a look at Wikipedia: *https://en.wikipedia.org/wiki/List_of_integrals_of_trigonometric_functions*

$$\int x \cdot cos(a \cdot x) dx = \frac{1}{a^2} \cdot cos(a \cdot x) + \frac{1}{a} \cdot x \cdot sin(a \cdot x)$$

We apply, omitting the factor 5:

$$\int_{-1}^{0} t \cdot cos(\omega_k \cdot t) dt = \left[\frac{1}{{\omega_k}^2} \cdot cos(\omega_k \cdot t) + \frac{1}{\omega_k} \cdot t \cdot sin(\omega_k \cdot t) \right]_{-1}^{0} =$$

$$\left(\frac{1}{{\omega_k}^2} \cdot cos(0) + \frac{1}{\omega_k} \cdot 0 \cdot sin(0) \right) - \left(\frac{1}{{\omega_k}^2} \cdot cos(-\omega_k) - \frac{1}{\omega_k} \cdot sin(-\omega_k) \right) =$$

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(-\omega_k) + \frac{1}{\omega_k} \cdot sin(-\omega_k)$$

We note that $cos(x) = cos(-x)$ and $sin(-x) = -sin(x)$

We get:

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k)$$

$$\int_{0}^{1} t \cdot cos(\omega_k \cdot t) dt = \left[\frac{1}{{\omega_k}^2} \cdot cos(\omega_k \cdot t) + \frac{1}{\omega_k} \cdot t \cdot sin(\omega_k \cdot t) \right]_{0}^{1} =$$

$$\frac{1}{{\omega_k}^2} \cdot cos(\omega_k) + \frac{1}{\omega_k} \cdot sin(\omega_k) - \frac{1}{{\omega_k}^2}$$

We build the difference:

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k) - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k) + \frac{1}{{\omega_k}^2} =$$

$$\frac{2}{{\omega_k}^2} - \frac{2}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{2}{\omega_k} \cdot sin(\omega_k) = \frac{2}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{2}{\omega_k} \cdot sin(\omega_k)$$

We multiply the factor 5 and get the result for part two:

$$5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{10}{\omega_k} \cdot sin(\omega_k)$$

We calculate part one:

$$5\int_{-1}^{1} cos(\omega_k \cdot t)dt = 5\left[\frac{1}{\omega_k} \cdot sin(\omega_k \cdot t)\right]_{-1}^{1} = 5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) - 5 \cdot \frac{1}{\omega_k} \cdot sin(-\omega_k) =$$

$$5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) + 5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) = \frac{10}{\omega_k} \cdot sin(\omega_k)$$

We build the sum of part one and part two:

$$5\int_{-1}^{1} cos(\omega_k \cdot t)dt + 5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt =$$

$$\frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{10}{\omega_k} \cdot sin(\omega_k) + \frac{10}{\omega_k} \cdot sin(\omega_k) = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big)$$

We get A_k:

$$A_k = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big)$$

We expand ω_k and use $T = 2$:

$$\omega_k = \pi k, k \in \{0,1,2,\ldots\}$$

We get:

$$A_k = \frac{10}{(\pi k)^2}\big(1 - cos(\pi k)\big)$$

For $k = \{2,4,6,\ldots\}$ we get $A_k = 0$

For $k = 1,3,5,\ldots$ we get $A_k = \frac{20}{(\pi k)^2}$ because the cos for these numbers is -1.

We need $A_{k=0}$:

$$A_{k=0} = \frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot cos(0 \cdot t)dt = \frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot dt$$

We use our function $f(t)$:

$$f(t) = \begin{cases} 5 \cdot (1 + t) & for -1 \le t \le 0 \\ 5 \cdot (1 - t) & for\ 0 \le t \le 1 \end{cases}$$

We calculate:

$$\frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot dt = \frac{5}{2} \cdot \left(\int_{-1}^{0} (1 + t)dt + \int_{0}^{1} (1 - t)dt\right) =$$

$$\frac{5}{2}\cdot\left(\left[t+\frac{t^2}{2}\right]_{-1}^{0}+\left[t-\frac{t^2}{2}\right]_{0}^{1}\right)=\frac{5}{2}\cdot\left(0-\left(-1+\frac{1}{2}\right)+1-\frac{1}{2}\right)=$$

$$\frac{5}{2}\cdot\left(1-\frac{1}{2}+1-\frac{1}{2}\right)=\frac{5}{2}$$

We get $A_{k=0}$:

$$A_{k=0}=\frac{5}{2}$$

We remember the beginning:

$$f(t)=\sum_{k=0}^{\infty}\left(A_k\cdot cos(\omega_k\cdot t)+B_k\cdot sin(\omega_k\cdot t)\right),k\in\{0,1,2,\ldots\}$$

We harvest the results:

$$f(t)=A_0+\sum_{k=1}^{\infty}A_k\cdot cos(\omega_k\cdot t),k\in\{1,2,\ldots\}$$

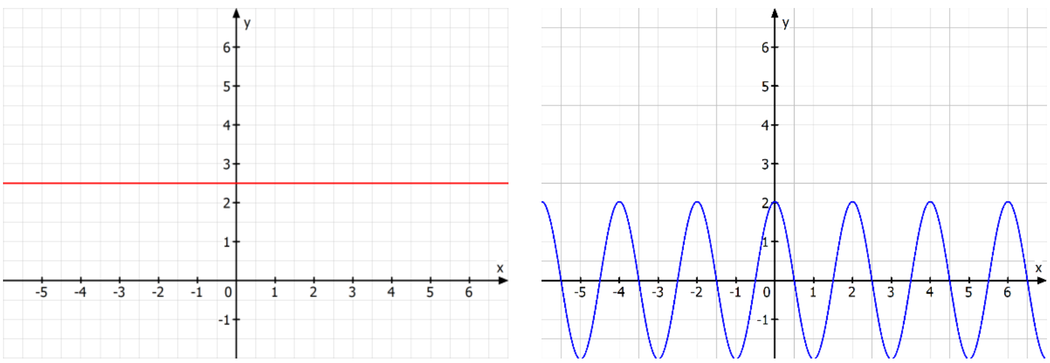
$$f(t)=\frac{5}{2}+\sum_{k=1}^{\infty}\frac{20}{(\pi k)^2}\cdot cos(\pi k\cdot t),k\in\{1,3,5,\ldots\}=$$

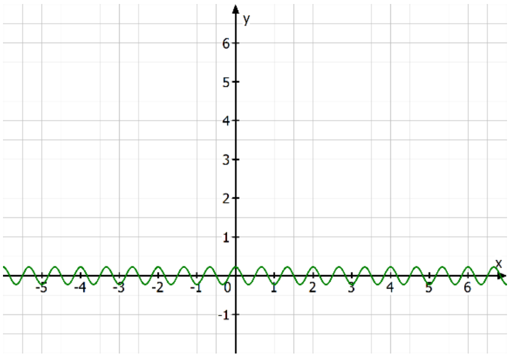
$$\frac{5}{2}+\frac{20}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot cos(\pi k\cdot t),k\in\{1,3,5,\ldots\}=$$

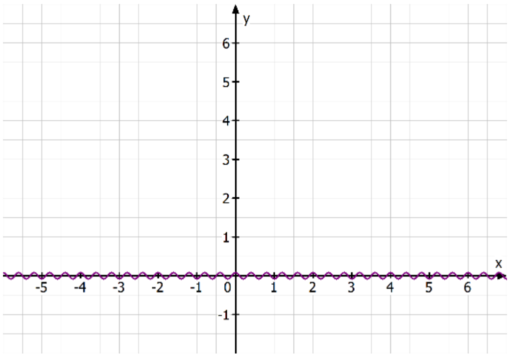
$$\frac{5}{2}+\frac{20}{\pi^2}\cdot\left(cos(\pi\cdot t)+\frac{1}{9}\cdot cos(3\pi\cdot t)+\frac{1}{25}\cdot cos(5\pi\cdot t)+\cdots\right)$$

We check graphically.

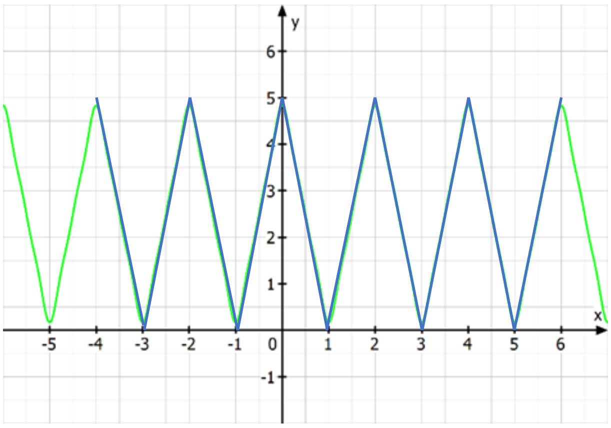
The individual functions $k=0,1,2,3$:







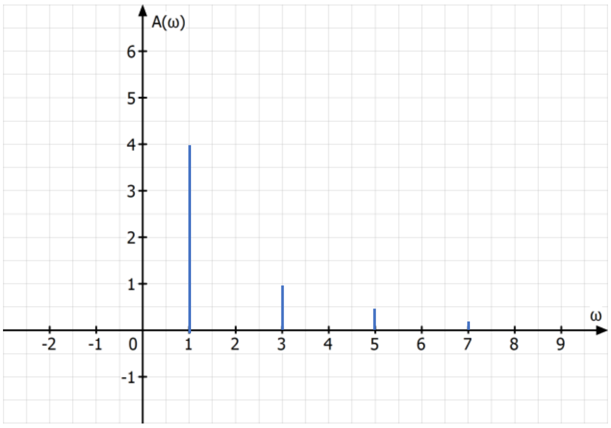
The sum of these functions compared with the original function:



By help of the Fourier series we can display the function in another format. Instead $x(t)$ we can display it by using $x(\omega_k)$

Necessary for this is that the amplitudes of the individual oscillations build a converging or Cauchy series.

We plot the spectra (not to scale …):



Note: For a complete description we need the phase spectrum too. Both together contain the complete information of the oscillation. For more information you may look at
https://www.sciencedirect.com/topics/engineering/phase-spectrum

Example 2

We use an odd function:

$$f(t) = t^3 - 3 \cdot t$$

Properties of the function:

$$f'(t) = 3 \cdot t^2 - 3$$

$$f''(t) = 6 \cdot t$$

$$f'''(t) = 6$$

Local extrema:

$$f'(t) = 0 \rightarrow t = \pm 1$$

Zeros:

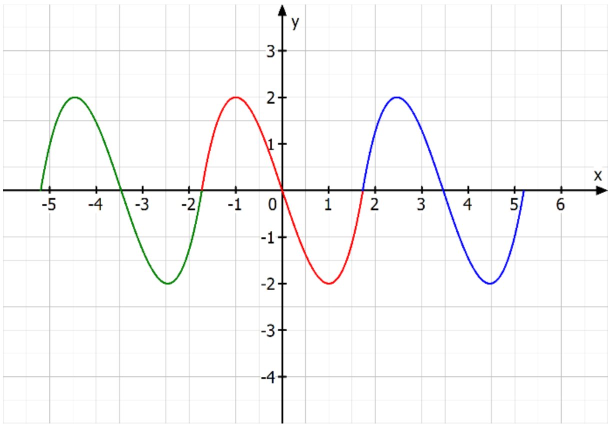
$$f(t) = 0 \rightarrow t = 0, t = \pm\sqrt{3}$$

Slope at zeros:

$$f'\left(-\sqrt{3}\right) = 6 = f'\left(\sqrt{3}\right)$$

The function has period $T = 2 \cdot \sqrt{3}$

The function has amplitude 2.



Note: The amplitude is half of the total swing of the function.

We need calculate B_k only because the function is odd.

We use:

$$\omega_k = \frac{2\pi}{T} k = \frac{\pi}{\sqrt{3}} k$$

We calculate B_k:

$$B_k = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_k \cdot t) dt \rightarrow$$

$$B_k = \frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3 \cdot t) \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt =$$

$$B_k = \frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt - \frac{3}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt$$

From Wikipedia we get (by replacing x with t and a with ω_k:

$$\int \quad t \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt = \frac{sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right)}{\left(\frac{\pi}{\sqrt{3}} k\right)^2} - \frac{t \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t\right)}{\frac{\pi}{\sqrt{3}} k} + C$$

$$\int \quad t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt$$

$$= -\left(\frac{t^3}{\frac{\pi}{\sqrt{3}} k} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) + 3 \cdot \frac{t^2}{\left(\frac{\pi}{\sqrt{3}} k\right)^2} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \frac{\pi}{2}\right) + 6 \cdot \frac{t}{\left(\frac{\pi}{\sqrt{3}} k\right)^3} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}} k\right)^4} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \frac{3}{2}\pi\right)\right) + C$$

We calculate the integrals one by one.

We take the scheme from Wikipedia.

Integral one:

$$\frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt =$$

$$-\frac{1}{\sqrt{3}} \cdot \left[\frac{t^3}{\frac{\pi}{\sqrt{3}} k} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) + 3 \cdot \frac{t^2}{\left(\frac{\pi}{\sqrt{3}} k\right)^2} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \frac{\pi}{2}\right) + 6 \cdot \frac{t}{\left(\frac{\pi}{\sqrt{3}} k\right)^3} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}} k\right)^4} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot t + \frac{3}{2}\pi\right)\right]_{-\sqrt{3}}^{\sqrt{3}} =$$

$$-\frac{1}{\sqrt{3}} \cdot \left(\left(\frac{\sqrt{3}^3}{\frac{\pi}{\sqrt{3}} k} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3}\right) + 3 \cdot \frac{\sqrt{3}^2}{\left(\frac{\pi}{\sqrt{3}} k\right)^2} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \frac{\pi}{2}\right) + 6 \cdot \frac{\sqrt{3}}{\left(\frac{\pi}{\sqrt{3}} k\right)^3} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}} k\right)^4} \cdot cos\left(\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \frac{3}{2}\pi\right)\right) - \left(-\frac{\sqrt{3}^3}{\frac{\pi}{\sqrt{3}} k} \cdot cos\left(-\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3}\right) + 3 \cdot \frac{\sqrt{3}^2}{\left(\frac{\pi}{\sqrt{3}} k\right)^2} \cdot cos\left(-\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \frac{\pi}{2}\right) - 6 \cdot \frac{\sqrt{3}}{\left(\frac{\pi}{\sqrt{3}} k\right)^3} \cdot cos\left(-\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}} k\right)^4} \cdot cos\left(-\frac{\pi}{\sqrt{3}} k \cdot \sqrt{3} + \frac{3}{2}\pi\right)\right)\right) =$$

$$-\frac{1}{\sqrt{3}}\cdot\left(\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)\right)-\left(-\frac{9}{\pi k}\cdot cos(-\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)-\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)\right)=$$

$$-\frac{1}{\sqrt{3}}\cdot\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)+\frac{9}{\pi k}\cdot cos(-\pi k)-\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)-\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)$$

We use $cos(x)=cos(-x)$:

$$-\frac{1}{\sqrt{3}}\cdot\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)+\frac{9}{\pi k}\cdot cos(-\pi k)-\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)-\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)$$

We use:

$cos\left(\pi k-\frac{\pi}{2}\right)=0$	$cos\left(\pi k+\frac{\pi}{2}\right)=0$	$cos\left(\pi k+\frac{3}{2}\pi\right)=0$	$cos\left(\pi k+\frac{5}{2}\pi\right)=0$
$cos(\pi k)=cos(-\pi k)$	$cos(\pi k+\pi)=\ cos(-\pi k+\pi)$		

We get:

$$-\frac{18}{\sqrt{3}}\cdot\left(\frac{1}{\pi k}\cdot cos(\pi k)+\frac{6}{(\pi k)^3}\cdot cos(\pi k+\pi)\right)$$

We calculate integral two:

$$-\frac{3}{\sqrt{3}}\cdot\int_{-\sqrt{3}}^{\sqrt{3}}t\cdot sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)dt=-\frac{3}{\sqrt{3}}\cdot\left[\frac{sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2}-\frac{t\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)}{\frac{\pi}{\sqrt{3}}k}\right]_{-\sqrt{3}}^{\sqrt{3}}=$$

$$-\frac{3}{\sqrt{3}}\cdot\left(\left(\frac{sin\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2}-\frac{\sqrt{3}\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)}{\frac{\pi}{\sqrt{3}}k}\right)-\left(\frac{sin\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2}+\frac{\sqrt{3}\cdot cos\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)}{\frac{\pi}{\sqrt{3}}k}\right)\right)=$$

$$-\frac{3}{\sqrt{3}}\cdot\left(\left(\frac{3\cdot sin(\pi k)}{(\pi k)^2}-\frac{3\cdot cos(\pi k)}{\pi k}\right)-\left(\frac{3\cdot sin(-\pi k)}{(\pi k)^2}+\frac{3\cdot cos(-\pi k)}{\pi k}\right)\right)=$$

$$-\frac{9}{\sqrt{3}}\cdot\left(\frac{sin(\pi k)}{(\pi k)^2}-\frac{cos(\pi k)}{\pi k}+\frac{sin(-\pi k)}{(\pi k)^2}-\frac{cos(-\pi k)}{\pi k}\right)$$

We use:

$$sin(\pi k)=sin(-\pi k)=0$$

We get:

$$-\frac{9}{\sqrt{3}}\cdot\left(-\frac{cos(\pi k)}{\pi k}-\frac{cos(-\pi k)}{\pi k}\right)=\frac{18}{\sqrt{3}}\cdot\frac{cos(\pi k)}{\pi k}$$

We remember:

$$B_k = \frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt - \frac{3}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) dt$$

We insert the integrals:

$$B_k = -\frac{18}{\sqrt{3}} \cdot \left(\frac{1}{\pi k} \cdot cos(\pi k) + \frac{6}{(\pi k)^3} \cdot cos(\pi k + \pi)\right) + \frac{18}{\sqrt{3}} \cdot \frac{cos(\pi k)}{\pi k} =$$

$$-\frac{18}{\sqrt{3}} \cdot \left(\frac{1}{\pi k} \cdot cos(\pi k) + \frac{6}{(\pi k)^3} \cdot cos(\pi k + \pi) - \frac{cos(\pi k)}{\pi k}\right) =$$

$$-\frac{108}{\sqrt{3}(\pi k)^3} \cdot cos(\pi k + \pi)$$

We use:

$$cos(\pi k + \pi) = -(-1)^k$$

We remember (for odd functions):

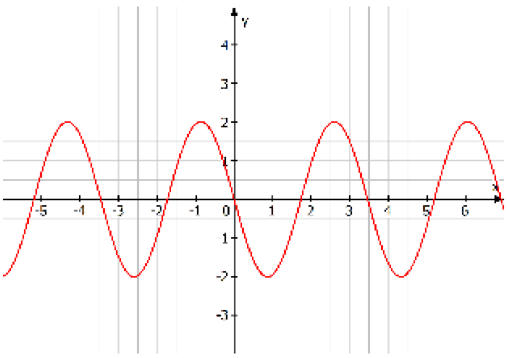
$$f(t) = \sum_{k=0}^{\infty} \left(B_k \cdot sin(\omega_k \cdot t)\right), k \in \{0,1,2, \dots\}$$

We harvest the results, using that $B_0 = 0$, $k = 1,2,3, \dots$, $\omega_k = \frac{\pi}{\sqrt{3}} k$.

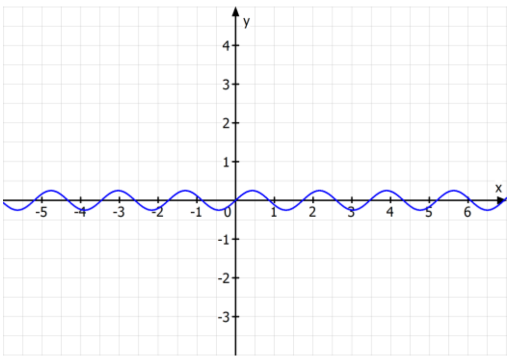
$$f(t) = \sum_{k=1}^{\infty} B_k \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right) \rightarrow$$

$$f(t) = \sum_{k=1}^{\infty} \left(\frac{108}{\sqrt{3}(\pi k)^3} \cdot (-1)^k \cdot sin\left(\frac{\pi}{\sqrt{3}} k \cdot t\right)\right)$$

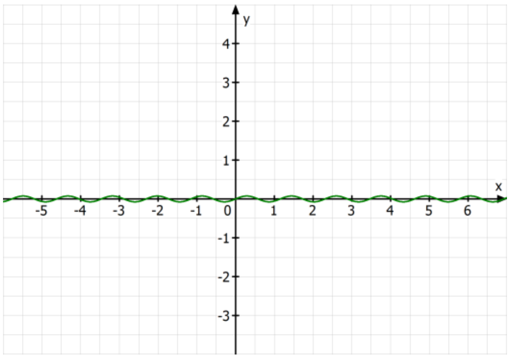
We plot the individual functions.

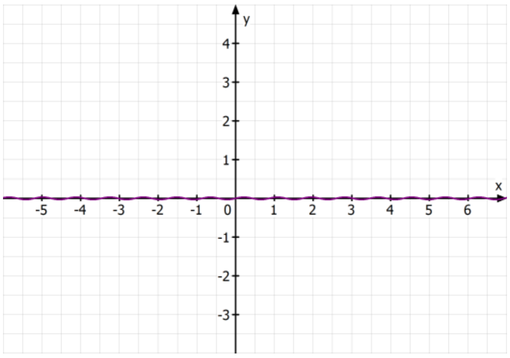


$k = 1$



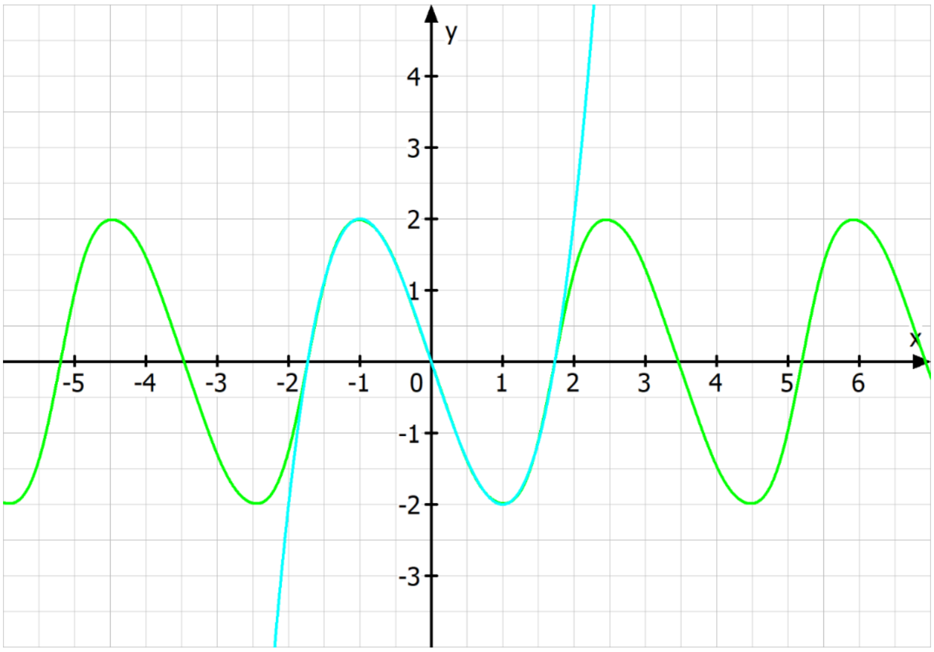
$k = 2$





$k = 3$ $k = 4$

We plot the sum (green) together with the original function (cyan, not clipped beyond the period interval):



Note: In contrast to the first example the approximation is smooth. The triangle function of the first example was continuous, but not differentiable at the vertices. This leads to irrepressible oscillations at these points.

We plot the spectra (not to scale …):