This paper introduces basic ideas of the Fourier transform. It follows:

https://math.stackexchange.com/questions/1002/fourier-transform-for-dummies

https://studyflix.de/mathematik/fourier-transformation-1471 (in German)

https://www.ocean.washington.edu/courses/ess522/lectures/02_fouriertransform.pdf

https://www.allaboutcircuits.com/technical-articles/an-introduction-to-the-discrete-fourier-transform/

Hope I can help you with learning Quantum mechanics.

Contents

General

The Fourier transform constructs (replaces)a periodic function $f(t)$ with period T by help of sin and cos functions:

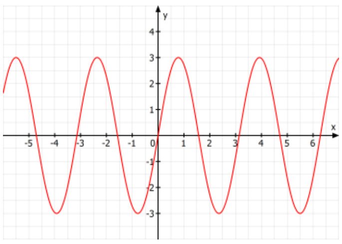
$$f(t) = \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t) \right), k \in \{0,1,2,\dots\}$$

A_k, B_k are amplitudes, intensities.

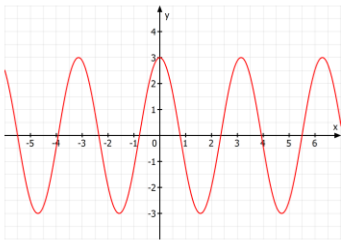
ω_k are frequencies:

$$\omega_k = \frac{2\pi}{T} k, k \in \{0,1,2,\dots\}$$

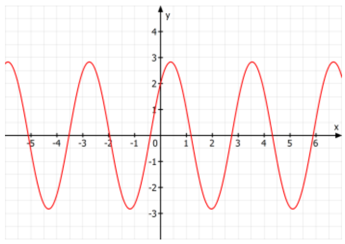
We distinguish between periodic functions:



odd



even



neither nor

Even or odd functions are easier to transform.

We want to calculate the amplitudes belonging to different numbers k.

We multiply $f(t)$ by $cos(\omega_{k'} \cdot t)$:

$$f(t) \cdot cos(\omega_{k'} \cdot t) = \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t) \right) \cdot cos(\omega_{k'} \cdot t), k \in \{0,1,2,\dots\}$$

We integrate:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos(\omega_{k'} \cdot t) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t) \right) \cdot cos(\omega_{k'} \cdot t)\, dt$$

We use orthogonality relations for sin and cos:

$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot cos(\omega_{k'} \cdot t) dt$	$=$	$\begin{cases} 0 \;\; for \quad k \neq k' \\ \frac{T}{2} \;\; for\; 0 \neq k = k' \\ T \;\; for\; 0 = k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} sin(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t) dt$	$=$	$\begin{cases} 0 \;\; for \quad k \neq k', k = 0 \; or \; k' = 0 \\ \frac{T}{2} \;\; for \quad 0 \neq k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t) dt$	$=$	0

Using the orthogonality relations, we see:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} A_k\cdot cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t) + B_k\cdot sin(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)\,dt$$

The sum vanishes because there is exactly one term that is not zero depending on k'.

The products of sin and cos are zero altogether.

We get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} A_k\cdot cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)dt$$

Result:

$k = k' \neq 0$	$k = k' = 0$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt = A_k\cdot\frac{T}{2}$	$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt = A_0\cdot T$

We get the amplitudes:

$A_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt$	$A_0 = \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$

Note: $k = k'$

The same way we calculate B_k using $sin(\omega_{k'}\cdot t)$.

We multiply $f(t)$ with $sin(\omega_{k'}\cdot t)$:

$$f(t)\cdot sin(\omega_{k'}\cdot t) = \sum_{k=0}^{\infty}\big(A_k\cdot cos(\omega_k\cdot t) + B_k\cdot sin(\omega_k\cdot t)\big)\cdot sin(\omega_{k'}\cdot t),\, k\in\{0,1,2,\dots\}$$

We integrate:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty}\big(A_k\cdot cos(\omega_k\cdot t) + B_k\cdot sin(\omega_k\cdot t)\big)\cdot sin(\omega_{k'}\cdot t)\,dt$$

We use orthogonality relations:

$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k\cdot t)\cdot cos(\omega_{k'}\cdot t)dt$	$=$	$\begin{cases} 0 \;\; for \;\; k\neq k' \\ \frac{T}{2} \;\; for \; 0\neq k = k' \\ T \;\; for \; 0 = k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} sin(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)dt$	$=$	$\begin{cases} 0 \;\; for \;\; k\neq k', k = 0 \; or \; k' = 0 \\ \frac{T}{2} \;\; for \;\; 0\neq k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)dt$	$=$	0

Using the orthogonality relations, we see:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=0}^{\infty} A_k\cdot cos(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t) + B_k\cdot sin(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)\, dt$$

The sum vanishes because there is exactly one term that is not zero depending on $k = k'$.

The products of sin and cos are zero altogether.

We get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_{k'}\cdot t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} B_k\cdot sin(\omega_k\cdot t)\cdot sin(\omega_{k'}\cdot t)dt$$

Result:

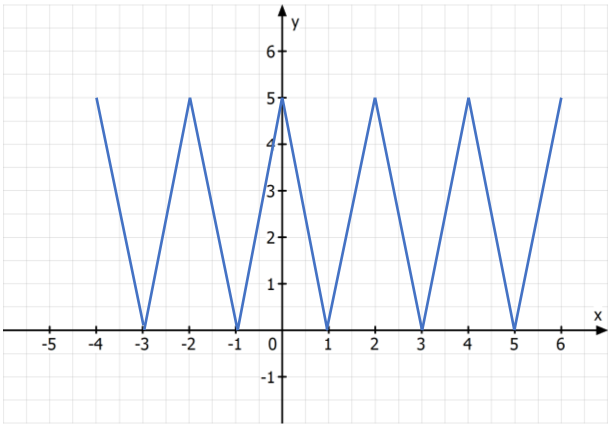
$k = k' \neq 0$	$k = k' = 0$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_k\cdot t)dt = B_k\cdot\frac{T}{2}$	0

We get the amplitudes:

$B_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_k\cdot t)dt$	$B_0 = 0$

Example 1: The triangle function

The even triangle function with period $T = 2$ and amplitude $\frac{5}{2}$.



Note: The amplitude is half of the total swing of the function.

We use the definition:

$$f(t) = \begin{cases} 5\cdot\left(1+\frac{2}{T}\cdot t\right) & for -\frac{T}{2}\leq t\leq 0 \\ 5\cdot\left(1-\frac{2}{T}\cdot t\right) & for\ 0\leq t\leq\frac{T}{2}\end{cases}$$

This is an even function.

We need calculate A_k only, because the product of an even and an odd function gives an odd function.

We use the triangle function with $T = 2$ to calculate A_k:

$$f(t) = \begin{cases} 5 \cdot (1+t) \; for -1 \le t \le 0 \\ 5 \cdot (1-t) \quad for \; 0 \le t \le 1 \end{cases}$$

We calculate A_k:

$$A_k = \frac{2}{2} \cdot \int_{-1}^{1} f(t) \cdot cos(\omega_k \cdot t)dt = \int_{-1}^{0} 5 \cdot (1+t) \cdot cos(\omega_k \cdot t)dt + \int_{0}^{1} 5 \cdot (1-t) \cdot cos(\omega_k \cdot t)dt =$$

$$5\int_{-1}^{0} cos(\omega_k \cdot t)dt + 5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt + 5\int_{0}^{1} cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt =$$

$$5\int_{-1}^{1} cos(\omega_k \cdot t)dt + 5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt$$

We calculate part two:

$$5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt$$

We take a look at Wikipedia: *https://en.wikipedia.org/wiki/List_of_integrals_of_trigonometric_functions*

$$\int x \cdot cos(a \cdot x)dx = \frac{1}{a^2} \cdot cos(a \cdot x) + \frac{1}{a} \cdot x \cdot sin(a \cdot x)$$

We apply, omitting the factor 5:

$$\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt = \left[\frac{1}{{\omega_k}^2} \cdot cos(\omega_k \cdot t) + \frac{1}{\omega_k} \cdot t \cdot sin(\omega_k \cdot t)\right]_{-1}^{0} =$$

$$\left(\frac{1}{{\omega_k}^2} \cdot cos(0) + \frac{1}{\omega_k} \cdot 0 \cdot sin(0)\right) - \left(\frac{1}{{\omega_k}^2} \cdot cos(-\omega_k) - \frac{1}{\omega_k} \cdot sin(-\omega_k)\right) =$$

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(-\omega_k) + \frac{1}{\omega_k} \cdot sin(-\omega_k)$$

We note that $cos(x) = cos(-x)$ and $sin(-x) = -sin(x)$

We get:

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k)$$

$$\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt = \left[\frac{1}{{\omega_k}^2} \cdot cos(\omega_k \cdot t) + \frac{1}{\omega_k} \cdot t \cdot sin(\omega_k \cdot t)\right]_{0}^{1} =$$

$$\frac{1}{{\omega_k}^2} \cdot cos(\omega_k) + \frac{1}{\omega_k} \cdot sin(\omega_k) - \frac{1}{{\omega_k}^2}$$

We build the difference:

$$\frac{1}{{\omega_k}^2} - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k) - \frac{1}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{1}{\omega_k} \cdot sin(\omega_k) + \frac{1}{{\omega_k}^2} =$$

$$\frac{2}{{\omega_k}^2} - \frac{2}{{\omega_k}^2} \cdot cos(\omega_k) - \frac{2}{\omega_k} \cdot sin(\omega_k) = \frac{2}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{2}{\omega_k} \cdot sin(\omega_k)$$

We multiply the factor 5 and get the result for part two:

$$5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{10}{\omega_k} \cdot sin(\omega_k)$$

We calculate part one:

$$5\int_{-1}^{1} cos(\omega_k \cdot t)dt = 5\left[\frac{1}{\omega_k} \cdot sin(\omega_k \cdot t)\right]_{-1}^{1} = 5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) - 5 \cdot \frac{1}{\omega_k} \cdot sin(-\omega_k) =$$

$$5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) + 5 \cdot \frac{1}{\omega_k} \cdot sin(\omega_k) = \frac{10}{\omega_k} \cdot sin(\omega_k)$$

We build the sum of part one and part two:

$$5\int_{-1}^{1} cos(\omega_k \cdot t)dt + 5\int_{-1}^{0} t \cdot cos(\omega_k \cdot t)dt - 5\int_{0}^{1} t \cdot cos(\omega_k \cdot t)dt =$$

$$\frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big) - \frac{10}{\omega_k} \cdot sin(\omega_k) + \frac{10}{\omega_k} \cdot sin(\omega_k) = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big)$$

We get A_k:

$$A_k = \frac{10}{{\omega_k}^2}\big(1 - cos(\omega_k)\big)$$

We expand ω_k and use $T = 2$:

$$\omega_k = \pi k, k \in \{0,1,2, \dots\}$$

We get:

$$A_k = \frac{10}{(\pi k)^2}\big(1 - cos(\pi k)\big)$$

For $k = \{2,4,6, \dots\}$ we get $A_k = 0$

For $k = 1,3,5, \dots$ we get $A_k = \frac{20}{(\pi k)^2}$ because the cos for these numbers is -1.

We need $A_{k=0}$:

$$A_{k=0} = \frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot cos(0 \cdot t)dt = \frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot dt$$

We use our function $f(t)$:

$$f(t) = \begin{cases} 5 \cdot (1 + t) & for -1 \le t \le 0 \\ 5 \cdot (1 - t) & for\ 0 \le t \le 1 \end{cases}$$

We calculate:

$$\frac{1}{2} \cdot \int_{-1}^{1} f(t) \cdot dt = \frac{5}{2} \cdot \left(\int_{-1}^{0} (1 + t)dt + \int_{0}^{1} (1 - t)dt\right) =$$

$$\frac{5}{2}\cdot\left(\left[t+\frac{t^2}{2}\right]_{-1}^{0}+\left[t-\frac{t^2}{2}\right]_{0}^{1}\right)=\frac{5}{2}\cdot\left(0-\left(-1+\frac{1}{2}\right)+1-\frac{1}{2}\right)=$$

$$\frac{5}{2}\cdot\left(1-\frac{1}{2}+1-\frac{1}{2}\right)=\frac{5}{2}$$

We get $A_{k=0}$:

$$A_{k=0}=\frac{5}{2}$$

We remember:

$$f(t)=\sum_{k=0}^{\infty}\big(A_k\cdot cos(\omega_k\cdot t)+B_k\cdot sin(\omega_k\cdot t)\big)\,,k\in\{0,1,2,\dots\}$$

We harvest the results:

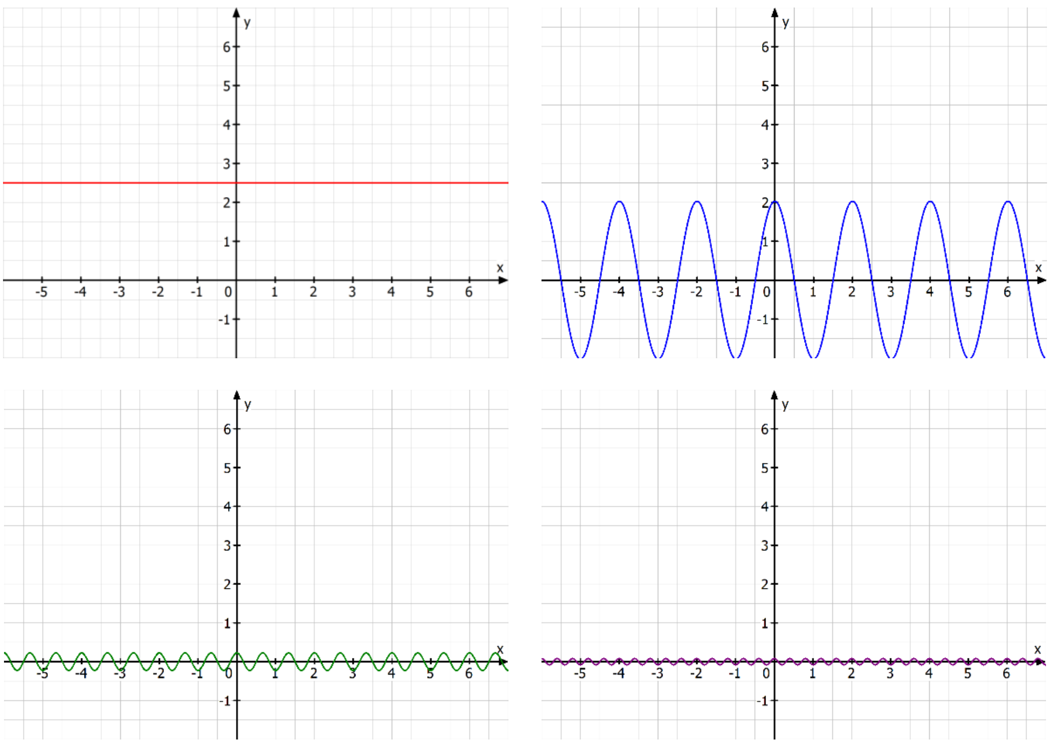
$$f(t)=A_0+\sum_{k=1}^{\infty}A_k\cdot cos(\omega_k\cdot t)\,,k\in\{1,2,\dots\}$$

$$f(t)=\frac{5}{2}+\sum_{k=1}^{\infty}\frac{20}{(\pi k)^2}\cdot cos(\pi k\cdot t)\,,k\in\{1,3,5,\dots\}=$$

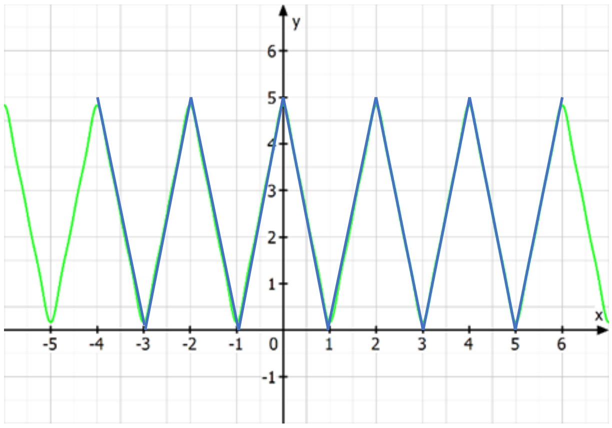
$$\frac{5}{2}+\frac{20}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot cos(\pi k\cdot t)\,,k\in\{1,3,5,\dots\}=$$

$$\frac{5}{2}+\frac{20}{\pi^2}\cdot\left(cos(\pi\cdot t)+\frac{1}{9}\cdot cos(3\pi\cdot t)+\frac{1}{25}\cdot cos(5\pi\cdot t)+\cdots\right)$$

The individual functions $k=0,1,2,3$:



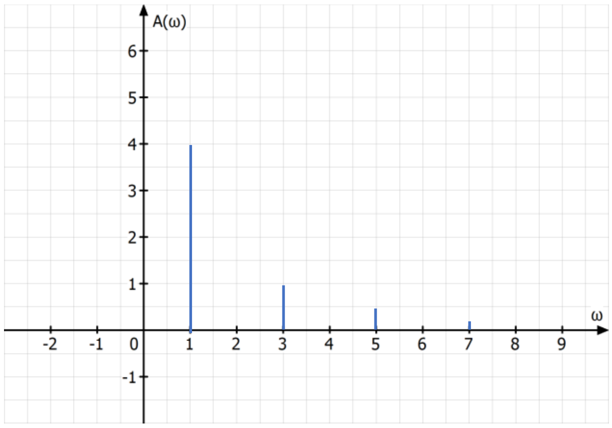
The sum of these functions compared with the original function:



By help of the Fourier series we can display the function in another format. Instead $x(t)$ we can display it by using $x(\omega_k)$

Necessary for this is that the amplitudes of the individual oscillations build a converging or Cauchy series.

We plot the spectra (not to scale …):



Note: For a complete description we need the phase spectrum too. Both together contain the complete information of the oscillation. For more information you may look at https://www.sciencedirect.com/topics/engineering/phase-spectrum

Example 2: An odd function

We use:

$$f(t) = t^3 - 3 \cdot t$$

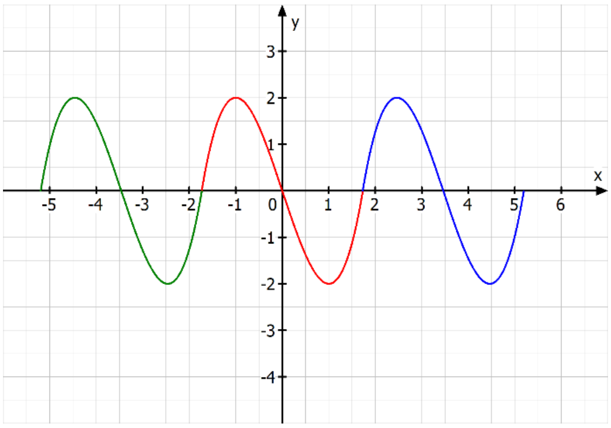
Zeros:

$$f(t) = 0 \rightarrow t = 0, t = \pm\sqrt{3}$$

Slope at zeros:

$$f'(-\sqrt{3}) = 6 = f'(\sqrt{3})$$

The function has period $T = 2 \cdot \sqrt{3}$ and amplitude 2.



Note: The amplitude is half of the total swing of the function.

We need calculate B_k only because the function is odd.

We use:

$$\omega_k = \frac{2\pi}{T}k = \frac{\pi}{\sqrt{3}}k$$

We calculate B_k:

$$B_k = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin(\omega_k \cdot t)dt \rightarrow$$

$$B_k = \frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3 \cdot t) \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) dt =$$

$$B_k = \frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) dt - \frac{3}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) dt$$

From Wikipedia we get (by replacing x with t and a with ω_k:

$$\int \quad t \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) dt = \frac{sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} - \frac{t \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)}{\frac{\pi}{\sqrt{3}}k} + C$$

$$\int t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)dt$$
$$= -\left(\frac{t^3}{\frac{\pi}{\sqrt{3}}k} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) + 3 \cdot \frac{t^2}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \frac{\pi}{2}\right) + 6 \cdot \frac{t}{\left(\frac{\pi}{\sqrt{3}}k\right)^3}\right.$$
$$\left. \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}}k\right)^4} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \frac{3}{2}\pi\right)\right) + C$$

We calculate the integrals one by one, using the scheme from Wikipedia.

We calculate integral one:

$$-\frac{3}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)dt = -\frac{3}{\sqrt{3}} \cdot \left[\frac{sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} - \frac{t \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)}{\frac{\pi}{\sqrt{3}}k}\right]_{-\sqrt{3}}^{\sqrt{3}} =$$

$$-\frac{3}{\sqrt{3}} \cdot \left(\left(\frac{sin\left(\frac{\pi}{\sqrt{3}}k \cdot \sqrt{3}\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} - \frac{\sqrt{3} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot \sqrt{3}\right)}{\frac{\pi}{\sqrt{3}}k}\right) - \left(\frac{sin\left(-\frac{\pi}{\sqrt{3}}k \cdot \sqrt{3}\right)}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} + \frac{\sqrt{3} \cdot cos\left(-\frac{\pi}{\sqrt{3}}k \cdot \sqrt{3}\right)}{\frac{\pi}{\sqrt{3}}k}\right)\right) =$$

$$-\frac{3}{\sqrt{3}} \cdot \left(\left(\frac{3 \cdot sin(\pi k)}{(\pi k)^2} - \frac{3 \cdot cos(\pi k)}{\pi k}\right) - \left(\frac{3 \cdot sin(-\pi k)}{(\pi k)^2} + \frac{3 \cdot cos(-\pi k)}{\pi k}\right)\right) =$$

$$-\frac{9}{\sqrt{3}} \cdot \left(\frac{sin(\pi k)}{(\pi k)^2} - \frac{cos(\pi k)}{\pi k} + \frac{sin(-\pi k)}{(\pi k)^2} - \frac{cos(-\pi k)}{\pi k}\right)$$

We use:

$$sin(\pi k) = sin(-\pi k) = 0$$

We get:

$$-\frac{9}{\sqrt{3}} \cdot \left(-\frac{cos(\pi k)}{\pi k} - \frac{cos(-\pi k)}{\pi k}\right) = \frac{18}{\sqrt{3}} \cdot \frac{cos(\pi k)}{\pi k}$$

We calculate integral two:

$$\frac{1}{\sqrt{3}} \cdot \int_{-\sqrt{3}}^{\sqrt{3}} t^3 \cdot sin\left(\frac{\pi}{\sqrt{3}}k \cdot t\right)dt =$$

$$-\frac{1}{\sqrt{3}} \cdot \left[\frac{t^3}{\frac{\pi}{\sqrt{3}}k} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t\right) + 3 \cdot \frac{t^2}{\left(\frac{\pi}{\sqrt{3}}k\right)^2} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \frac{\pi}{2}\right) + 6 \cdot \frac{t}{\left(\frac{\pi}{\sqrt{3}}k\right)^3} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \pi\right) + \frac{1}{\left(\frac{\pi}{\sqrt{3}}k\right)^4} \cdot cos\left(\frac{\pi}{\sqrt{3}}k \cdot t + \frac{3}{2}\pi\right)\right]_{-\sqrt{3}}^{\sqrt{3}} =$$

$$-\frac{1}{\sqrt{3}}\cdot\left(\left(\frac{\sqrt{3}^3}{\frac{\pi}{\sqrt{3}}k}\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)+3\cdot\frac{\sqrt{3}^2}{\left(\frac{\pi}{\sqrt{3}}k\right)^2}\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\frac{\pi}{2}\right)+6\cdot\frac{\sqrt{3}}{\left(\frac{\pi}{\sqrt{3}}k\right)^3}\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\pi\right)+\frac{1}{\left(\frac{\pi}{\sqrt{3}}k\right)^4}\cdot cos\left(\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\frac{3}{2}\pi\right)\right)\right.$$
$$\left.-\left(-\frac{\sqrt{3}^3}{\frac{\pi}{\sqrt{3}}k}\cdot cos\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}\right)+3\cdot\frac{\sqrt{3}^2}{\left(\frac{\pi}{\sqrt{3}}k\right)^2}\cdot cos\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\frac{\pi}{2}\right)-6\cdot\frac{\sqrt{3}}{\left(\frac{\pi}{\sqrt{3}}k\right)^3}\cdot cos\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\pi\right)+\frac{1}{\left(\frac{\pi}{\sqrt{3}}k\right)^4}\cdot cos\left(-\frac{\pi}{\sqrt{3}}k\cdot\sqrt{3}+\frac{3}{2}\pi\right)\right)\right)=$$

$$-\frac{1}{\sqrt{3}}\cdot\left(\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)\right)\right.$$
$$\left.-\left(-\frac{9}{\pi k}\cdot cos(-\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)-\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)\right)=$$

$$-\frac{1}{\sqrt{3}}\cdot\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)+\frac{9}{\pi k}\cdot cos(-\pi k)-\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)-\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)$$

We use $cos(x) = cos(-x)$:

$$-\frac{1}{\sqrt{3}}\cdot\left(\frac{9}{\pi k}\cdot cos(\pi k)+\frac{27}{(\pi k)^2}\cdot cos\left(\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(\pi k+\pi)+\frac{9}{(\pi k)^4}\cdot cos\left(\pi k+\frac{3}{2}\pi\right)+\frac{9}{\pi k}\cdot cos(-\pi k)-\frac{27}{(\pi k)^2}\cdot cos\left(-\pi k+\frac{\pi}{2}\right)+\frac{54}{(\pi k)^3}\cdot cos(-\pi k+\pi)-\frac{9}{(\pi k)^4}\cdot cos\left(-\pi k+\frac{3}{2}\pi\right)\right)$$

We use:

$cos\left(\pi k-\frac{\pi}{2}\right)=0$	$cos\left(\pi k+\frac{\pi}{2}\right)=0$	$cos\left(\pi k+\frac{3}{2}\pi\right)=0$	$cos\left(\pi k+\frac{5}{2}\pi\right)=0$
$cos(\pi k)=cos(-\pi k)$	$cos(\pi k+\pi)=\ cos(-\pi k+\pi)$		

We get:

$$-\frac{18}{\sqrt{3}}\cdot\left(\frac{1}{\pi k}\cdot cos(\pi k)+\frac{6}{(\pi k)^3}\cdot cos(\pi k+\pi)\right)$$

We remember:

$$B_k=\frac{1}{\sqrt{3}}\cdot\int_{-\sqrt{3}}^{\sqrt{3}}t^3\cdot sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)dt-\frac{3}{\sqrt{3}}\cdot\int_{-\sqrt{3}}^{\sqrt{3}}t\cdot sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)dt$$

We insert the integrals:

$$B_k=-\frac{18}{\sqrt{3}}\cdot\left(\frac{1}{\pi k}\cdot cos(\pi k)+\frac{6}{(\pi k)^3}\cdot cos(\pi k+\pi)\right)+\frac{18}{\sqrt{3}}\cdot\frac{cos(\pi k)}{\pi k}=$$

$$-\frac{18}{\sqrt{3}}\cdot\left(\frac{1}{\pi k}\cdot cos(\pi k)+\frac{6}{(\pi k)^3}\cdot cos(\pi k+\pi)-\frac{cos(\pi k)}{\pi k}\right)=$$

$$-\frac{108}{\sqrt{3}(\pi k)^3}\cdot cos(\pi k+\pi)$$

We use:

$$cos(\pi k+\pi)=-(-1)^k$$

We remember (for odd functions):

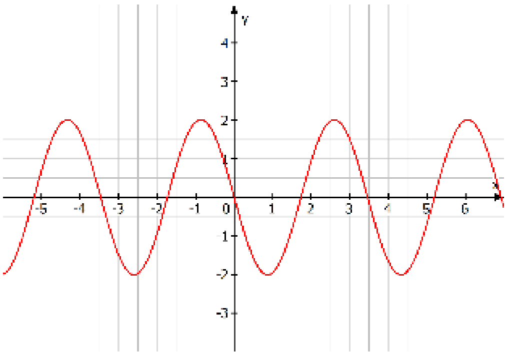
$$f(t)=\sum_{k=0}^{\infty}\left(B_k\cdot sin(\omega_k\cdot t)\right),k\in\{0,1,2,\dots\}$$

We harvest the results, using that $B_0=0$, $k=1,2,3,\dots$, $\omega_k=\frac{\pi}{\sqrt{3}}k$.

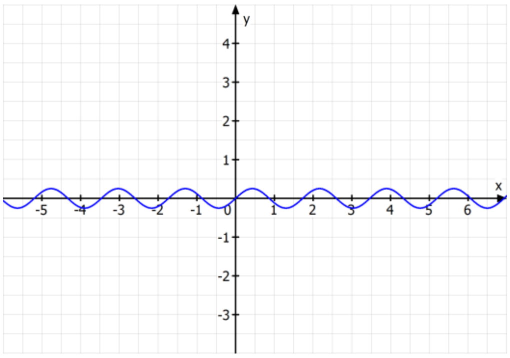
$$f(t)=\sum_{k=1}^{\infty}B_k\cdot sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)\rightarrow$$

$$f(t)=\sum_{k=1}^{\infty}\left(\frac{108}{\sqrt{3}(\pi k)^3}\cdot(-1)^k\cdot sin\left(\frac{\pi}{\sqrt{3}}k\cdot t\right)\right)$$

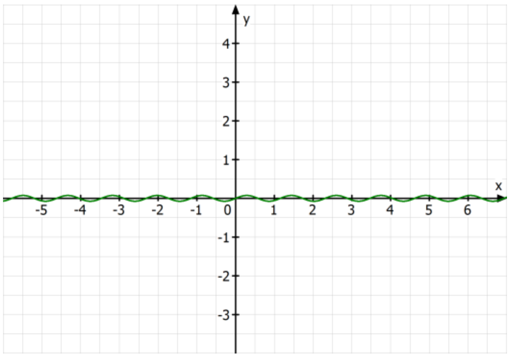
We plot the individual functions.



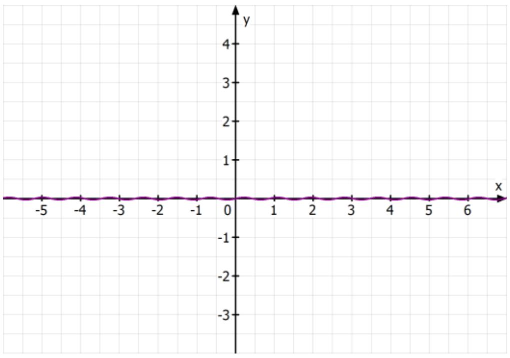
$k=1$



$k=2$

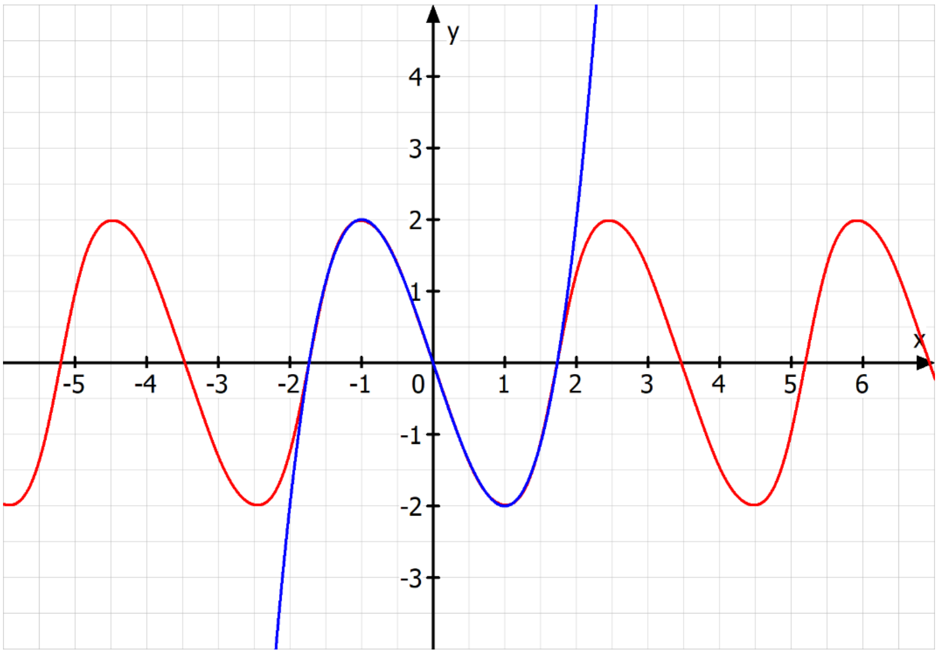


$k=3$



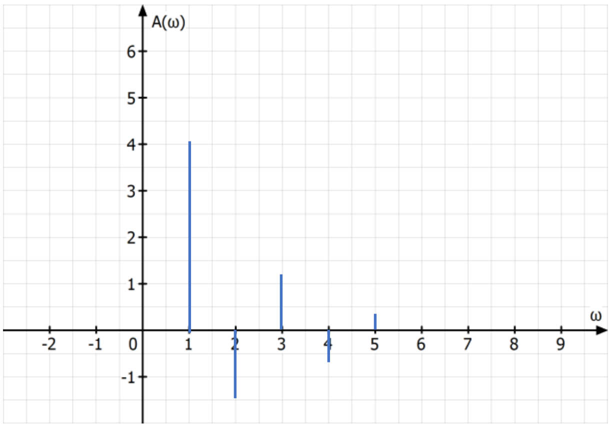
$k=4$

We plot the sum (red) together with the original function (blue, not clipped beyond the period interval):



Note: In contrast to the first example the approximation is smooth. The triangle function of the first example was continuous, but not differentiable at the vertices. This leads to irrepressible oscillations at these points.

We plot the spectra (not to scale …):



Exponential style

We remember we want to disassemble a function into series of sin and cos functions:

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(A_k \cdot cos(\omega_k \cdot t) + B_k \cdot sin(\omega_k \cdot t)\right), k \in \{0,1,2,\ldots\}$$

We remember ω_k:

$$\omega_k = \frac{2\pi}{T} \cdot k$$

We replace sin and cos by their complex counterparts:

$$sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Omitting the term $\frac{A_0}{2}$ we get the sum:

$$\sum_{k=1}^{\infty} \left(A_k \cdot \frac{e^{i\omega_k \cdot t} + e^{-i\omega_k \cdot t}}{2} + B_k \cdot \frac{e^{i\omega_k \cdot t} - e^{-i\omega_k \cdot t}}{2i}\right) =$$

$$\sum_{k=1}^{\infty} \left(A_k \cdot \frac{e^{i\omega_k \cdot t} + e^{-i\omega_k \cdot t}}{2} - i \cdot B_k \cdot \frac{e^{i\omega_k \cdot t} - e^{-i\omega_k \cdot t}}{2}\right) =$$

$$\sum_{k=1}^{\infty} \left(A_k \cdot \frac{e^{i\omega_k \cdot t}}{2} + A_k \cdot \frac{e^{-i\omega_k \cdot t}}{2} - i \cdot B_k \cdot \frac{e^{i\omega_k \cdot t}}{2} + i \cdot B_k \cdot \frac{e^{-i\omega_k \cdot t}}{2}\right) =$$

$$\frac{1}{2} \cdot \sum_{k=1}^{\infty} \left((A_k - i \cdot B_k) \cdot e^{i\omega_k \cdot t} + (A_k + i \cdot B_k) \cdot e^{-i\omega_k \cdot t}\right) =$$

$$\frac{1}{2} \cdot \sum_{k=1}^{\infty} \left((A_k - i \cdot B_k) \cdot e^{i\omega_k \cdot t}\right) + \frac{1}{2} \cdot \sum_{k=1}^{\infty} \left((A_k + i \cdot B_k) \cdot e^{-i\omega_k \cdot t}\right) =$$

$$\frac{1}{2} \cdot \sum_{k=1}^{\infty} \left((A_k - i \cdot B_k) \cdot e^{i\omega_k \cdot t}\right) + \frac{1}{2} \cdot \sum_{k=-\infty}^{-1} \left((A_{-k} + i \cdot B_{-k}) \cdot e^{i\omega_k \cdot t}\right)$$

Note: With $k \rightarrow -k$ we have $\omega_k \rightarrow -\omega_k$.

We can combine both sums into one, using $\frac{A_0}{2}$ for $k = 0$ and write:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{i\omega_k \cdot t}$$

$$C_k = \begin{cases} \frac{1}{2}(A_k - i \cdot B_k), & k > 0 \\ \frac{A_0}{2}, & k = 0 \\ \frac{1}{2}(A_k + i \cdot B_k), & k < 0 \end{cases}$$

We check whether the orthogonality relations for sin and cos are valid for their exponential counterparts:

$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot cos(\omega_{k'} \cdot t)dt$	=	$\begin{cases} 0 \ for \quad k \neq k' \\ \frac{T}{2} \ for\ 0 \neq k = k' \\ T \ for\ 0 = k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} sin(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t)dt$	=	$\begin{cases} 0 \ for \quad k \neq k', k = 0\ or\ k' = 0 \\ \frac{T}{2} \ for \quad 0 \neq k = k' \end{cases}$
$\int_{-\frac{T}{2}}^{\frac{T}{2}} cos(\omega_k \cdot t) \cdot sin(\omega_{k'} \cdot t)dt$	=	0

We only check the first one:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{e^{i\omega_k \cdot t} + e^{-i\omega_k \cdot t}}{2} \cdot \frac{e^{i\omega_{k'} \cdot t} + e^{-i\omega_{k'} \cdot t}}{2} dt =$$

$$\frac{1}{4} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(e^{i\omega_k \cdot t} + e^{-i\omega_k \cdot t}\right) \cdot \left(e^{i\omega_{k'} \cdot t} + e^{-i\omega_{k'} \cdot t}\right) dt =$$

$$\frac{1}{4} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot (\omega_k + \omega_{k'}) \cdot t} + e^{i \cdot (\omega_k - \omega_{k'}) \cdot t} + e^{-i \cdot (\omega_k - \omega_{k'}) \cdot t} + e^{-i \cdot (\omega_k + \omega_{k'}) \cdot t} dt =$$

We remember ω_k:

$$\omega_k = \frac{2\pi}{T} \cdot k$$

We omit the factor $1/_4$ and get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot (k+k') \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot (k-k') \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i \cdot (k-k') \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i \cdot (k+k') \cdot \frac{2\pi}{T} \cdot t} dt =;$$

We use that k and k' are integers and want to examine the cases $k = k'$ and $k \neq k'$.

Case $k = k' \rightarrow k + k' \sim 2 \cdot n, k - k' = 0$.

We get:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot (2 \cdot n) \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot 0 \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i \cdot 0 \cdot \frac{2\pi}{T} \cdot t} dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i \cdot (2 \cdot n) \cdot \frac{2\pi}{T} \cdot t} dt$$

We calculate the integrals:

Integral one:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i \cdot (2n) \cdot \frac{2\pi}{T} \cdot t} dt = \frac{T}{i \cdot 2\pi \cdot 2n} \left[e^{i \cdot (2n) \cdot \frac{2\pi}{T} \cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{T}{i \cdot 2\pi \cdot 2n} \left(e^{i2n\pi} - e^{-i2n\pi}\right) = 0$$

Integral two:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\cdot 0\cdot\frac{2\pi}{T}\cdot t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{0}dt = T$$

Integral three:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot 0\cdot\frac{2\pi}{T}\cdot t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{0}dt = T$$

Integral four:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot(2n)\cdot\frac{2\pi}{T}\cdot t}dt = -\frac{T}{i\cdot 2\pi\cdot 2n}\left[e^{-i\cdot(2n)\cdot\frac{2\pi}{T}\cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{T}{i\cdot 2\pi\cdot 2n}\left(e^{-i2n\pi} - e^{i2n\pi}\right) = 0$$

We get the sum (adding the factor $1/_4$):

$$\frac{T}{2}$$

Case $k \neq k' \to k + k' \sim n,\ k - k' = m \neq 0$.

This leads to the following behavior:

$k + k' = 2n - 1 = odd$	$k + k' = 2n = even$
$k' = 2n - 1 - k$	$k' = 2n - k$
$k - k' = k - 2n + 1 + k = 2(k - n) + 1$	$k - k' = k - 2n + k = 2(k - n)$
$k - k' = m = odd$	$k - k' = m = even$

Case $\boldsymbol{k + k' = odd}$:

We get the integrals:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\cdot m\cdot\frac{2\pi}{T}\cdot t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot m\cdot\frac{2\pi}{T}\cdot t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot n\cdot\frac{2\pi}{T}\cdot t}dt =$$

We calculate integrals:

Integral one:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}dt = \frac{T}{i\cdot 2\pi\cdot n}\left[e^{i\cdot n\cdot\frac{2\pi}{T}\cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{T}{i\cdot 2\pi\cdot n}\left(e^{in\pi} - e^{-in\pi}\right)$$

Integral four:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot n\cdot\frac{2\pi}{T}\cdot t}dt = -\frac{T}{i\cdot 2\pi\cdot n}\left[e^{-i\cdot n\cdot\frac{2\pi}{T}\cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{T}{i\cdot 2\pi\cdot n}\left(e^{-in\pi} - e^{in\pi}\right)$$

Integral two:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i\cdot m\cdot\frac{2\pi}{T}\cdot t}dt = \frac{T}{i\cdot 2\pi\cdot m}\left[e^{i\cdot m\cdot\frac{2\pi}{T}\cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{T}{i\cdot 2\pi\cdot m}\left(e^{im\pi} - e^{-im\pi}\right)$$

Integral three:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot m\cdot\frac{2\pi}{T}\cdot t}dt = -\frac{T}{i\cdot 2\pi\cdot m}\left[e^{-i\cdot m\cdot\frac{2\pi}{T}\cdot t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{T}{i\cdot 2\pi\cdot m}\left(e^{-im\pi} - e^{im\pi}\right)$$

We remember: $e^{in\pi} = e^{-in\pi} = e^{im\pi} = e^{-im\pi} = -1$ for odd numbers n, m. All integrals are zero.

Case $k + k' = even$

For even numbers we have $e^{in\pi} = e^{-in\pi} = e^{im\pi} = e^{-im\pi} = +1$. The integrals are zero too. From this we conclude without further proof that the orthogonality relations for sin and cos hold if we express the trigonometric functions by their exponential counterpart:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\cdot n\cdot\frac{2\pi}{T}\cdot t}e^{i\cdot m\cdot\frac{2\pi}{T}\cdot t}dt = T\delta_{mn}$$

We write $f(t)$ as a complex function:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k\cdot e^{i\omega_k\cdot t}$$

$$C_k = \begin{cases} \frac{1}{2}(A_k - i\cdot B_k), k > 0 \\ \frac{A_0}{2}, \quad k = 0 \\ \frac{1}{2}(A_k + i\cdot B_k), k < 0 \end{cases}$$

We remember the coefficients A_k and B_k:

$A_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt$	$A_0 = \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$
$B_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_k\cdot t)dt$	$B_0 = 0$

We replace the expressions for sin and cos by the complex exponential functions.

We remember:

$$\omega_k = \frac{2\pi}{T}\cdot k$$

We get:

$A_k = \frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot cos(\omega_k\cdot t)dt \rightarrow$	$i\cdot B_k = i\cdot\frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot sin(\omega_k\cdot t)dt \rightarrow$
$\frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\frac{e^{i\omega_k\cdot t} + e^{-i\omega_k\cdot t}}{2}dt =$	$i\cdot\frac{2}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\frac{e^{i\omega_k\cdot t} - e^{-i\omega_k\cdot t}}{2i}dt =$
$\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{i\frac{2\pi}{T}\cdot k\cdot t}dt + \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-i\frac{2\pi}{T}\cdot k\cdot t}dt$	$\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{i\frac{2\pi}{T}\cdot k\cdot t}dt - \frac{1}{T}\cdot\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot e^{-i\frac{2\pi}{T}\cdot k\cdot t}dt$

We calculate $A_k - i \cdot B_k$:

$$A_k - i \cdot B_k = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{i\frac{2\pi}{T} \cdot k \cdot t} dt + \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt - \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{i\frac{2\pi}{T} \cdot k \cdot t} dt + \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt =$$

$$\frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt$$

We calculate $A_k + i \cdot B_k$:

$$A_k + i \cdot B_k = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{i\frac{2\pi}{T} \cdot k \cdot t} dt + \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt + \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{i\frac{2\pi}{T} \cdot k \cdot t} dt - \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt =$$

$$\frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{i\frac{2\pi}{T} \cdot k \cdot t} dt$$

We see that this fits with $k > 0$ and $k < 0$ and can write C_k:

$$C_k = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\frac{2\pi}{T} \cdot k \cdot t} dt$$

Replacing $\frac{2\pi}{T} \cdot k$ by ω_k we get:

$$C_k = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\omega_k \cdot t} dt$$

The use of exponential functions instead of sin and cos allows us to write the deconstruction in one term:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{i\omega_k \cdot t} = \frac{1}{T} \cdot \sum_{k=-\infty}^{\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\omega_k \cdot t} dt \right) \cdot e^{i\omega_k \cdot t}$$

Aperiodic function

We increase the period length $T \to \infty$:

$$f(t) = \lim_{T \to \infty} \frac{1}{T} \cdot \sum_{k=-\infty}^{\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-i\omega_k \cdot t} dt \right) \cdot e^{i\omega_k \cdot t}$$

With increasing period length, the distance between two spectral lines becomes smaller:

$$\Delta\omega = \omega_{k+1} - \omega_k = \frac{2\pi}{T} \cdot (k+1) - \frac{2\pi}{T} \cdot k = \frac{2\pi}{T}$$

Note: $\omega_k = \frac{2\pi}{T} \cdot k$. Keeping the ratio $\frac{k}{T}$ fixed, ω can take any value between $-\infty$ and ∞.

We express $\frac{1}{T}$ by $\Delta\omega$:

$$\frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

We replace $\frac{1}{T}$:

$$\lim_{T\to\infty} \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} e^{i\omega_k \cdot t} \cdot \Delta\omega \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau) \cdot e^{-i\omega_k \cdot \tau} d\tau =;$$

With limes $T \to \infty$ $\Delta\omega \to d\omega$, $\omega_k \to \omega$, $\sum \to \int$:

$$f(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t}\, d\omega \cdot \int_{-\infty}^{\infty} f(\tau) \cdot e^{-i\omega \cdot \tau} d\tau$$

Usually the factor $\frac{1}{2\pi}$ is split up into $\sqrt{\frac{1}{2\pi}} \cdot \sqrt{\frac{1}{2\pi}}$ to keep things symmetric:

$$f(t) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t}\, d\omega \cdot \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} f(\tau) \cdot e^{-i\omega \cdot \tau} d\tau$$

Note: It is possible to hold $\frac{1}{2\pi}$ on either side of the product but this changes the way to work with the Fourier transform.

We work with:

$$\sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} f(\tau) \cdot e^{-i\omega \cdot \tau} d\tau$$

We rename τ to t and get a function of ω:

$$\sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega \cdot t} dt \coloneqq \tilde{f}(\omega)$$

We name it $\tilde{f}(\omega)$.

$\tilde{f}(\omega)$ is the Fourier transform of $f(t)$. It is a function of frequencies.

Having the Fourier transform $\tilde{f}(\omega)$ we can build the original function $f(t)$ by:	Having the original function $f(t)$ we can build the Fourier transform $\tilde{f}(\omega)$ by:
$f(t) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t} \cdot \tilde{f}(\omega) d\omega$	$\tilde{f}(\omega) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega \cdot t} dt$

Note: The sign of the exponentials must be opposite but can be defined the other way around. This reflects if we go through the unit circle clockwise or counterclockwise or waves propagating to the left or to the right. In physical scenarios this may become important.

The Fourier transform has some useful properties.

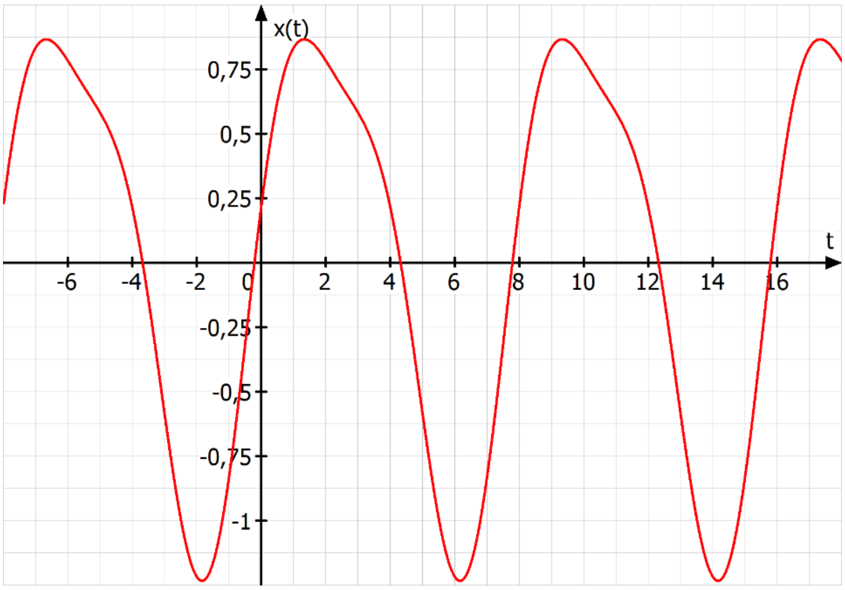
Differentiation results in a multiplication with $i\omega$:	$\frac{d}{dt}f(t) = i\omega \cdot \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t} \cdot \tilde{f}(\omega) d\omega$
The Fourier transform of $\frac{d}{dt}f(t)$ then is	$i\omega \cdot \tilde{f}(\omega)$

Integration results in a division through $i\omega$:	$\int_t f(t)dt = \frac{\sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t} \cdot \tilde{f}(\omega) d\omega}{i\omega}$
The Fourier transform of $\int_t f(t)dt$ then is:	$\frac{\tilde{f}(\omega)}{i\omega} + c$

Translation results in multiplying with $e^{i\omega a}$:	$f(t+a) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot (t+a)} \cdot \tilde{f}(\omega) d\omega =$ $e^{i\omega a} \cdot \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega \cdot t} \cdot \tilde{f}(\omega) d\omega$
The Fourier transform of $f(t+a)$ then is:	$e^{i\omega a} \cdot \tilde{f}(\omega)$

Discrete Fourier Transformation

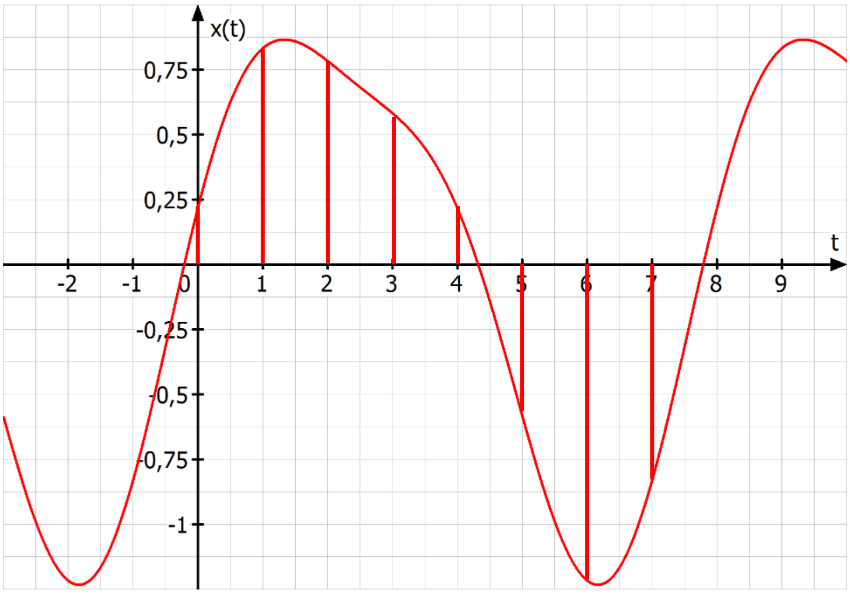
Assume that $x(t)$ is given by the following waveform:



Note: The t-axis in units of ${}^{1}/{}_{8000}\,s$.

We use a sampling rate of $8{,}000\ Hz$ taking $L = 8$ samples.

We get:



The values sampled might be:

n	0	1	2	3	4	5	6	7
$x(n)$	0.2165	0.8321	0.7835	0.5821	0.2165	-0.5821	-1.2165	-0.8321

Trigonometric functions have a period of 2π.

We took 8 samples so the space between two frequency points will be:

$$\frac{2\pi}{8} = \frac{\pi}{4}$$

The frequency set ω of trigonometric functions is of form:

$$\frac{\pi}{4} \cdot k, k = 0,1, \ldots ,7$$

We express all trigonometric functions by Euler's identity:

$$e^{i \cdot \frac{\pi}{4} \cdot k \cdot n}, n = 0,1, \ldots ,7$$

The weighted sum $x(n)$ representing the original signal:

$$x(n) = \sum_{k=0}^{7} X'(k) \cdot e^{i \cdot \frac{\pi}{4} \cdot k \cdot n}$$

We decompose the sum:

$x(0) = X'(0) + X'(1) + X'(2) + X'(3) + X'(4) + X'(5) + X'(6) + X'(7)$

$x(1) = X'(0) + X'(1) \cdot e^{i \cdot \frac{\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{2\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{3\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{4\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{5\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{6\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{7\pi}{4}}$

$x(2) = X'(0) + X'(1) \cdot e^{i \cdot \frac{2\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{4\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{6\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{8\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{10\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{12\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{14\pi}{4}}$

$x(3) = X'(0) + X'(1) \cdot e^{i \cdot \frac{3\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{6\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{9\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{12\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{15\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{18\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{21\pi}{4}}$

$x(4) = X'(0) + X'(1) \cdot e^{i \cdot \frac{4\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{8\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{12\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{16\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{20\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{24\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{28\pi}{4}}$

$x(5) = X'(0) + X'(1) \cdot e^{i \cdot \frac{5\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{10\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{15\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{20\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{25\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{30\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{35\pi}{4}}$

$x(6) = X'(0) + X'(1) \cdot e^{i \cdot \frac{6\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{12\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{18\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{24\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{30\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{36\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{42\pi}{4}}$

$x(7) = X'(0) + X'(1) \cdot e^{i \cdot \frac{7\pi}{4}} + X'(2) \cdot e^{i \cdot \frac{14\pi}{4}} + X'(3) \cdot e^{i \cdot \frac{21\pi}{4}} + X'(4) \cdot e^{i \cdot \frac{28\pi}{4}} + X'(5) \cdot e^{i \cdot \frac{35\pi}{4}} + X'(6) \cdot e^{i \cdot \frac{42\pi}{4}} + X'(7) \cdot e^{i \cdot \frac{49\pi}{4}}$

These are eight equations with eight variables we can solve to calculate the coefficients $X'(n)$.

Rounded to three decimals we get from wxmaxima:

$X'(0) = 0.244,$	$X'(1) = 0.244 - 0.500i,$
$X'(2) = 0.352 - 0.063i,$	$X'(3) = 0.244$
$X'(4) = 0.244,$	$X'(5) = 0.244,$
$X'(6) = 0.352 + 0.063i,$	$X'(7) = 0.244 + 0.500i$

We eliminate the constant 0.244:

$X'(0) = 0,$	$X'(1) = -0.500i,$	$X'(2) = 0.108 - 0.063i,$	$X'(3) = 0$
$X'(4) = 0,$	$X'(5) = 0,$	$X'(6) = 0.108 + 0.063i,$	$X'(7) = 0.500i$

Note: I don't know where wxmaxima takes the constant factor 0.244 from. matlab produces the correct solution directly. For your convenience I added the sheets of wxmaxima and matlab in the appendix.

We take a look at the sum:

$$x(n) = \sum_{k=0}^{7} X'(k) \cdot e^{i \cdot \frac{\pi}{4} \cdot k \cdot n}$$

We omit the $X'(n)$ that are zero and rewrite:

$x(0) = X'(1) + X'(2) + X'(6) + X'(7)$

$x(1) = X'(1) \cdot e^{i \cdot \frac{1\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{1\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{1\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{1\pi}{4} \cdot 7}$

$x(2) = X'(1) \cdot e^{i \cdot \frac{2\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{2\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{2\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{2\pi}{4} \cdot 7}$

$x(3) = X'(1) \cdot e^{i \cdot \frac{3\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{3\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{3\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{3\pi}{4} \cdot 7}$

$x(4) = X'(1) \cdot e^{i \cdot \frac{4\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{4\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{4\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{4\pi}{4} \cdot 7}$

$x(5) = X'(1) \cdot e^{i \cdot \frac{5\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{5\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{5\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{5\pi}{4} \cdot 7}$

$x(6) = X'(1) \cdot e^{i \cdot \frac{6\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{6\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{6\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{6\pi}{4} \cdot 7}$

$x(7) = X'(1) \cdot e^{i \cdot \frac{7\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{7\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{7\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{7\pi}{4} \cdot 7}$

We get $x(n)$:

$$x(n) = X'(1) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 2} + X'(6) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 6} + X'(7) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 7}$$

We get rid of the factors 6 and 7 by using the periodicity of 8:

$$e^{i \cdot \frac{n\pi}{4} \cdot 7} = e^{i \cdot \frac{n\pi}{4} \cdot (8-1)} = e^{i \cdot \frac{n\pi}{4} \cdot 8} \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 1} = e^{-i \cdot \frac{n\pi}{4} \cdot 1}$$

$$e^{i \cdot \frac{n\pi}{4} \cdot 6} = e^{i \cdot \frac{n\pi}{4} \cdot (8-2)} = e^{i \cdot \frac{n\pi}{4} \cdot 8} \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 2} = e^{-i \cdot \frac{n\pi}{4} \cdot 2}$$

We get:

$$x(n) = X'(1) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 2} + X'(6) \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 2} + X'(7) \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 1} \rightarrow$$

$$x(n) = X'(1) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 1} + X'(7) \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 1} + X'(2) \cdot e^{i \cdot \frac{n\pi}{4} \cdot 2} + X'(6) \cdot e^{-i \cdot \frac{n\pi}{4} \cdot 2}$$

We use that $X'(1)$ and $X'(7)$ as well as $X'(2)$ and $X'(6)$ are complex conjugated

$X'(1) = -0.500i$	$X'(7) = 0.500i$
$X'(2) = 0.108 - 0.063i,$	$X'(6) = 0.108 + 0.063i,$

We get:

$$x(n) = \left[-0.500i \cdot e^{i \cdot \frac{n\pi}{4}} + 0.500i \cdot e^{-i \cdot \frac{n\pi}{4}}\right] + \left[(0.108 - 0.063i) \cdot e^{i \cdot \frac{n\pi}{2}} + (0.108 + 0.063i) \cdot e^{-i \cdot \frac{n\pi}{2}}\right]$$

We use that for complex numbers $u = (x + iy)$ and $v = (a + ib)$ holds:

$$u^*v + uv^* = (x - iy)(a + ib) + (x + iy)(a - ib) =$$

$$xa + ixb - iya + yb + xa - ixb + iya + yb = 2xa + 2yb = 2Re(u)Re(v) + 2Im(u)Im(u)$$

$Re(0.500i) = 0$	$Im(0.500i) = 0.5$
$Re\left(e^{i \cdot \frac{n\pi}{4} \cdot 1}\right) = cos\left(n\frac{\pi}{4}\right)$	$Im\left(e^{i \cdot \frac{n\pi}{4} \cdot 1}\right) = sin\left(n\frac{\pi}{4}\right)$

$Re(0.108 + 0.063i) = 0.108$	$Im(0.108 + 0.063i) = 0.063$
$Re\left(e^{i \cdot \frac{n\pi}{2}}\right) = cos\left(n\frac{\pi}{2}\right)$	$Im\left(e^{i \cdot \frac{n\pi}{2}}\right) = sin\left(n\frac{\pi}{2}\right)$

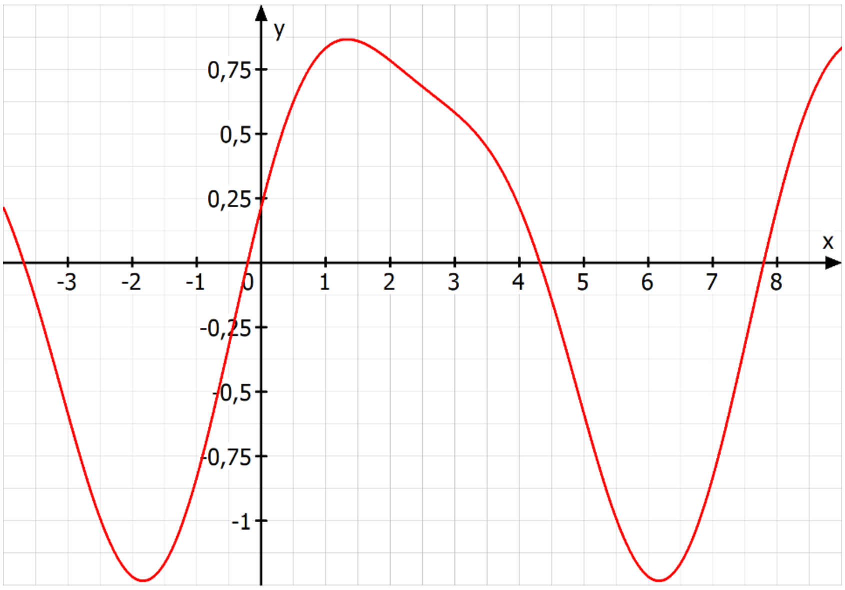
Note: Don't confuse complex and complex conjugated.

We get:

$$x(n) = \left[sin\left(n\frac{\pi}{4}\right)\right] + \left[0.216 \cdot cos\left(n\frac{\pi}{2}\right) + 0.126 \cdot sin\left(n\frac{\pi}{2}\right)\right]$$

For $n = 0,1,\dots,7$ we get back the values measured.

We replace the discrete n by the continuous variable t and get:

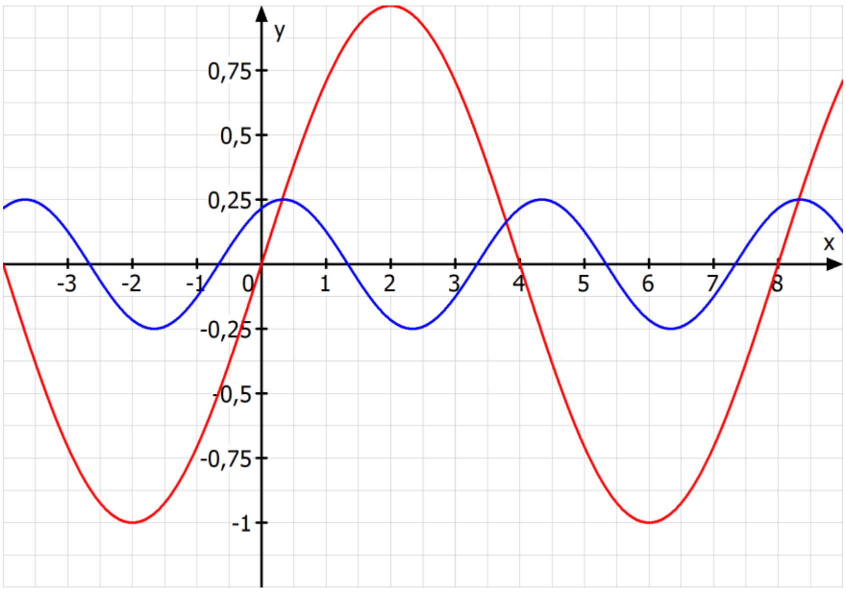


This is the wave we came from.

With a sampling frequency $f_s = 8{,}000\ Hz$ we obtain the frequencies of the two compounds:

$f_1 = \frac{\pi}{4} \cdot \frac{f_s}{2\pi} = 1{,}000\ Hz$ and $f_2 = \frac{\pi}{2} \cdot \frac{f_s}{2\pi} = 2{,}000\ Hz$

The compound signal is a sum of two waves:

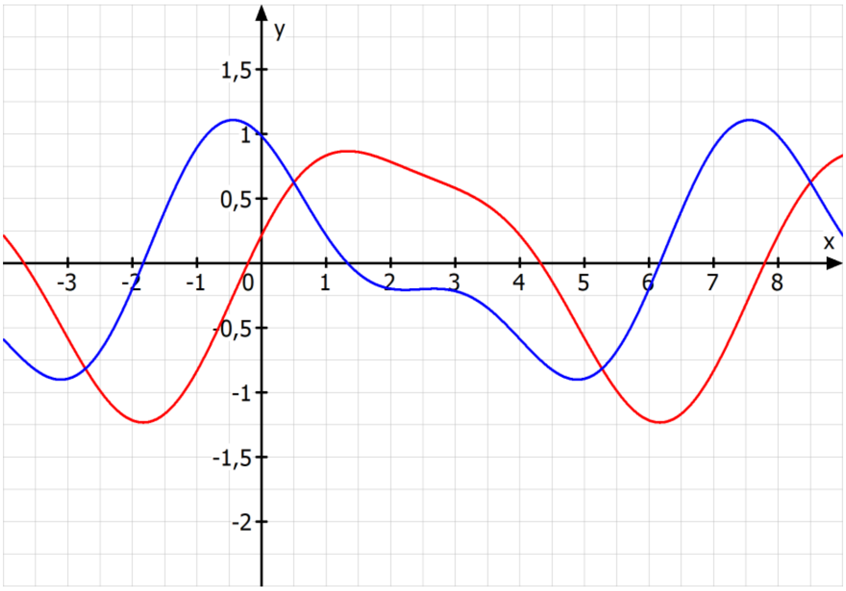


We now can differentiate the original wave by differentiating its components.

$$x(t) = \left[sin\left(t\frac{\pi}{4}\right)\right] + \left[0.216 \cdot cos\left(t\frac{\pi}{2}\right) + 0.126 \cdot sin\left(t\frac{\pi}{2}\right)\right]$$

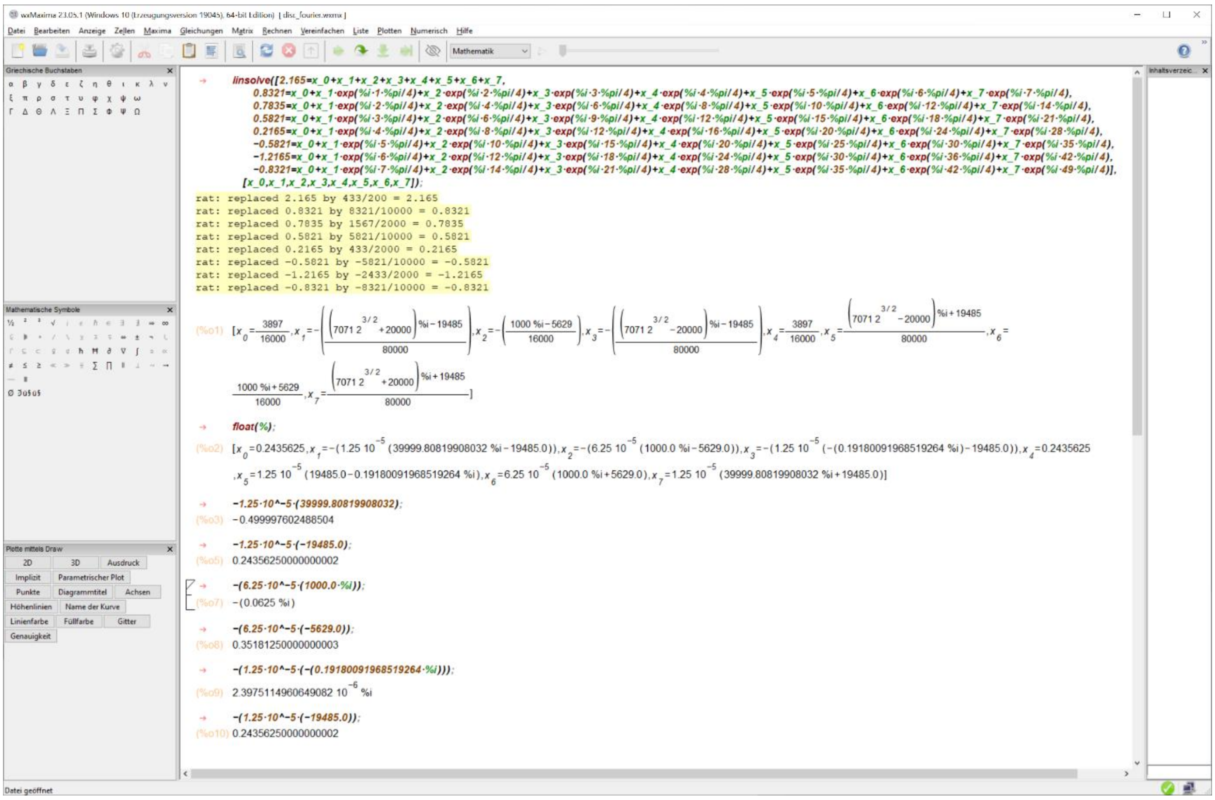
$$\frac{d}{dt}x(t) = \left[\frac{\pi}{4} \cdot cos\left(t\frac{\pi}{4}\right)\right] + \left[-0.216 \cdot \frac{\pi}{2} \cdot sin\left(t\frac{\pi}{2}\right) + 0.126 \cdot \frac{\pi}{2} \cdot cos\left(t\frac{\pi}{2}\right)\right]$$

We plot the function (red) and its derivative (blue):



Appendix

wxmaxima input



matlab input

```
R=[1 1 1 1 1 1 1 1;
1 exp(1i*pi/4) exp(2i*pi/4) exp(3i*pi/4) exp(4i*pi/4) exp(5i*pi/4) exp(6i*pi/4) exp(7i*pi/4);
1 exp(2i*pi/4) exp(4i*pi/4) exp(6i*pi/4) exp(8i*pi/4) exp(10i*pi/4) exp(12i*pi/4) exp(14i*pi/4);
1 exp(3i*pi/4) exp(6i*pi/4) exp(9i*pi/4) exp(12i*pi/4) exp(15i*pi/4) exp(18i*pi/4) exp(21i*pi/4);
1 exp(4i*pi/4) exp(8i*pi/4) exp(12i*pi/4) exp(16i*pi/4) exp(20i*pi/4) exp(24i*pi/4) exp(28i*pi/4);
1 exp(5i*pi/4) exp(10i*pi/4) exp(15i*pi/4) exp(20i*pi/4) exp(25i*pi/4) exp(30i*pi/4) exp(35i*pi/4);
1 exp(6i*pi/4) exp(12i*pi/4) exp(18i*pi/4) exp(24i*pi/4) exp(30i*pi/4) exp(36i*pi/4) exp(42i*pi/4);
1 exp(7i*pi/4) exp(14i*pi/4) exp(21i*pi/4) exp(28i*pi/4) exp(35i*pi/4) exp(42i*pi/4) exp(49i*pi/4)]
V=[0.2165;0.8321;0.7835;0.5821;0.2165;-0.5821;-1.2165;-0.8321]

linsolve(R,V)

ans =
   0.0000 - 0.0000i
   0.0000 - 0.5000i
   0.1083 - 0.0625i
   0.0000 + 0.0000i
   0.0000 + 0.0000i
  -0.0000 - 0.0000i
   0.1082 + 0.0625i
  -0.0000 + 0.5000i
```