This paper deals with Fourier series.

Fourier series are a prerequisite for Fourier transformation.

It follows:

Fourier-Analysis, Glatz, Grieb, Hohloch, Kümmerer, Mohr, Cornelsen 1996, ISBN 3-464-41327-6

Fourier series

Periodic functions

Definition

A function $f(x)$ is periodic if $\forall x \in \mathbb{R}: f(x+P) = f(x)$.

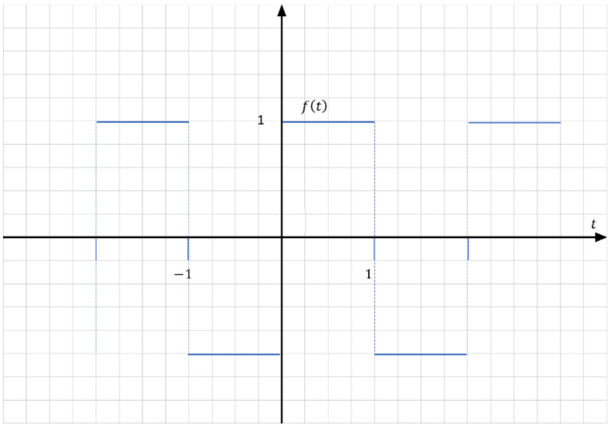
The smallest positive value P is the period of the function.

Examples:

a) $sin(x) = sin(x + 2\pi) \dots period\ 2\pi$
b) $sin(kx) = sin(kx + 2\pi) = sin\left(k\left(x + \frac{2\pi}{k}\right)\right) \dots period\ \frac{2\pi}{k}$
c) $sin(\omega t) = sin(\omega t + 2\pi) = sin\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right) \dots period\ \frac{2\pi}{\omega}$

Note: If the function is time-dependent, the period is called "oscillation duration" and ω “angular frequency”.

d) $f(x) = f(x+2)\ and\ f(x) = \begin{cases} -1\ for -1 \le x < 0 \\ 1\quad for\quad 0 < x < 1 \end{cases}$



The term $f(x+2) = f(x)$ periodically continues the function defined $[-1,1)$.

We get a square wave oscillation.

Piecewise continuous functions

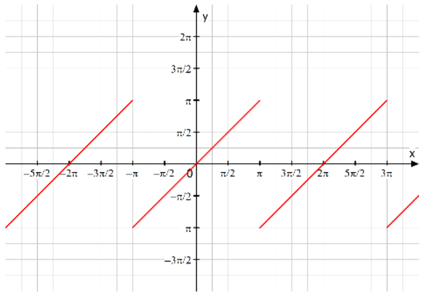
Definition:

$f(x)$ is piecewise continuous in $[a,b]$ if $f(x)$ is continuous except for a finite number of discontinuities (jump points).

Characteristic for a discontinuity x_0 are the one-sided limits left and right:

$$f(x_0 +), (x_0 -)$$

Example: Sawtooth wave



$$f(x) = f(x + 2\pi) \; and \; f(x) = x \; for - \pi < x < \pi$$

The discontinuities $x_0 = \pi$:

$$f(\pi +) = -\pi, f(\pi -) = \pi$$

Trigonometric polynomials

We use harmonic oscillations with period $p = \frac{2\pi}{k}$:

$$f_k(x) = a \; cos(kx) + b \; sin(kx)$$

$$f_k(x) = A \; cos(kx + \varphi) \tag{1}$$

$f_k(x) = a \; cos(kx) + b \; sin(kx)$ is the term without phase angle, mathematically easier to handle.

$f_k(x) = A \; cos(kx + \varphi)$ is the term with amplitude A and phase angle φ, more suitable for practical applications.

Both terms can be converted into each other:

$$a = A \; cos(\varphi), b = -A \; sin(\varphi)$$

$$A = \sqrt{a^2 + b^2}, tan(\varphi) = -\frac{b}{a}$$

If we superimpose harmonic oscillations of different periods, we generally obtain non-periodic functions. However, if the partial oscillations have periods $p_1 = \frac{2\pi}{k_1}$ and $p_2 = \frac{2\pi}{k_2}$ with least common multiple p, we get a nonharmonic oscillation with period p.

If k_1 and k_2 are integer positive numbers, the resulting oscillation has the period 2π.

Example:

$f(x) = 2 \; sin(x) - cos(2x)$	$f_1(x) = 2 \; sin(x) \ldots \; P_1 = 2\pi$	$f_2(x) = cos(2x) \ldots \; P_2 = \pi$

$f(x)$ has the period $p = 2\pi$.

Overlaps of harmonic oscillations are important for applications:

$$f_k(x) = a_k \; cos(kx) + b_k \; sin(kx)$$

resp.

$$f_k(x) = A_k cos(kx + \varphi_k) \tag{2}$$

Definition trigonometric polynomial:

> Any superposition of harmonic oscillations of (2) is called a trigonometric polynomial with the fundamental period 2π.

We get the trigonometric polynomial of degree n:

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \; cos(kx) + b_k \; sin(kx) \tag{3}$$

Fourier series for 2π periodic functions

Every function $T_n(x)$ according to (3) is a 2π periodic function. We can

- approximate any reasonable nonperiodic function by polynomials,
- approximate any reasonable periodic function by trigonometric polynomials.

We can approximate a general periodic motion with oscillation duration $T = \frac{2\pi}{\omega}$ by superimposing harmonic oscillations with angular frequencies $\omega, 2\omega, 3\omega$. We decompose a general oscillation into its fundamental oscillation and harmonics. This procedure is called harmonic analysis.

Below we consider 2π periodic functions. They can be converted by substitution to the case of other time-dependent oscillations.

The harmonic analysis of a 2π periodic function $f(x)$ is based on the following mathematical consideration. The coefficients a_k, b_k of the trigonometric polynomial $T_n(x)$ according to (3) should be determined in such a way that the mean square of error (note: variance) becomes as small as possible:

$$\int_{-\pi}^{\pi} \big(f(x) - T_n(x)\big)^2 dx \Rightarrow minimum$$

This results in the following theorem:

Fourier's theorem:

Every "reasonable", piecewise defined continuous function $f(x)$ in the interval $-\pi < x < \pi$ can be represented as a convergent trigonometric series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, cos(kx) + b_k\, sin(kx) \tag{4}$$

with the coefficients:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx, k = 0,1,2,\dots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx, k = 1,2,\dots \tag{5}$$

The Fourier series (4) converges for each x to:

(a) $f(x)$ at each point the function is continuous,

(b) $\frac{1}{2}\big(f(x_0+) + (x_0-)\big)$ at each discontinuity x_0.

Note: In literature often a_0 instead of $\frac{a_0}{2}$ is used.

We often calculate a_0 explicitly:

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

We calculate the mean value m of a function over the interval $-\pi < x < \pi$:

$$m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

If we compare m and a_0 we see that the absolute member $\frac{a_0}{2}$ in the Fourier series (4) represents the mean value m of the periodic function. In many cases, this value can be specified directly.

With s_n we refer to the approximation of the function $f(x)$ by a trigonometric polynomial of degree n:

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \tag{6}$$

At discontinuities the Fourier series converges against the arithmetic mean of the left- and right-sided limit of $f(x)$.

Example: Triangle oscillation

$$f(x) = |x|, \qquad -\pi < x < \pi$$

$$f(x + 2\pi) = f(x)$$

From the coefficient formulas (5) we obtain for the coefficients of the cosine series:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| cos(kx)dx = \frac{2}{\pi}\int_0^{\pi} xcos(kx)dx \qquad k = 0,1,2,\dots$$

For $k = 0$ we get:

$$\frac{2}{\pi}\int_0^{\pi} xcos(0)dx = \pi$$

Note: This is the mean value.

We integrate by parts: $f' = cos(kx), f = \frac{sin(kx)}{k}, g = x, g' = 1$

$$a_k = \frac{2}{\pi}\left(\left[\frac{xsin(kx)}{k}\right]_0^{\pi} - \int_0^{\pi} \frac{sin(kx)}{k} dx\right) =$$

$$\frac{2}{\pi}\left(\left[\frac{xsin(kx)}{k}\right]_0^{\pi} + \left[\frac{cos(kx)}{k^2}\right]_0^{\pi}\right) = \frac{2}{\pi}\left(\left[\frac{xsin(kx)}{k} + \frac{cos(kx)}{k^2}\right]_0^{\pi}\right) =$$

$$\frac{2}{k^2\pi}([cos(kx)]_0^{\pi}) = \frac{2}{k^2\pi}(cos(k\pi) - 1) =$$

Note: $k = 0,1,2,\dots$

$$\frac{2}{k^2\pi}\left((-1)^k - 1\right)$$

$$a_1 = -\frac{4}{\pi}, a_2 = 0, a_3 = -\frac{4}{\pi}\cdot\frac{1}{9}, a_4 = 0, a_5 = -\frac{4}{\pi}\cdot\frac{1}{25}$$

For k even, $a_k = 0$.

For k odd, $a_k = -\frac{4}{\pi}\cdot\frac{1}{k^2}$

From the coefficient formulas (5) we obtain for the coefficients of the sine series:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| sin(kx)dx = 0$$

The sine function is an odd function. The integration over a symmetrical interval around the origin therefore returns zero.

With this, we get the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, cos(kx) + b_k\, sin(kx) =$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, cos(kx) =$$

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} -\frac{4}{\pi} \cdot \frac{1}{k^2}\, cos(kx) =$$

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2}\, cos(kx) =$$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(cos(x) + \frac{1}{9} cos(3x) + \frac{1}{25} cos(5x) + \cdots\right) =$$

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{1}{(2r+1)^2}\, cos\big((2r+1)x\big)$$

Note: The last transformation needed to have the index in steps of one.

Example: Sawtooth wave

$$f(x) = x, \qquad -\pi < x < \pi$$

$$f(x + 2\pi) = f(x)$$

From the coefficient formulas (5) we obtain for the coefficients of the cosine series:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot cos(kx) dx \quad , k = 0,1,2, \ldots$$

Note: The integrand is the product of an even function cos and an odd function x. Thus, the value of the integral over a symmetric interval becomes zero:

$$a_k = 0\ \forall k$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot sin(kx) dx = \frac{2}{\pi} \int_{0}^{\pi} x \cdot sin(kx) dx \quad k = 0,1,2, \ldots$$

Note: The product of two odd functions is even.

We integrate the coefficients b_k by parts.

$$f' = sin(kx), f = -\frac{cos(kx)}{k}, g = x, g' = 1$$

$$b_k = \frac{2}{\pi}\left(\left[-\frac{x cos(kx)}{k}\right]_0^{\pi} + \int_0^{\pi} \frac{cos(kx)}{k} dx\right) =$$

$$\frac{2}{\pi}\left(\left[-\frac{x\cos(kx)}{k}\right]_0^{\pi}+\left[\frac{\sin(kx)}{k^2}\right]_0^{\pi}\right)=\frac{2}{\pi}\left(\left[-\frac{x\cos(kx)}{k}\right]_0^{\pi}\right)=$$

$$\frac{2}{\pi}\left(-\frac{\pi}{k}\right)\cos(k\pi)=-\frac{2}{k}\cos(k\pi)=\frac{2}{k}\cdot(-1)^{k+1}$$

We specify the Fourier series:

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(kx)+b_k\sin(kx)=$$

$$\sum_{k=1}^{\infty}b_k\sin(kx)=\sum_{k=1}^{\infty}\frac{2}{k}\cdot(-1)^{k+1}\sin(kx)=$$

$$\frac{2}{k}\sum_{k=1}^{\infty}(-1)^{k+1}\sin(kx)=$$

$$2\left[\sin(x)-\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)-\frac{1}{4}\sin(4x)\pm\cdots\right] \qquad 7$$

Differences between polynomial and Fourier series

The triangle function and the sawtooth function coincide in the interval $[0;\pi)$

The two series:

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{1}{(2r+1)^2}\cos\big((2r+1)x\big)$$

$$f(x)=2\left[\sin(x)-\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)-\frac{1}{4}\sin(4x)\pm\cdots\right]=$$

converge in interval $[0;\pi)$ but represent two completely different series.

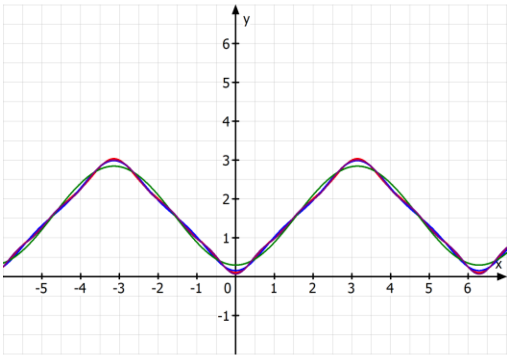
The approximation by trigonometric polynomials and the convergence speed of the Fourier series depend heavily on how the function is defined on the entire fundamental interval. In contrast, a power series is uniquely determined by the behavior of the function at the development point x_0 .

Convergence behavior of Fourier series

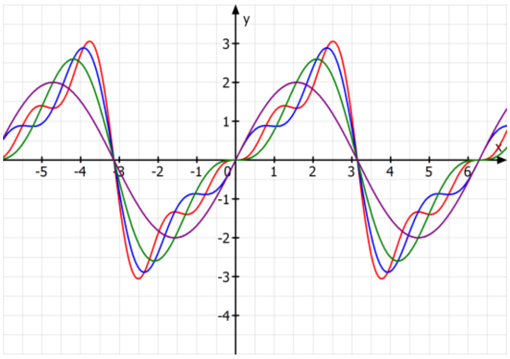
We see a much better convergence of the Fourier series in the continuous triangle oscillation than in the sawtooth oscillation, which is only piecewise continuous. The smoother the periodically continued function, the faster its Fourier series converges. For the decay of the Fourier coefficients it can generally be shown:

$$f(x)\ continuous\ \Longrightarrow a_k,b_k\sim\frac{1}{k^2}$$

$$f(x)\ has\ discontinuities\ \rightarrow a_k,b_k\sim\frac{1}{k}$$



convergence triangle function



convergence sawtooth function

Fourier series for oscillations

General formulas – harmonic analysis

In physics we often use functions that depend on time. We therefore transform the 2π -periodic spatial functions.

The oscillation duration:

$$\omega = \frac{2\pi}{T} \Longrightarrow T = \frac{2\pi}{\omega}$$

We substitute:

$$x = \omega \cdot t, dx = \omega\, dt \tag{8}$$

We transform the basic interval of length 2π to an interval of length T:

$$-\pi < x < \pi$$

$$-\frac{T}{2} = -\frac{\pi}{\omega} < t < \frac{\pi}{\omega} = \frac{T}{2}$$

The 2π-periodic function $f(x)$ becomes the T-periodic function $f(t)$:

$f(x)$ with $f(x + 2\pi) = f(x)$ becomes $f[x(t)] = f(t)$ with $f(t + T) = f(t)$

From (4), (5) we get the theorem:

Any wave $f(t)$ with angular frequency ω and oscillation duration $T = \frac{2\pi}{\omega}$ can be represented as a convergent trigonometric series of the form:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \tag{9}$$

We get the coefficients:

$$a_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(k\omega t) dt \quad k = 0,1,2, \ldots$$

$$b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(k\omega t) dt \quad k = 1,2, \ldots \tag{10}$$

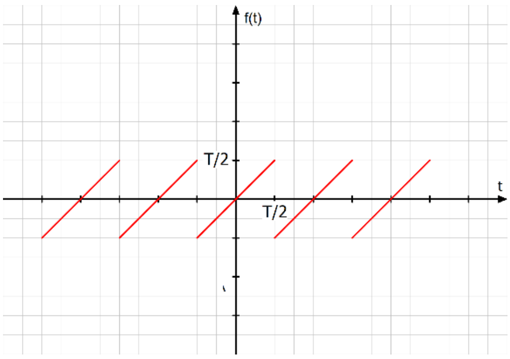
The series converges against $f(t)$ at each point of continuity and against the arithmetic mean of the left-side and right-side limit at discontinuities.

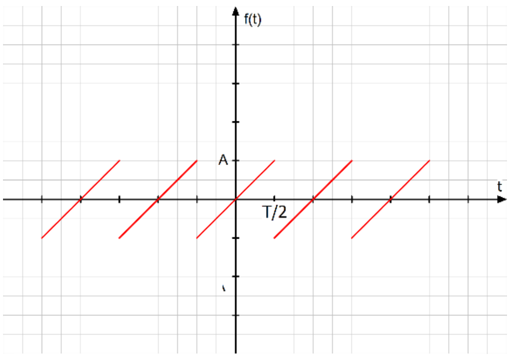
By s_n (6) we denote the approximation of the function $f(t)$ by the Fourier series (9):

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\, cos(k\omega t) + b_k\, sin(k\omega t)$$

If we know the Fourier series of a 2π-periodic function we can calculate the range of a corresponding oscillation with the oscillation period T by means of substitution (8).

Example sawtooth oscillation with $T = \frac{2\pi}{\omega}$:





$$f_1(t) = t \qquad -\frac{T}{2} < t < \frac{T}{2}$$

$$f_2(t) = \frac{2A}{T} \cdot t \qquad -\frac{T}{2} < t < \frac{T}{2}$$

From (7) we get the series:

$$f_1(t) = \frac{2}{\omega}\left[sin(\omega t) - \frac{1}{2} sin(2\omega t) + \frac{1}{3} sin(3\omega t) - \frac{1}{4} sin(4\omega t) \pm \cdots\right]$$

$$f_2(t) = \frac{2A}{\pi}\left[sin(\omega t) - \frac{1}{2} sin(2\omega t) + \frac{1}{3} sin(3\omega t) - \frac{1}{4} sin(4\omega t) \pm \cdots\right]$$

We can write the harmonics in phase form (1). (9) then becomes:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k\, cos(k\omega t + \varphi_k) \qquad (11)$$

The terms in (9) and (11) denote:

$\frac{a_0}{2}$ mean

$A_1\, cos(\omega t + \varphi_1)$ or $a_1\, cos(\omega t) + b_1\, sin(\omega t)$ 1st harmonic

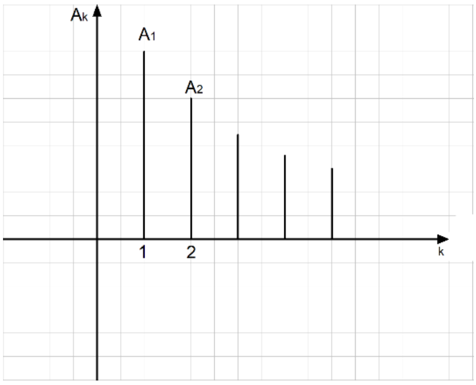
$A_2\, cos(2\omega t + \varphi_2)$ or $a_2\, cos(2\omega t) + b_1\, sin(2\omega t)$ 2nd harmonic etc.

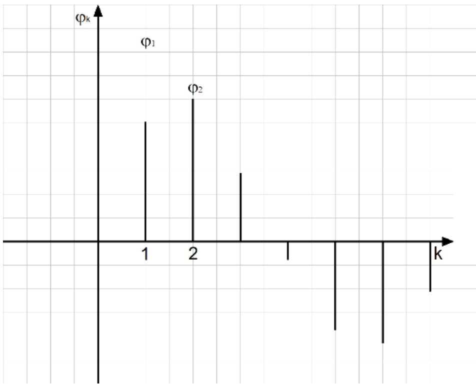
The 1st harmonic corresponds to the fundamental oscillation.

The decomposition of an oscillation into its harmonics is called harmonic analysis.

If we plot the amplitudes A_k or the phase angles φ_k against the frequencies $k \cdot \omega$ resp. the frequency parameter k, we get the amplitude spectrum or the phase spectrum of the oscillation. Both are discrete line spectra.

The amplitude spectrum provides a statement about the proportions of the fundamental oscillation and the individual harmonics of the oscillation under consideration.





According to Fourier's theorem, a periodic time function $f(t)$ can be represented by a Fourier series with amplitudes A_k and phases φ_k.	Conversely, for every spectrum $\{A_k; \varphi_k\}$ in the frequency domain exists a periodic time function $f(t)$.

We have two equivalent representations of a periodic signal, a representation in the time domain and a representation through the spectrum $\{A_k; \varphi_k\}$ in the frequency domain.

The amplitude spectrum does not contain any information about the phase position of the individual harmonics.

Simplification in the calculation of Fourier series.

Often, we get the mean $\frac{a_0}{2}$ without further calculation.

Symmetry:

<table>
<tr><td>For even functions holds:
$f(-t) = f(t)$</td><td>For odd functions holds:
$f(-t) = -f(t)$</td></tr>
<tr><td colspan="2">We use $cos(-\omega t) = cos(\omega t)$ and $sin(-\omega t) = -sin(\omega t)$.</td></tr>
<tr><td colspan="2">Thus, from the coefficient formulas (10) we get:</td></tr>
<tr><td>$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)cos(k\omega t)dt$$ $$= \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)cos(k\omega t)dt$$</td><td>$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)cos(k\omega t)dt = 0$$</td></tr>
<tr><td>$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)sin(k\omega t)dt = 0$$</td><td>$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)sin(k\omega t)dt$$ $$= \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)sin(k\omega t)dt$$</td></tr>
<tr><td>For even functions the result is a pure cosine series:</td><td>For odd functions this results in pure sine series:</td></tr>
</table>

$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega t)$	$f(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega t)$
$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)dt \quad k = 0,1,2,\ldots$	$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)dt \quad k = 1,2,\ldots$

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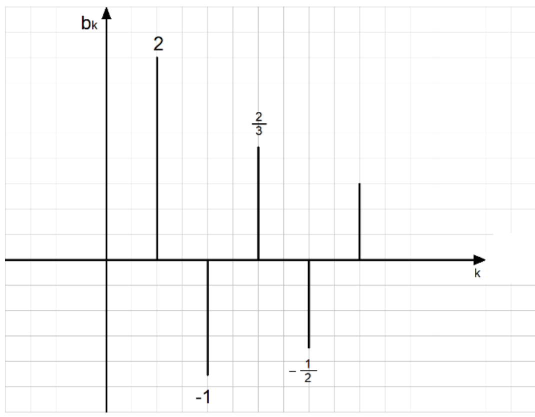
In case of pure cosine series, we use only the line spectrum a_k and omit the complete spectrum $\{A_k = |a_k|\}, \{\varphi_k = 0 \; für \; a_k > 0, \varphi_k = \pi \; for \; a_k < 0\}$.

The same holds for pure sine series. For the sawtooth wave we use

$$f_1(t) = \frac{2}{\omega}\left[\sin(\omega t) - \frac{1}{2}\sin(2\omega t) + \frac{1}{3}\sin(3\omega t) - \frac{1}{4}\sin(4\omega t) \pm \cdots\right]$$

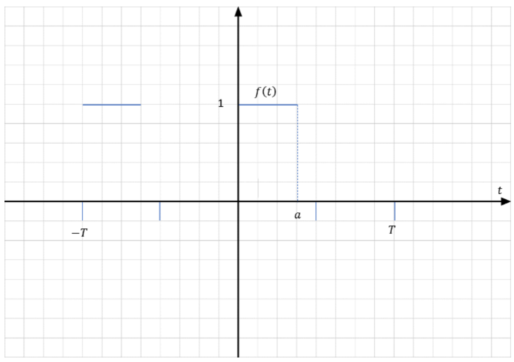
and get this picture:



Example square wave oscillation:

We want to determine the Fourier series for the square wave oscillation:

$$f(t) = \begin{cases} 1 \; for \; 0 \le t < a \\ 0 \; for \; a \le t < T \end{cases} \quad f(t+T) = f(t), \qquad 0 < a < T$$



We use the coefficient formulas (12):

$$a_k = \frac{2}{T}\int_0^a f(t)\cos(k\omega t)dt \quad k = 0,1,2,\ldots$$

$$b_k = \frac{2}{T}\int_0^a f(t)\sin(k\omega t)dt \quad k = 1,2,\ldots$$

$$a_0 = \frac{2}{T}\int_0^a dt = \frac{2a}{T} \to mean \; m = \frac{a}{T}$$

$$a_k = \frac{2}{T}\int_0^a \cos(k\omega t)dt = \left[\frac{2}{T}\cdot\frac{1}{k\omega}\cdot\sin(k\omega t)\right]_0^a = \frac{2}{T}\cdot\frac{1}{k\omega}\cdot\sin(k\omega a)$$

$$b_k = \frac{2}{T}\int_0^a sin(k\omega t)dt = \left[-\frac{2}{T}\cdot\frac{1}{k\omega}\cdot cos(k\omega t)\right]_0^a = \frac{2}{T}\cdot\frac{1}{k\omega}\cdot\big(1 - cos(k\omega a)\big) \qquad k = 1,2,\dots$$

We set $\omega = \frac{2\pi}{T}$ and get:

$$a_k = \frac{1}{k\pi}\cdot sin\left(k\frac{2\pi}{T}a\right)$$
$$b_k = \frac{1}{k\pi}\cdot\left(1 - cos\left(k\frac{2\pi}{T}a\right)\right) \qquad k = 1,2,\dots$$

For the case $a = \frac{T}{2}$ we get:

$$a_0 = 1 \Longrightarrow m = \frac{1}{2}$$

$$a_k = \frac{1}{k\pi}\cdot sin(k\pi) = 0, \qquad k = 1,2,\dots$$

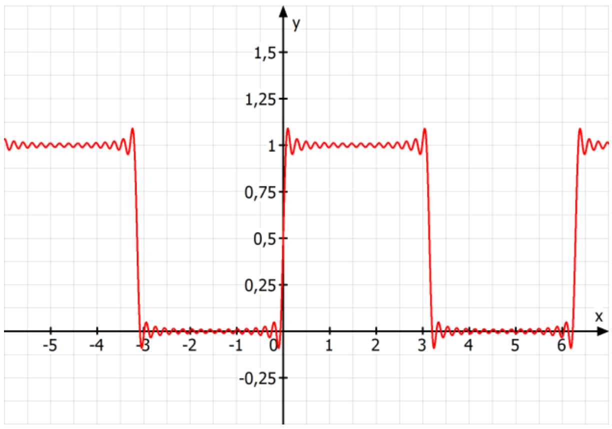
$$b_k = \frac{1}{k\pi}\cdot\big(1 - cos(k\pi)\big) = \frac{1}{k\pi}\cdot\big(1-(-1)^k\big)$$

$$b_1 = \frac{2}{\pi}, b_3 = \frac{1}{3}\cdot\frac{2}{\pi}, b_5 = \frac{1}{5}\cdot\frac{2}{\pi}, \dots$$

$$b_2 = b_4 = b_4 = \cdots = 0$$

We get:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(sin(\omega t) + \frac{1}{3}sin(3\omega t) + \frac{1}{5}sin(5\omega t) + \cdots\right)$$



The figure shows the Fourier approximation s_{31} of the square wave oscillation.

For large values of n, the Gibbs phenomenon occurs: Near the discontinuities an overshoot of the Fourier series occurs by about the 9% jump height. The area in which this overshoot occurs becomes smaller and smaller with growing n while the height remains unchanged.

Complex representation

By help of Euler's formulas we convert the trigonometric series (4):

$e^{ikx} = cos(kx) + i \cdot sin(kx)$	$cos(kx) = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$
$e^{-ikx} = cos(kx) - i \cdot sin(kx)$	$sin(kx) = \frac{1}{2}\left(e^{ikx} - e^{-ikx}\right)$

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The coefficients a_k and b_k can be the complex numbers.

We write the associated complex conjugated coefficients as $a^*{}_k$ and $b^*{}_k$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, cos(kx) + b_k\, sin(kx) \rightarrow$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right) + b_k\, s\frac{1}{2}\left(e^{ikx} - e^{-ikx}\right) =$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k - i \cdot b_k}{2} e^{ikx} + \frac{a_k + i \cdot b_k}{2} e^{-ikx}$$

We rewrite the coefficients:

$$c_0 = \frac{a_0}{2}$$

$$c_k = \frac{a_k - i \cdot b_k}{2} \qquad k = 1,2, \ldots \qquad (14)$$

$$c_{-k} \coloneqq c^*{}_k = \frac{a_k + i \cdot b_k}{2}$$

We get:

$$f(x) = c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k} e^{-ikx} = c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + \sum_{k=1}^{\infty} c_{-k} e^{-ikx}$$

By renaming $c_{-k} \coloneqq c^*{}_k$ we can transform this into a two-sided sum:

$$\sum_{k=1}^{\infty} c_{-k} e^{-ikx} = \sum_{k=-1}^{-\infty} c_k e^{ikx}$$

We get the complex Fourier series:

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} \qquad (15)$$

The summation index ranges from $-\infty$ to ∞ resp. $k = \pm 1, \pm 2, \ldots$

The complex Fourier series (15) contains complex sums. We can trace this back to the real version (4) by summarizing (13) the $c_k e^{ikx}$ and $c_{-k} e^{-ikx}$ pairs:

$$c_{-k} e^{-ikx} + c_k e^{ikx} = c^*{}_k \left(e^{ikx}\right)^* + c_k e^{ikx} =$$

$$2Re\left(c_k e^{ikx}\right) = 2Re\left(\frac{1}{2}(a_k - i \cdot b_k)\left(cos(kx) + i \cdot\ sin(kx)\right)\right) =$$

$$a_k \, cos(kx) + b_k \, sin(kx)$$

The complex quantities $c_k, k \in \mathbb{Z}$ of (14) are the complex Fourier coefficients.

From these we can restore the real Fourier coefficients:

$$a_k = 2 \cdot Re(c_k) = c_k + c^*{}_k, k = 0,1,2,\ldots$$

$$b_k = -2 \cdot Im(c_k) = i \cdot (c_k - c^*{}_k), k = 1,2,\ldots$$

For symmetric functions the complex Fourier series becomes:

$f(t)$ even $\Leftrightarrow$ real cos series $\Leftrightarrow$ c_k real

$f(t)$ odd $\Leftrightarrow$ real sin series $\Leftrightarrow$ c_k purely imaginary

We determine the complex Fourier coefficients using the integrals:

$$c_0 = \frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$c_k = \frac{a_k - i \cdot b_k}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) cos(kx) dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) sin(kx) dx =$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \big(cos(kx) - i sin(kx)\big) dx =^{6.1}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

We get the following theorem:

For each 2π periodic function $f(x)$, we get the Fourier series in complex form:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

The coefficients are:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \quad k = 0, \pm 1, \pm 2, \ldots$$

For time-dependent functions, we use the substitution $x = \omega t$ and get the theorem:

Any oscillation $f(t)$ with angular frequency ω and oscillation duration $T = \frac{2\pi}{\omega}$ can be represented as a convergent series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t} \, , \omega = \frac{2\pi}{T}$$

The coefficients are:

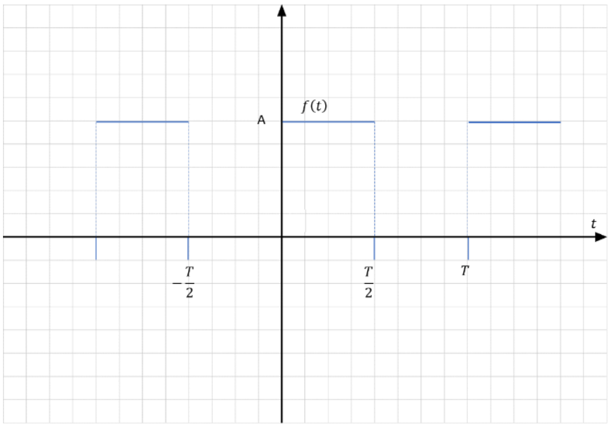
$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ik\omega t} dt \quad k = 0, \pm 1, \pm 2, \ldots \qquad (16)$$

Note: If we integrate we treat the imaginary unit i like a constant.

Example:

$$f(t) = \begin{cases} 0\ for\ -\frac{T}{2} \leq t < 0 \\ A \quad for \quad 0 < t < \frac{T}{2} \end{cases}$$

$$f(T+t) = f(t)$$



We calculate:

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{1}{T}\int_0^{\frac{T}{2}} Adt = \frac{A}{2}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-i\cdot k\omega t}dt = \frac{1}{T}\int_0^{\frac{T}{2}} Ae^{-i\cdot k\omega t}dt = -\frac{A}{ik\omega T}\left[e^{-i\cdot k\omega t}\right]_0^{\frac{T}{2}} =$$

$$-\frac{A}{i\cdot k\omega T}\left(e^{-i\cdot k\omega\frac{T}{2}} - 1\right) = \frac{i\cdot A}{2\pi k}\left(e^{-i\cdot k\pi} - 1\right)$$

We rewrite $e^{-ik\pi}$:

$$e^{-i\cdot k\pi} = cos(k\pi) - i\cdot sin(k\pi) = (-1)^k$$

We get:

$$c_k = \frac{i\cdot A}{2\pi k}\left((-1)^k - 1\right) = \begin{cases} 0 \quad for\ k = \pm 2, \pm 4, \dots \\ -\frac{i\cdot A}{\pi k} \quad for\ k = \pm 1, \pm 3, \dots \end{cases}$$

The result is the complex Fourier series:

$$f(t) = \frac{A}{2} - \frac{i\cdot A}{\pi}e^{i\omega t} + \frac{i\cdot A}{\pi}e^{-i\omega t} - \frac{i\cdot A}{3\pi}e^{i3\omega t} + \frac{i\cdot A}{3\pi}e^{-i3\omega t} - \frac{i\cdot A}{5\pi}e^{i5\omega t} + \frac{i\cdot A}{5\pi}e^{-i5\omega t} \mp \dots$$

$$= \frac{A}{2} + \frac{A}{\pi}\sum_{k=1}^{\infty}\frac{i\cdot\left(-e^{i(2k-1)\omega t} + e^{-i(2k-1)\omega t}\right)}{(2k-1)}$$

From the complex Fourier series, we get the real Fourier series by transforming the complex summands:

$$i\cdot\left(-e^{i(2k-1)\omega t} + e^{-i(2k-1)\omega t}\right) =$$

$$i \cdot \left(-cos\left((2k-1)\omega t\right) - i \cdot sin\left((2k-1)\omega t\right) + cos\left((2k-1)\omega t\right) - i \cdot sin\left((2k-1)\omega t\right)\right) =$$

$$2sin\left((2k-1)\omega t\right)$$

We get:

$$f(t) = \frac{A}{2} + \frac{A}{\pi}\sum_{k=1}^{\infty} \frac{i \cdot \left(-e^{i(2k-1)\omega t} + e^{-i(2k-1)\omega t}\right)}{(2k-1)} =$$

$$\frac{A}{2} + \frac{2A}{\pi}\sum_{k=1}^{\infty} \frac{(2k-1)\omega t}{(2k-1)} =$$

$$\frac{A}{2} + \frac{2A}{\pi}\left(sin(\omega t) + \frac{sin(3\omega t)}{3} + \frac{sin(5\omega t)}{5} + \cdots\right)$$

We compare with the square wave oscillation on S. 12, there we had $A = 1$:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(sin(\omega t) + \frac{1}{3}sin(3\omega t) + \frac{1}{5}sin(5\omega t) + \cdots\right)$$

Both calculation methods give the same result.

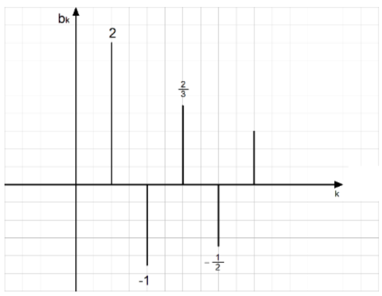
Fourier-Integral

Periodic oscillations can be described as Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t} \;, \omega_0 = \frac{2\pi}{T}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ik\omega_0 t} dt \quad k = 0, \pm 1, \pm 2, \ldots$$

In the spectra, the fundamental frequency corresponds to ω_0 the distance between successive spectral lines.



We set for the distance $\omega_0 = \Delta\omega$ and rewrite:

$$\omega_0 = \frac{2\pi}{T} \rightarrow \frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

$$\omega_k = k\omega_0$$

We get equivalent equations:

$$f(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (c_k \cdot T) e^{i\omega_k t}\, \Delta\omega \qquad (17)$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_k t} dt \quad k = 0, \pm 1, \pm 2, \ldots \qquad (18)$$

We increase the period of the periodic function and finally get a nonperiodic function, which we can describe with a Fourier series.

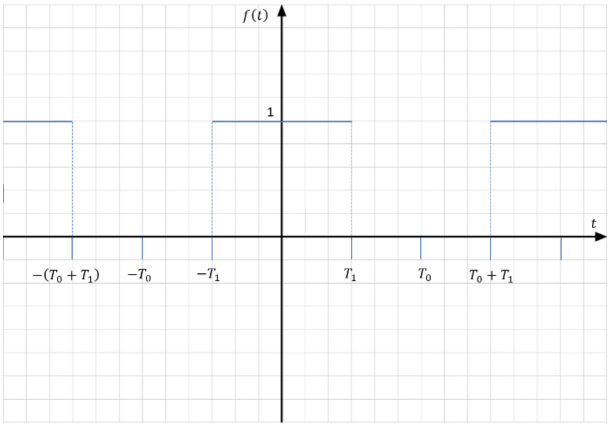
$$T \rightarrow \infty, \Delta\omega \rightarrow 0$$

Formally, we obtain from (17) and (18):

$c_k \cdot T \to \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_k t} dt$	$S(\omega) \to \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	19
$f_T(\tau) \to \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (c_k \cdot T) e^{i\omega_k t} \, \Delta\omega$	$f(t) \to \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} dt$	20

Note: The method corresponds to the method used in developing of the integral. From a sum of rectangles, the continuous integral is created by refining the rectangles.

Example.



The graph shows a periodic square wave of duration $T_i = 2T_1$ and the period $T = 2T_0$.

$$f(t) = \begin{cases} 1 & for \quad 0 \le |t| < T_1 \\ 0 & for \quad T_1 < |t| < T_0 \end{cases}$$

$$f(t+T) = f(t)$$

From (18) we determine the Fourier coefficients:

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_k t} dt \quad k = 0, \pm 1, \pm 2, \ldots$$

For c_0 we obtain:

$$c_0 = \frac{1}{2T_0} \int_{-T_0}^{T_0} f(t) dt = \frac{1}{2T_0} [t]_{-T_1}^{T_1} = \frac{2T_1}{2T_0} = \frac{T_1}{T_0} \qquad (21)$$

We calculate the coefficients c_k. We use:

$$T = 2T_0, \omega_k = k\omega_0, \omega = \frac{\pi}{T_0}, f(t) = 1 \text{ für } 0 \le |t| < T_1$$

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i\omega_k t} dt \Longrightarrow$$

$$c_k = \frac{1}{2T_0} \int_{-T_1}^{T_1} e^{-ik\omega_0 t} dt = -\frac{1}{2T_0 ik\omega_0} [e^{-ik\omega_0 t}]_{-T_1}^{T_1} =$$

$$-\frac{1}{2T_0ik\omega_0}\left(e^{-i\cdot k\omega_0T_1}-e^{i\cdot k\omega_0T_1}\right)=\frac{2i}{2T_0ik\omega_0}sin(k\omega_0T_1)=$$

$$\frac{1}{T_0k\omega_0}sin\left(k\frac{2\pi}{2T_0}T_1\right)=\frac{1}{T_0k\frac{2\pi}{2T_0}}sin\left(k\pi\frac{T_1}{T_0}\right)=$$

$$=\frac{1}{\pi k}sin\left(k\pi\frac{T_1}{T_0}\right)$$

We get:

$$c_k=\frac{1}{\pi k}sin\left(k\pi\frac{T_1}{T_0}\right) \tag{22}$$

Using the coefficients, we write the Fourier series:

$$f(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}c_ke^{i\omega_kt}\,\Delta\omega=$$

$$\frac{T_1}{T_0}+\frac{2T_0}{2\pi}\sum_{k=-\infty}^{\infty}\frac{1}{\pi k}sin\left(k\pi\frac{T_1}{T_0}\right)e^{i\omega_kt}\omega_0=$$

$$\frac{T_1}{T_0}+\frac{T_0\omega_0}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{k}sin\left(k\pi\frac{T_1}{T_0}\right)e^{ik\omega_0t}\;=$$

$$\frac{T_1}{T_0}+\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{k}sin\left(k\pi\frac{T_1}{T_0}\right)e^{ik\omega_0t}\;=$$

$$\frac{T_1}{T_0}+\frac{1}{\pi}\left(sin\left(\pi\frac{T_1}{T_0}\right)e^{i\omega_0t}-sin\left(-\pi\frac{T_1}{T_0}\right)e^{-i\omega_0t}+\frac{1}{2}sin\left(2\pi\frac{T_1}{T_0}\right)e^{i2\omega_0t}-\frac{1}{2}sin\left(-2\pi\frac{T_1}{T_0}\right)e^{-i2\omega_0t}\right.$$
$$\left.+\frac{1}{3}sin\left(3\pi\frac{T_1}{T_0}\right)e^{i3\omega_0t}-\frac{1}{2}sin\left(-3\pi\frac{T_1}{T_0}\right)e^{-i3\omega_0t}\pm\cdots\right)=$$

$$\frac{T_1}{T_0}+\frac{1}{\pi}\left(sin\left(\pi\frac{T_1}{T_0}\right)e^{i\omega_0t}+sin\left(\pi\frac{T_1}{T_0}\right)e^{-i\omega_0t}+\frac{1}{2}sin\left(2\pi\frac{T_1}{T_0}\right)e^{i2\omega_0t}+\frac{1}{2}sin\left(2\pi\frac{T_1}{T_0}\right)e^{-i2\omega_0t}\right.$$
$$\left.+\frac{1}{3}sin\left(3\pi\frac{T_1}{T_0}\right)e^{i3\omega_0t}+\frac{1}{2}sin\left(3\pi\frac{T_1}{T_0}\right)e^{-i3\omega_0t}\pm\cdots\right)=$$

$$\frac{T_1}{T_0}+\frac{1}{\pi}\left(sin\left(\pi\frac{T_1}{T_0}\right)\left(e^{i\omega_0t}+e^{-i\omega_0t}\right)+\frac{1}{2}sin\left(2\pi\frac{T_1}{T_0}\right)\left(e^{i2\omega_0t}+e^{-i2\omega_0t}\right)\right.$$
$$\left.+\frac{1}{3}sin\left(3\pi\frac{T_1}{T_0}\right)\left(e^{i3\omega_0t}+e^{-i3\omega_0t}\right)+\cdots\right)=$$

$$\frac{T_1}{T_0}+\frac{1}{\pi}\left(\frac{2}{1}sin\left(\pi\frac{T_1}{T_0}\right)cos(\omega_0t)+\frac{2}{2}sin\left(2\pi\frac{T_1}{T_0}\right)cos(2\omega_0t)+\frac{2}{3}sin\left(3\pi\frac{T_1}{T_0}\right)cos(3\omega_0t)+\cdots\right)=$$

$$\frac{T_1}{T_0}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}sin\left(k\pi\frac{T_1}{T_0}\right)cos(k\omega_0t)$$

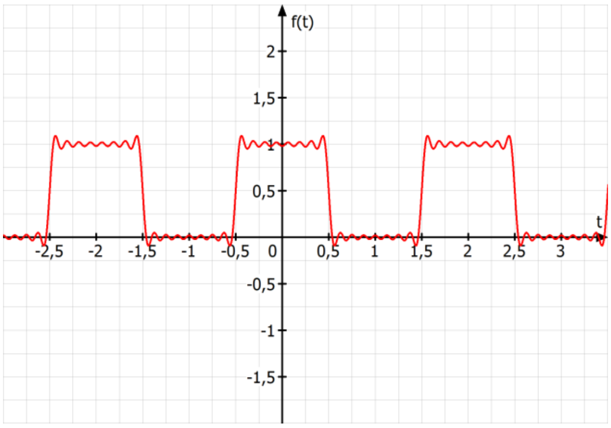
We take a look at the approximation s_{15} for the ratio $\frac{T_1}{T_0} = \frac{1}{2}$ and $T_0 = 1$.

We get ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2T_0} = \pi$$

$$s_{15} = \frac{T_1}{T_0} + \frac{2}{\pi}\sum_{k=1}^{15} \frac{1}{k} sin\left(k\pi\frac{1}{2}\right) cos(k\omega_0 t) =$$

$$\frac{1}{2} + \frac{2}{\pi}\left(cos(\omega_0 t) - \frac{1}{3}cos(3\omega_0 t) + \frac{1}{5}cos(5\omega_0 t) - \frac{1}{7}cos(7\omega_0 t) + \cdots - \frac{1}{15}cos(15\omega_0 t)\right)$$

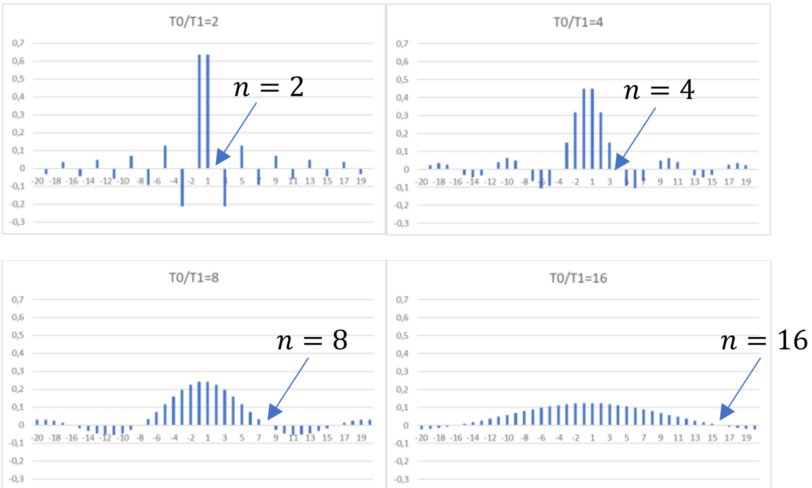


In order to prepare for the limes $T \to \infty$, we take a look at the spectrum for different ratios of $n = \frac{T_0}{T_1}$. T_0 is the duration of the period, if T_0 it becomes larger, n grows also.

(21) and (2 2) become:

$$c_0 = \frac{1}{n}, c_k = \frac{1}{\pi k} sin\left(k\pi\frac{T_1}{T_0}\right)$$

The graphs have the following appearance for different n:



The larger n, the wider and flatter the graphs.

We normalize the spectra by multiplying with the period duration and determine $c_k \cdot T$.

Again, we have $T = 2T_0, \omega_k = k\omega_0, \omega = \frac{\pi}{T_0}$ and get:

$$c_k \cdot T = \frac{T}{\pi k} sin\left(k\pi \frac{T_1}{T_0}\right) = \frac{2T_0}{\pi k} sin(k\omega_0 T_1) =$$

$$\frac{2}{\omega_k} sin(\omega_k T_1) =;$$

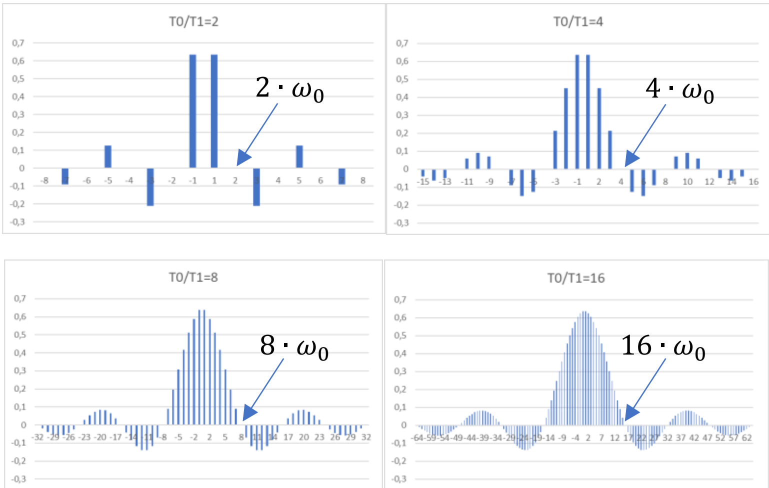
All spectral lines have the same envelope if we use the parameter $u \coloneqq \omega_k T_1$:

$$2T_1 \cdot \frac{sin(\omega_k T_1)}{\omega_k T_1}$$ 23

$$F(u) = 2T_1 \frac{sin(u)}{u}$$

The envelope has the zeros $u = \pm\pi, \pm 2\pi$, ...

We get the following representation on $k\omega_0$ the abscissa (x-axis).



The number of spectral lines between the zero crossings is growing. The first zero digit belongs to n-th fundamental frequency ω_0.

Limes

We now consider $T \to \infty$. The discrete sequence of spectral lines $\{c_k \cdot T\}$ changes to the continuous function $F(u)$:

$$u = \omega_k T_1 \to \omega T_1$$

$$\{c_k \cdot T\}_{T \to \infty} \to F(u) = 2T_1 \frac{sin(u)}{u}$$

$F(u) = F(\omega T_1)$ is called spectral density and denoted by $S(\omega)$.

We go back to (19) and (20).

We define the relationship between the function $S(\omega)$ and $f(t)$:

$$S(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} dt$$

We call the spectral density $S(\omega)$ of the time signal $f(t)$.

The Integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega t}dt$$

is called the Fourier integral.

We refer to the relationship between $f(t)$ and $S(\omega)$ as the Fourier transform. The function in the time domain $f(t)$ (original) corresponds to the function $S(\omega)$ in the frequency domain (image). Such mappings between functions we call functional transformations.

Up to now we have the following results:

- If a function $f(t)$ is periodic, we get a discrete spectrum, $\{c_k\}$
- If a function is not periodic, we get a continuous spectrum $S(\omega)$.

The spectral density of our example:

$$S(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

The definition (square wave momentum of height A, aperiodic function):

$$f(t) = \begin{cases} A \quad for & 0 \le |t| < T_1 \\ 0 & else \end{cases}$$

We calculate $S(\omega)$:

$$S(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt =$$

$$\int_{-T_1}^{T_1} Ae^{-i\omega t}dt = -\frac{A}{i\omega}\left[e^{-i\omega t}\right]_{-T_1}^{T_1} =$$

$$-\frac{A}{i\omega}\left(e^{-i\omega T_1} - e^{i\omega T_1}\right) =$$

$$\frac{A}{i\omega}\left(2i \cdot sin(\omega T_1)\right) =$$

$$\frac{2A}{\omega} sin(\omega T_1)$$

We extend numerator and nominator by T_1 and get:

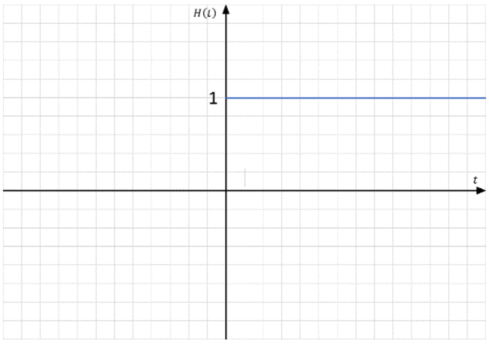
$$2AT_1 \frac{sin(\omega T_1)}{\omega T_1}$$

For $A = 1$ this is the same function we got with (23) by making the transition $T \to \infty$.

Generalized function

Heaviside step function

We define the Heaviside step function:



$$H(t) = \begin{cases} 1 \; for \; t > 0 \\ 0 \; for \; t < 0 \end{cases}$$

The Heaviside step function allows us to represent piecewise defined functions in closed form.

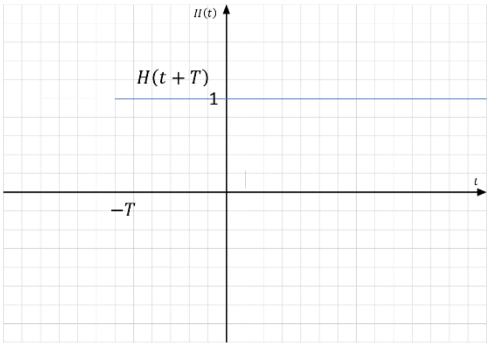
Example

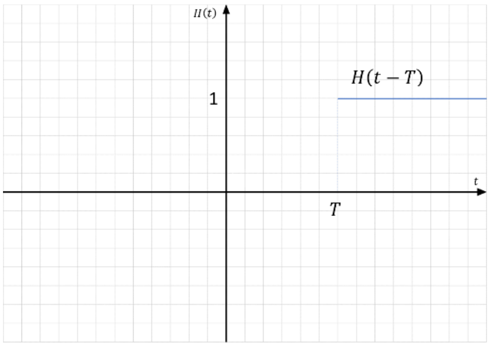
The rectangle function is defined as:

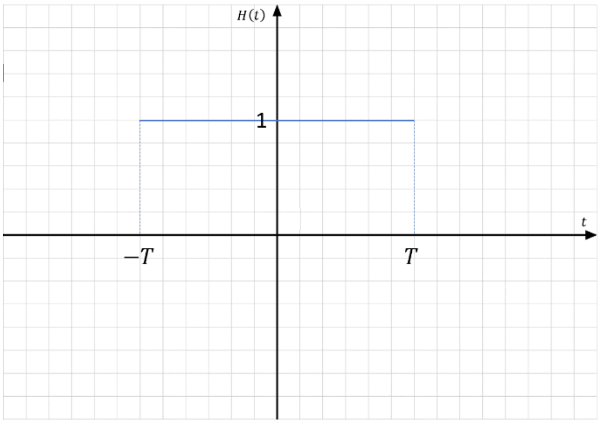
$$f(t) = \begin{cases} 1 \quad for \quad & |t| < T_1 \\ \;\, 0 & else \end{cases}$$

We write this as:

$$f(t) = H(t+T) - H(t-T)$$

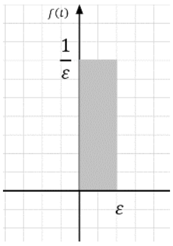






Dirac δ function

Note: Dirac momentum or the Dirac function is not a mathematically precise definition, not a classical function.



We start with a rectangle function $d_\varepsilon(t)$ of time duration ε and height $\frac{1}{\varepsilon}$.

The area of the function $d_\varepsilon(t)$ is 1.

We get the Dirac function by the limes:

$$\delta(t) = \lim_{\varepsilon \to 0} d_\varepsilon(t)$$

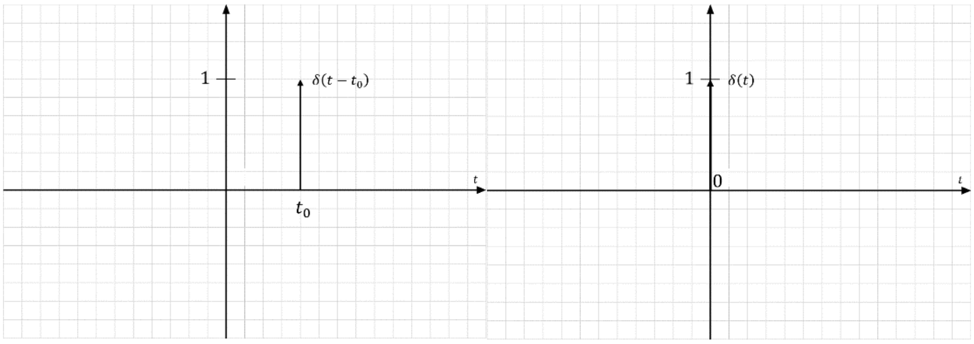
Since the area remains the same, we get the Dirac "function" $\delta(t)$:

$$\delta(t) = \begin{cases} \infty & for \;\; t = 0 \\ 0 & else \end{cases}$$

If we need the Dirac function for $t_0 \neq 0$, we write:

$$\delta(t - t_0) = \begin{cases} \infty & for \;\; t = t_0 \\ 0 & else \end{cases}$$

Usually we represent the Dirac function by an arrow of length 1 .



The Dirac function has the property:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

By help of the Dirac function we can extract specific values of other functions:

$$f(t) \cdot \delta(t - t_0) = f(t_0) \cdot \delta(t - t_0)$$

$$\int_{-\infty}^{\infty} f(t) \cdot \delta(t - t_0) dt = f(t_0)$$ 24

Example:

$$\int_{-\infty}^{\infty} sin(t) \cdot \delta\left(t - \frac{\pi}{2}\right) dt = sin\left(\frac{\pi}{2}\right)$$

$$\int_{-\infty}^{\infty} e^{-at} \cdot \delta(t) dt = e^0$$

We have a relationship between Heaviside and Dirac function, even though if it is not precise in the classical sense:

$$\delta(t) \approx \frac{dH(t)}{dt}$$

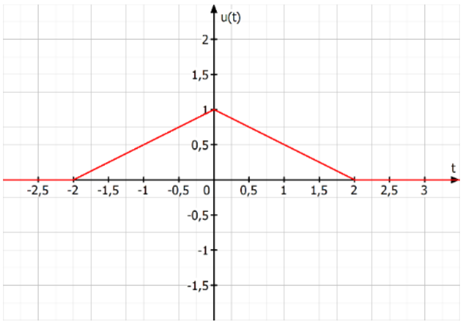
$$H(t) \approx \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Example:

We use the function $u(t)$:

$$u(t) = \left(\left(1 + \frac{t}{T}\right)\left(H(t+T) - H(t)\right)\right) + \left(\left(1 - \frac{t}{T}\right)\left(H(t) - H(t-T)\right)\right)$$

For value $T = 2$ we get the following picture:



We obtain the derivatives by differentiating with respect to time.

$$\dot{u}(t) = \frac{d}{dt}u(t) = \frac{d}{dt}\left(\left(1+\frac{t}{T}\right)\big(H(t+T)-H(t)\big)+\left(1-\frac{t}{T}\right)\big(H(t)-H(t-T)\big)\right) =$$

$$\frac{1}{T}\big(H(t+T)-H(t)\big)+\left(1+\frac{t}{T}\right)\big(\delta(t+T)-\delta(t)\big)-\frac{1}{T}\big(H(t)-H(t-T)\big)+\left(1-\frac{t}{T}\right)\big(\delta(t)-\delta(t-T)\big) =$$

$$\frac{1}{T}H(t+T)-\frac{1}{T}H(t)+\delta(t+T)-\delta(t)+\frac{t}{T}\delta(t+T)-\frac{t}{T}\delta(t)-\frac{1}{T}H(t)+\frac{1}{T}H(t-T)+\delta(t)-\delta(t-T)-\frac{t}{T}\delta(t)+\frac{t}{T}\delta(t-T) =$$

$$\frac{1}{T}\big(H(t+T)-H(t)-H(t)+H(t-T)\big)+\frac{t}{T}\big(\delta(t+T)-\delta(t)-\delta(t)+\delta(t-T)\big)+\delta(t+T)-\delta(t)+\delta(t)-\delta(t-T) =$$

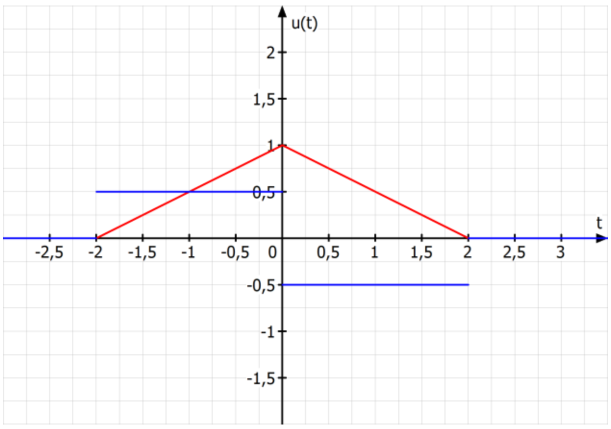
$$\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)\big)+\frac{t}{T}\big(\delta(t)(t+T)-2\delta(t)+\delta(t-T)\big)+\delta(t+T)-\delta(t-T) =$$

$$\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)\big)+\frac{1}{T}\big(t\delta(t)(t+T)-2t\delta(t)+t\delta(t-T)\big)+\delta(t+T)-\delta(t-T) =$$

$$\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)\big)+\frac{1}{T}\big(-T\delta(t)(t+T)-2\cdot 0\cdot\delta(t)+T\delta(t)(t-T)\big)+\delta(t+T)-\delta(t-T) =$$

$$\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)-T\delta(t)(t+T)+T\delta(t)(t-T)+T\delta(t+T)-T\delta(t-T)\big) =$$

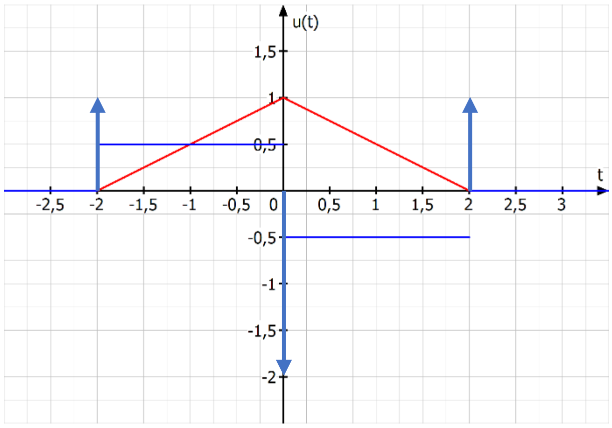
$$\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)\big)$$



The second derivative $\ddot{u}(t)$:

$$\ddot{u}(t) = \frac{d}{dt}\left(\frac{1}{T}\big(H(t+T)-2H(t)+H(t-T)\big)\right) =$$

$$\frac{1}{T}\big(\delta(t+T)-2\delta(t)+\delta(t-T)\big)$$



The first derivative has discontinuities where the function has edges.

The discontinuities of the first derivative have "infinite" slope.