Continuous functions can be represented as vectors. This paper takes a polynomial function, quantize it with ε and shows that the building of derivatives and integration can be done by using matrices.

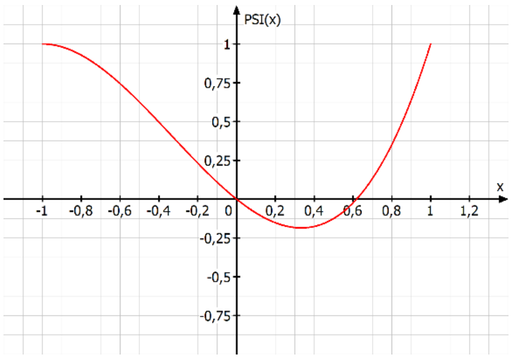
The commutator relation for position operator $\hat{x}$ and momentum operator $\hat{p}$ is valid for the quantized function too, the unavoidable error made by numeric is controllable by appropriate refining of ε.

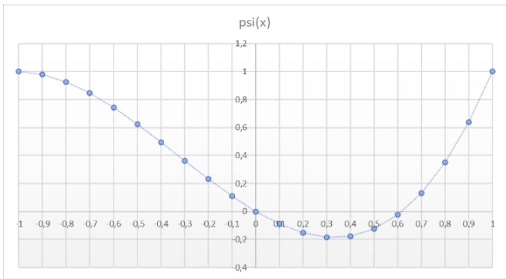
In Griffiths you might find this problem dealt with in "1.3 Probability".

Hope I can help you with learning quantum mechanics.

One basic concept of quantum mechanics works with the possibility that functions can be thought of as (infinite dimensional) vectors.

We take the function $\psi(x) = x^3 + x^2 - x$ and plot it and its quantized version:





We divide the x-axis into small units of ε. For every value of ε we get an infinite vector. Depending on the size of ε the vector changes its appearance but represents the same function. We write the vector symmetrically:

$$|\psi\rangle = \begin{pmatrix} \psi(-4\varepsilon) \\ \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \\ \psi(4\varepsilon) \end{pmatrix}$$

Note: we omit the “infinite rest” and work with the nine dimensions shown.

Note: all calculations are done by the help of maxima, working with 15×15- matrices and column vectors of dimension 15. The results shown are restricted to dimension 9. The border values shown therefore often does not fit with the results shown but are results of the full 15×15 calculations.

Note: we use a “symmetric” version of the vector with $\psi(0)$ in the middle to handle “infinity” better.

We can build the derivative of a function. Basically, we calculate the limit:

$$\psi'(x) = \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon) - \psi(x)}{\varepsilon}$$

According to the derivation rule we get the derived vector:

$$|\psi'\rangle = \frac{1}{\varepsilon} \cdot \begin{pmatrix} \psi(-3\varepsilon) - \psi(-4\varepsilon) \\ \psi(-2\varepsilon) - \psi(-3\varepsilon) \\ \psi(-\varepsilon) - \psi(-2\varepsilon) \\ \psi(0) - \psi(-\varepsilon) \\ \psi(\varepsilon) - \psi(0) \\ \psi(2\varepsilon) - \psi(\varepsilon) \\ \psi(3\varepsilon) - \psi(2\varepsilon) \\ \psi(4\varepsilon) - \psi(3\varepsilon) \\ \psi(5\varepsilon) - \psi(4\varepsilon) \end{pmatrix}$$

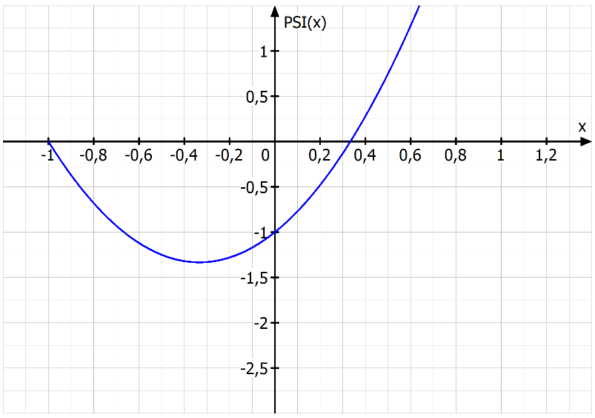
There must be a matrix D that performs this process:

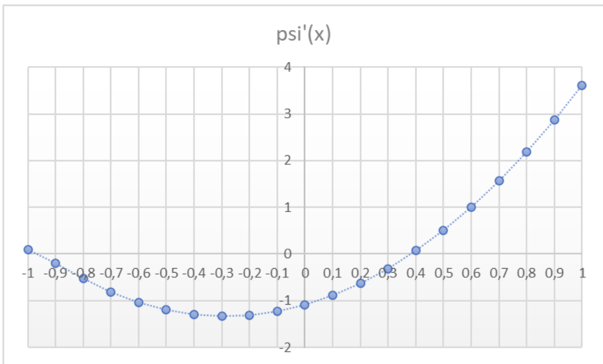
$$D := \frac{1}{\varepsilon} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We apply the matrix to $|\psi\rangle$:

$$\frac{1}{\varepsilon} \cdot \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \psi(-4\varepsilon) \\ \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \\ \psi(4\varepsilon) \end{pmatrix} = \frac{1}{\varepsilon} \cdot \begin{pmatrix} \psi(-3\varepsilon) - \psi(-4\varepsilon) \\ \psi(-2\varepsilon) - \psi(-3\varepsilon) \\ \psi(-\varepsilon) - \psi(-2\varepsilon) \\ \psi(0) - \psi(-\varepsilon) \\ \psi(\varepsilon) - \psi(0) \\ \psi(2\varepsilon) - \psi(\varepsilon) \\ \psi(3\varepsilon) - \psi(2\varepsilon) \\ \psi(4\varepsilon) - \psi(3\varepsilon) \\ \psi(5\varepsilon) - \psi(4\varepsilon) \end{pmatrix}$$

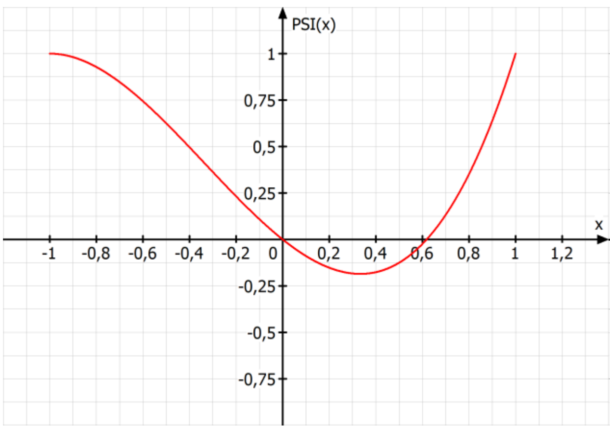
We check this graphically. The continuous function $\psi(x) = x^3 + x^2 - x$ has the derivative $\psi'(x) = 3x^2 + x - 1$ and shows the following picture:

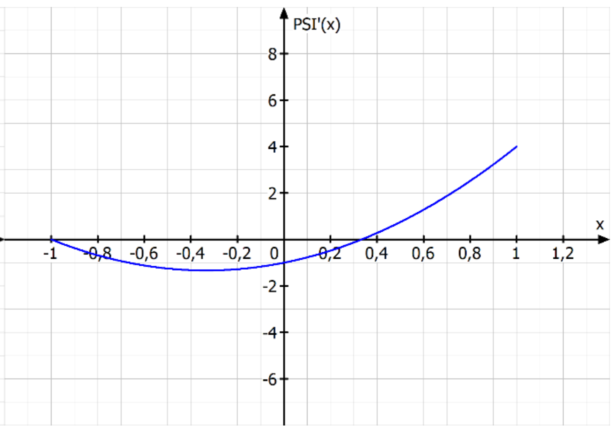


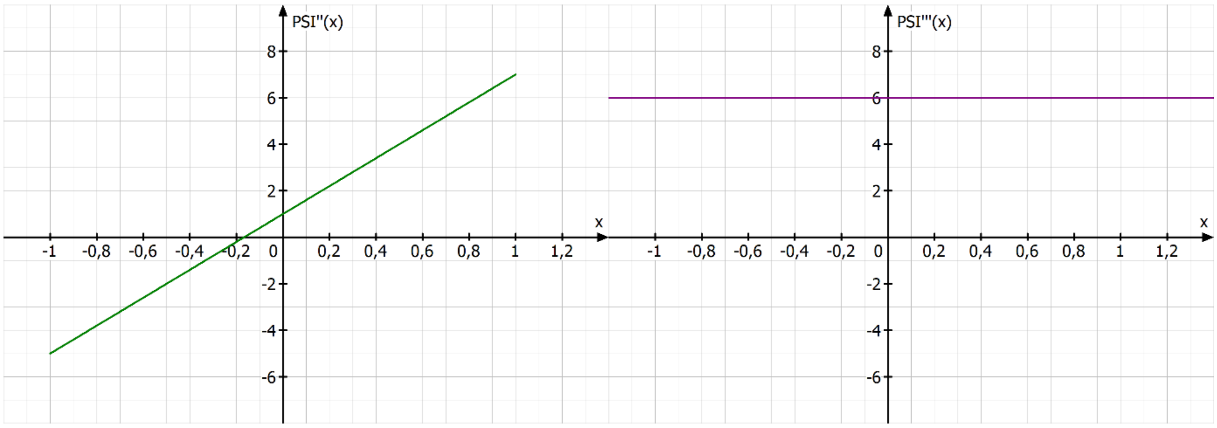


Perfect!

We calculate the other derivatives too. The function and its derivatives look like this:





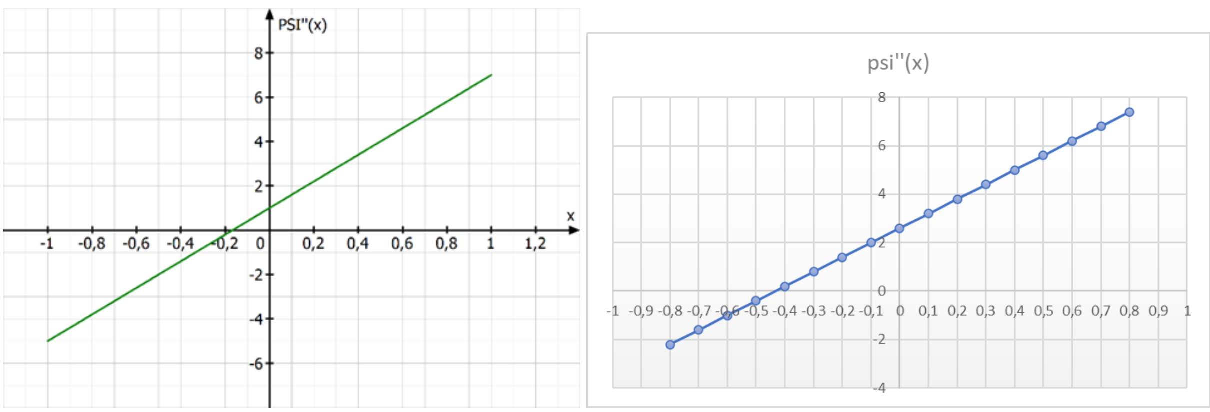


We calculate the second derivative by applying the matrix D to $|\psi'\rangle$:

$$\frac{1}{\varepsilon}\cdot\begin{pmatrix}-1&1&0&0&0&0&0&0&0\\0&-1&1&0&0&0&0&0&0\\0&0&-1&1&0&0&0&0&0\\0&0&0&-1&1&0&0&0&0\\0&0&0&0&-1&1&0&0&0\\0&0&0&0&0&-1&1&0&0\\0&0&0&0&0&0&-1&1&0\\0&0&0&0&0&0&0&-1&1\\0&0&0&0&0&0&0&0&-1\end{pmatrix}\cdot\frac{1}{\varepsilon}\cdot\begin{pmatrix}\psi(-3\varepsilon)-\psi(-4\varepsilon)\\\psi(-2\varepsilon)-\psi(-3\varepsilon)\\\psi(-\varepsilon)-\psi(-2\varepsilon)\\\psi(0)-\psi(-\varepsilon)\\\psi(\varepsilon)-\psi(0)\\\psi(2\varepsilon)-\psi(\varepsilon)\\\psi(3\varepsilon)-\psi(2\varepsilon)\\\psi(4\varepsilon)-\psi(3\varepsilon)\\\psi(5\varepsilon)-\psi(4\varepsilon)\end{pmatrix}=$$

$$\frac{1}{\varepsilon^2}\cdot\begin{pmatrix}\psi(-2\varepsilon)-2\psi(-3\varepsilon)+\psi(-4\varepsilon)\\\psi(-\varepsilon)-2\psi(-2\varepsilon)+\psi(-3\varepsilon)\\\psi(0)-2\psi(-\varepsilon)+\psi(-2\varepsilon)\\\psi(\varepsilon)-2\psi(0)+\psi(-\varepsilon)\\\psi(2\varepsilon)-2\psi(\varepsilon)+\psi(0)\\\psi(3\varepsilon)-2\psi(2\varepsilon)+\psi(\varepsilon)\\\psi(4\varepsilon)-2\psi(3\varepsilon)+\psi(2\varepsilon)\\\psi(5\varepsilon)-2\psi(4\varepsilon)+\psi(3\varepsilon)\\\psi(6\varepsilon)-2\psi(5\varepsilon)+\psi(4\varepsilon)\end{pmatrix}$$

We compare graphically continuous and quantized version:



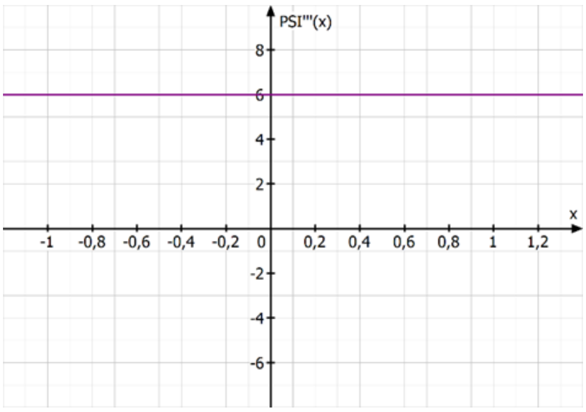
We get a line that drifts upwards. The slope of the drifted line is correct 6. The absolute value is too high, it seems that the line needs to be shifted two units to the right.

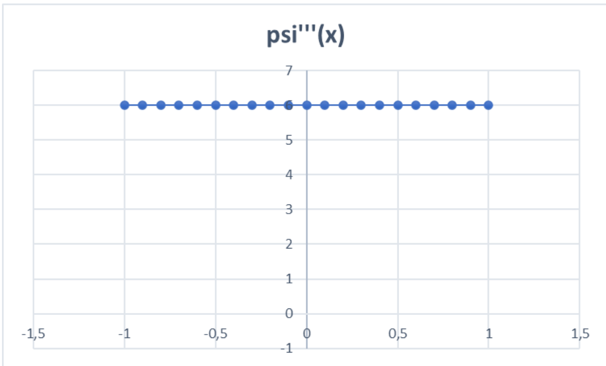
We calculate the third derivative by applying the matrix D to $|\psi''\rangle$:

$$\frac{1}{\varepsilon}\cdot\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\cdot\frac{1}{\varepsilon^2}\cdot\begin{pmatrix} \psi(-2\varepsilon)-2\psi(-3\varepsilon)+\psi(-4\varepsilon) \\ \psi(-\varepsilon)-2\psi(-2\varepsilon)+\psi(-3\varepsilon) \\ \psi(0)-2\psi(-\varepsilon)+\psi(-2\varepsilon) \\ \psi(\varepsilon)-2\psi(0)+\psi(-\varepsilon) \\ \psi(2\varepsilon)-2\psi(\varepsilon)+\psi(0) \\ \psi(3\varepsilon)-2\psi(2\varepsilon)+\psi(\varepsilon) \\ \psi(4\varepsilon)-2\psi(3\varepsilon)+\psi(2\varepsilon) \\ \psi(5\varepsilon)-2\psi(4\varepsilon)+\psi(3\varepsilon) \\ \psi(6\varepsilon)-2\psi(5\varepsilon)+\psi(4\varepsilon) \end{pmatrix} =$$

$$\frac{1}{\varepsilon^3}\cdot\begin{pmatrix} \psi(-\varepsilon)-3\psi(-2\varepsilon)+3\psi(-3\varepsilon)-\psi(-4\varepsilon) \\ \psi(0)-3\psi(-\varepsilon)+3\psi(-2\varepsilon)-\psi(-3\varepsilon) \\ \psi(\varepsilon)-3\psi(0)+3\psi(-\varepsilon)-\psi(-2\varepsilon) \\ \psi(2\varepsilon)-3\psi(\varepsilon)+3\psi(0)-\psi(-\varepsilon) \\ \psi(3\varepsilon)-3\psi(2\varepsilon)+3\psi(\varepsilon)-\psi(0) \\ \psi(4\varepsilon)-3\psi(3\varepsilon)+3\psi(2\varepsilon)-\psi(\varepsilon) \\ \psi(5\varepsilon)-3\psi(4\varepsilon)+3\psi(3\varepsilon)-\psi(2\varepsilon) \\ \psi(6\varepsilon)-3\psi(5\varepsilon)+3\psi(4\varepsilon)-\psi(3\varepsilon) \\ \psi(7\varepsilon)-3\psi(6\varepsilon)+3\psi(5\varepsilon)-\psi(4\varepsilon) \end{pmatrix}$$

We compare graphically the continuous and the quantized version:





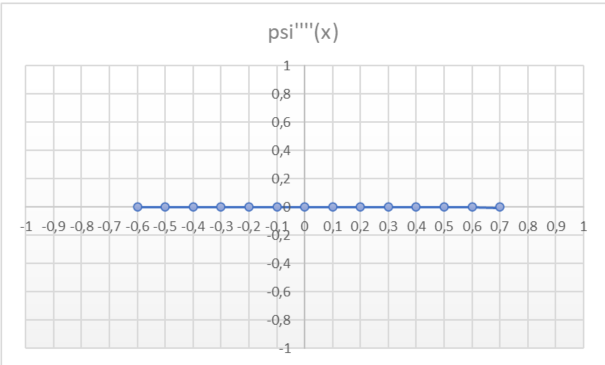
Perfect!

We calculate the fourth derivative by applying the matrix D to $|\psi'''\rangle$:

$$\frac{1}{\varepsilon}\cdot\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\cdot\frac{1}{\varepsilon^3}\cdot\begin{pmatrix} \psi(-\varepsilon)-3\psi(-2\varepsilon)+3\psi(-3\varepsilon)-\psi(-4\varepsilon) \\ \psi(0)-3\psi(-\varepsilon)+3\psi(-2\varepsilon)-\psi(-3\varepsilon) \\ \psi(\varepsilon)-3\psi(0)+3\psi(-\varepsilon)-\psi(-2\varepsilon) \\ \psi(2\varepsilon)-3\psi(\varepsilon)+3\psi(0)-\psi(-\varepsilon) \\ \psi(3\varepsilon)-3\psi(2\varepsilon)+3\psi(\varepsilon)-\psi(0) \\ \psi(4\varepsilon)-3\psi(3\varepsilon)+3\psi(2\varepsilon)-\psi(\varepsilon) \\ \psi(5\varepsilon)-3\psi(4\varepsilon)+3\psi(3\varepsilon)-\psi(2\varepsilon) \\ \psi(6\varepsilon)-3\psi(5\varepsilon)+3\psi(4\varepsilon)-\psi(3\varepsilon) \\ \psi(7\varepsilon)-3\psi(6\varepsilon)+3\psi(5\varepsilon)-\psi(4\varepsilon) \end{pmatrix} =$$

$$\frac{1}{\varepsilon^4}\cdot\begin{pmatrix} \psi(0)-4\psi(-\varepsilon)+6\psi(-2\varepsilon)-4\psi(-3\varepsilon)+\psi(-4\varepsilon) \\ \psi(\varepsilon)-4\psi(0)+6\psi(-\varepsilon)-4\psi(-2\varepsilon)+\psi(-3\varepsilon) \\ \psi(2\varepsilon)-4\psi(\varepsilon)+6\psi(0)-4\psi(-\varepsilon)+\psi(-2\varepsilon) \\ \psi(3\varepsilon)-4\psi(2\varepsilon)+6\psi(\varepsilon)-4\psi(0)+\psi(-\varepsilon) \\ \psi(4\varepsilon)-4\psi(3\varepsilon)+6\psi(2\varepsilon)-4\psi(\varepsilon)+\psi(0) \\ \psi(5\varepsilon)-4\psi(4\varepsilon)+6\psi(3\varepsilon)-4\psi(2\varepsilon)+\psi(\varepsilon) \\ \psi(6\varepsilon)-4\psi(5\varepsilon)+6\psi(4\varepsilon)-4\psi(3\varepsilon)+\psi(2\varepsilon) \\ \psi(7\varepsilon)-4\psi(6\varepsilon)+6\psi(5\varepsilon)-4\psi(4\varepsilon)+\psi(3\varepsilon) \\ \psi(8\varepsilon)-4\psi(7\varepsilon)+6\psi(6\varepsilon)-4\psi(5\varepsilon)+\psi(4\varepsilon) \end{pmatrix}$$

As it should be, we get a flat line:



The numeric values are about 10^{-12}, we interpret this as zero.

We observe a pattern in the process of building the derivatives. The matrices for building the derivatives evolve as follows:

The first derivative:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The second derivative:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The patterns for the original function and the derivatives are:

$$\begin{array}{ccccccc} 1 & & & & & & \\ -1 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ -1 & 3 & -3 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ -1 & 5 & -10 & 10 & -5 & 1 & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{array}$$

The coefficients follow the rule of Pascal's triangle with alternating signs. This leads to the following behavior:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The sum of the n^{th} and $(n+1)^{th}$ derivative shifts the n^{th} derivative on position to the right:

$$f^{(n)}(\varepsilon) + f^{(n+1)}(\varepsilon) = f^{(n)}(\varepsilon + 1)$$

This is according to the derivation limit:

$$f^{(n+1)}(\varepsilon) \approx \lim_{\varepsilon \to 0} \frac{f^{(n)}(\varepsilon + 1) - f^{(n)}(\varepsilon)}{\varepsilon}$$

Integration

We try the same procedure with the integration process.

We take the derivation matrix D

$$D := \frac{1}{\varepsilon} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and build the inverse matrix Int:

$$Int := \varepsilon \cdot \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We check whether the matrix Int applied to the first derivative $|\psi'\rangle$ gives back $|\psi\rangle$:

$$Int|\psi'\rangle = \varepsilon \cdot \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \frac{1}{\varepsilon} \cdot \begin{pmatrix} \psi(-3\varepsilon) - \psi(-4\varepsilon) \\ \psi(-2\varepsilon) - \psi(-3\varepsilon) \\ \psi(-\varepsilon) - \psi(-2\varepsilon) \\ \psi(0) - \psi(-\varepsilon) \\ \psi(\varepsilon) - \psi(0) \\ \psi(2\varepsilon) - \psi(\varepsilon) \\ \psi(3\varepsilon) - \psi(2\varepsilon) \\ \psi(4\varepsilon) - \psi(3\varepsilon) \\ \psi(5\varepsilon) - \psi(4\varepsilon) \end{pmatrix} =$$

$$\begin{pmatrix} \psi(-4\varepsilon) \\ \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \\ \psi(4\varepsilon) \end{pmatrix}$$

Perfect!

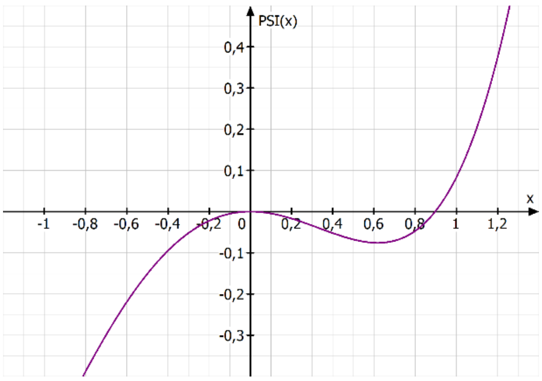
We apply the integration matrix to $|\psi\rangle$:

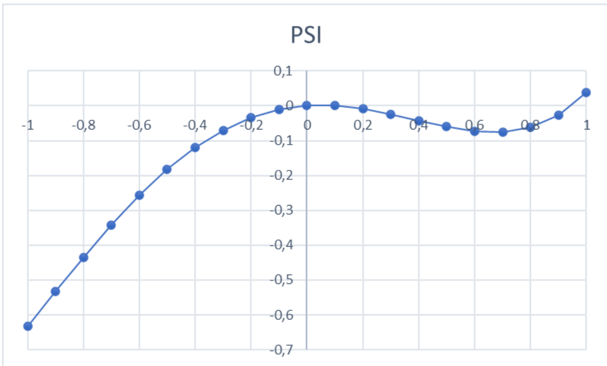
$$Int|\psi\rangle = \varepsilon \cdot \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \psi(-4\varepsilon) \\ \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \\ \psi(4\varepsilon) \end{pmatrix} =$$

$$-\varepsilon \cdot \begin{pmatrix} \psi(-4\varepsilon)+\psi(-3\varepsilon)+\psi(-2\varepsilon)+\psi(-\varepsilon)+\psi(0)+\psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon) \\ \psi(-3\varepsilon)+\psi(-2\varepsilon)+\psi(-\varepsilon)+\psi(0)+\psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon) \\ \psi(-2\varepsilon)+\psi(-\varepsilon)+\psi(0)+\psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon) \\ \psi(-\varepsilon)+\psi(0)+\psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon) \\ \psi(0)+\psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon)+\psi(8\varepsilon) \\ \psi(\varepsilon)+\psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon)+\psi(8\varepsilon)+\psi(9\varepsilon) \\ \psi(2\varepsilon)+\psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon)+\psi(8\varepsilon)+\psi(9\varepsilon)+\psi(10\varepsilon) \\ \psi(3\varepsilon)+\psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon)+\psi(8\varepsilon)+\psi(9\varepsilon)+\psi(10\varepsilon)+\psi(11\varepsilon) \\ \psi(4\varepsilon)+\psi(5\varepsilon)+\psi(6\varepsilon)+\psi(7\varepsilon)+\psi(8\varepsilon)+\psi(9\varepsilon)+\psi(10\varepsilon)+\psi(11\varepsilon)+\psi(12\varepsilon) \end{pmatrix}$$

Note: the matrix Int has in contrast to matrix D an infinite number of entries. For this reason, we must add the red elements coming from "beyond the border".

We compare graphically continuous and quantized version:





Note: the quantized version shifted by the constant 0.1375, the result: perfect!

In quantum mechanics we work with the position operator $\hat{x}$ and the momentum operator $\hat{p}$. The position operator $\hat{x}$ applied to a wave function $|\psi\rangle$ multiplies it with the position x:

$$\hat{x}|\psi\rangle = x|\psi\rangle$$

The matrix for the position operator $\hat{x}$:

$$\hat{x} \coloneqq \varepsilon \cdot \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

We apply $\hat{x}$ to $|\psi\rangle$:

$$\hat{x}|\psi\rangle = \varepsilon \cdot \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} \psi(-4\varepsilon) \\ \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \\ \psi(4\varepsilon) \end{pmatrix} =$$

$$\begin{pmatrix} -4\varepsilon \cdot \psi(-4\varepsilon) \\ -3\varepsilon \cdot \psi(-3\varepsilon) \\ -2\varepsilon \cdot \psi(-2\varepsilon) \\ -\varepsilon \cdot \psi(-\varepsilon) \\ 0 \cdot \psi(0) \\ \varepsilon \cdot \psi(\varepsilon) \\ 2\varepsilon \cdot \psi(2\varepsilon) \\ 3\varepsilon \cdot \psi(3\varepsilon) \\ 4\varepsilon \cdot \psi(4\varepsilon) \end{pmatrix}$$

Each entry is multiplied by its argument.

The momentum operator $\hat{p}$ applied to a wave function $\psi(x)$ derivates it:

$$\hat{p}\psi(x) = -i\hbar \frac{d}{dx}\psi(x)$$

The momentum operator:

$$\hat{p} = \frac{-i\hbar}{\varepsilon} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The quantum mechanical operators $\hat{p}$ and $\hat{x}$ fulfill the commutator relation:

$$[\hat{x},\hat{p}] = (\hat{x}\hat{p} - \hat{p}\hat{x}) = i\hbar I$$

Note: I is the matrix representation of the unit 1.

We take the operators and calculate:

$$(\hat{x}\hat{p} - \hat{p}\hat{x}) =$$

$$\varepsilon\cdot\begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}\cdot\frac{-i\hbar}{\varepsilon}\cdot\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} -$$

$$\frac{-i\hbar}{\varepsilon}\cdot\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\cdot\varepsilon\cdot\begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} =$$

$$i\hbar\begin{pmatrix} -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} - i\hbar\begin{pmatrix} -4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix} =$$

$$i\hbar\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Obviously, this is not the identity matrix.

We calculate the error we have, leaving the factor $i\hbar$ out. We subtract the identity matrix I to get the error:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We apply the error to an arbitrary $|\psi\rangle$:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \psi(-3\varepsilon) \\ \psi(-2\varepsilon) \\ \psi(-\varepsilon) \\ \psi(0) \\ \psi(\varepsilon) \\ \psi(2\varepsilon) \\ \psi(3\varepsilon) \end{pmatrix} =$$

$$\begin{pmatrix} \psi(-2\varepsilon) - \psi(-3\varepsilon) \\ \psi(-\varepsilon) - \psi(-2\varepsilon) \\ \psi(0) - \psi(-\varepsilon) \\ \psi(\varepsilon) - \psi(0) \\ \psi(2\varepsilon) - \psi(\varepsilon) \\ \psi(3\varepsilon) - \psi(2\varepsilon) \\ \psi(4\varepsilon) - \psi(3\varepsilon) \end{pmatrix}$$

Note: $\psi(4\varepsilon)$ comes from outside the border.

Developing ψ by a Taylor series at (-3ε):

$$\psi(-2\varepsilon) = \psi(-3\varepsilon) + \varepsilon\psi'(-3\varepsilon) + \frac{1}{2}\varepsilon^2\psi''(-3\varepsilon) + \frac{1}{6}\varepsilon^3\psi''(-3\varepsilon) + \cdots$$

$$\psi(-2\varepsilon) - \psi(-3\varepsilon) = \varepsilon\psi'(-3\varepsilon) + \frac{1}{2}\varepsilon^2\psi''(-3\varepsilon) + \frac{1}{6}\varepsilon^3\psi''(-3\varepsilon) + \cdots$$

The error we get out of quantization depends on ε and ψ'. For continuous functions with a bound first derivative the error can be controlled by holding ε small enough.