This paper shows proofs related to the Gauss Integrals:

$$\int_{-\infty}^{\infty} exp(-x^2)dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} exp\left(-\frac{(x-x_0)^2}{a}\right)dx = \sqrt{\pi a}$$

$$\int_{-\infty}^{\infty} x \cdot exp\left(-\frac{(x-x_0)^2}{a}\right)dx = x_0 \cdot \sqrt{\pi a}$$

$$\int_{-\infty}^{\infty} x^2 \cdot exp\frac{(x-x_0)^2}{a}dx = \left(\frac{a}{2} + x_0\right) \cdot \sqrt{\pi a}$$

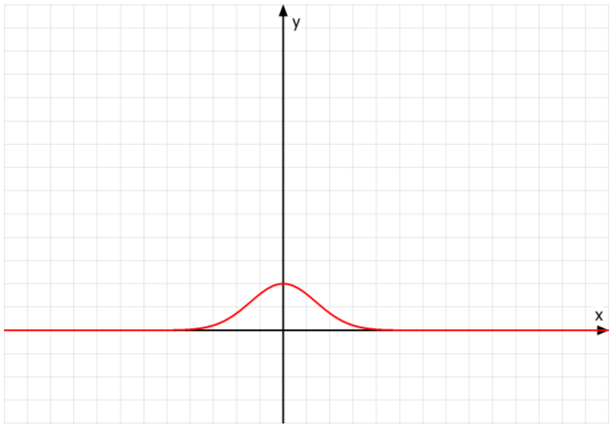
It follows:

Jan-Markus Schwindt, Conceptual Basis of Quantum Mechanics, ISBN 978-3-319-24524-9

Hope I can help you with learning quantum mechanics.

Proof one: $\int_{-\infty}^{\infty} exp(-x^2)dx = \sqrt{\pi}$

The function looks like:



We need:

$$\left(\int_{-\infty}^{\infty} exp(-x^2)dx\right)^2 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} exp\big(-(x^2+y^2)\big)dx\right)dy$$

We check:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} exp\big(-(x^2+y^2)\big)dx\right)dy =$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} exp(-x^2)exp(-y^2)dx\right)dy =$$

$$\int_{-\infty}^{\infty}\left(exp(-y^2)\int_{-\infty}^{\infty} exp(-x^2)dx\right)dy =$$

$$\int_{-\infty}^{\infty} exp(-x^2)\,dx \int_{-\infty}^{\infty} exp(-y^2)dy =$$

$$\int_{-\infty}^{\infty} exp(-x^2)\,dx \int_{-\infty}^{\infty} exp(-x^2)dx =$$

$$\left(\int_{-\infty}^{\infty} exp(-x^2)dx\right)^2$$

We rewrite in polar coordinates:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} exp\big(-(x^2+y^2)\big)dx\right)dy =$$

$$\int_{0}^{2\pi}\left(\int_{0}^{\infty} r\cdot exp(-r^2)dr\right)d\varphi =;$$

Note: r is a correction factor needed because the circumference with radius r is r-times the circumference of the unit circle. Integration along the angle φ gives r-times the contribution.

The inner integral is independent of the angle φ and gives 2π:

$$2\pi\int_0^\infty r\cdot exp(-r^2)dr$$

We substitute $u \coloneqq r^2$:

$$u \coloneqq r^2 \;\rightarrow\; \frac{du}{dr} = 2r \rightarrow dr = \frac{du}{2r}$$

We rewrite the integral:

$$2\pi\int_0^\infty r\cdot exp(-u)\frac{du}{2r} =$$

$$\pi\int_0^\infty exp(-u)du =$$

$$-\pi\cdot exp(-u)_0^\infty =$$

$$-\pi\cdot(0-1) = \pi$$

We get:

$$\left(\int_{-\infty}^\infty exp(-x^2)dx\right)^2 = \pi^2$$

It follows:

$$\int_{-\infty}^\infty exp(-x^2)dx = \sqrt{\pi}$$

Proof two: $\int_{-\infty}^\infty exp\left(-\frac{(x-x_0)^2}{a}\right)dx = \sqrt{\pi a}$

We substitute:

$$u \coloneqq \frac{x-x_0}{\sqrt{a}}$$

$$u \coloneqq \frac{x-x_0}{\sqrt{a}} \;\rightarrow\; \frac{du}{dx} = \frac{1}{\sqrt{a}}; dx = \sqrt{a}\cdot du$$

$$\int_{-\infty}^\infty exp\left(-\frac{(x-x_0)^2}{a}\right)dx \rightarrow$$

$$\int_{-\infty}^\infty exp(-u^2)\sqrt{a}\cdot du =$$

$$\sqrt{a}\cdot\int_{-\infty}^\infty exp(-u^2)du =$$

$$\sqrt{a}\cdot\sqrt{\pi}$$

Note: this holds for complex valued x_0 resp. $a > 0$ too.

Proof three: $\int_{-\infty}^{\infty} x \cdot exp\left(-\frac{(x-x_0)^2}{a}\right) dx = x_0 \cdot \sqrt{\pi a}$

We substitute:

$$u := \frac{x - x_0}{\sqrt{a}}$$

$$u := \frac{x - x_0}{\sqrt{a}} \rightarrow \frac{du}{dx} = \frac{1}{\sqrt{a}}; dx = \sqrt{a} \cdot du; x = u\sqrt{a} + x_0$$

$$\int_{-\infty}^{\infty} x \cdot exp\left(-\frac{(x - x_0)^2}{a}\right) dx \rightarrow$$

$$\int_{-\infty}^{\infty} \left(u\sqrt{a} + x_0\right) \cdot exp(-u^2)\sqrt{a} \cdot du =$$

$$\int_{-\infty}^{\infty} u \cdot a \cdot exp(-u^2)\sqrt{a} \cdot du + \int_{-\infty}^{\infty} x_0 \cdot exp(-u^2)\sqrt{a} \cdot du =$$

$$0 + \int_{-\infty}^{\infty} x_0 \cdot exp(-u^2) \cdot \sqrt{a} \cdot du =$$

$$x_0 \cdot \sqrt{a} \int_{-\infty}^{\infty} exp(-u^2) \cdot du =$$

$$x_0 \cdot \sqrt{a} \cdot \sqrt{\pi}$$

Proof four: $\int_{-\infty}^{\infty} x^2 \cdot exp \frac{(x-x_0)^2}{a} dx = \left(\frac{a}{2} + x_0\right) \cdot \sqrt{\pi a}$

We substitute:

$$u := \frac{x - x_0}{\sqrt{a}}$$

$$u := \frac{x - x_0}{\sqrt{a}} \rightarrow \frac{du}{dx} = \frac{1}{\sqrt{a}}; dx = \sqrt{a} \cdot du; x = u\sqrt{a} + x_0$$

$$\int_{-\infty}^{\infty} x^2 \cdot exp\left(-\frac{(x - x_0)^2}{a}\right) dx \rightarrow$$

$$\int_{-\infty}^{\infty} \left(u\sqrt{a} + x_0\right)^2 \cdot \left(exp(-u^2)\right)\sqrt{a}du =$$

$$\int_{-\infty}^{\infty} \left(u^2 a + 2u\sqrt{a}x_0 + {x_0}^2\right) \cdot \left(exp(-u^2)\right) \cdot \sqrt{a}du =$$

$$\int_{-\infty}^{\infty} u^2 a\left(exp(-u^2)\right)\sqrt{a}du + \int_{-\infty}^{\infty} 2u\sqrt{a}x_0\left(exp(-u^2)\right)\sqrt{a}du + \int_{-\infty}^{\infty} {x_0}^2\left(exp(-u^2)\right)\sqrt{a}du =$$

$$a\sqrt{a} \cdot \int_{-\infty}^{\infty} u^2 \cdot exp(-u^2)du + 0 + {x_0}^2\sqrt{a} \int_{-\infty}^{\infty} exp(-u^2)du =$$

$$a\sqrt{a} \cdot \int_{-\infty}^{\infty} u^2 \cdot exp(-u^2)du + 0 + {x_0}^2\sqrt{a\pi} =;$$

We integrate by parts:

$$\int_{-\infty}^{\infty} f' \cdot g = f \cdot g - \int_{-\infty}^{\infty} f \cdot g'$$

We set:

$$f' \coloneqq u \cdot exp(-u^2), g \coloneqq u \to f = -\frac{exp(-u^2)}{2}, g' = 1$$

$$\int_{-\infty}^{\infty} u^2 \cdot exp(-u^2) du \to$$

$$= \left[-\frac{exp(-u^2)}{2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{exp(-u^2)}{2} du =$$

$$= 0 - \int_{-\infty}^{\infty} -\frac{exp(-u^2)}{2} du =$$

$$\frac{1}{2}\int_{-\infty}^{\infty} exp(-u^2) du =$$

$$\frac{\sqrt{\pi}}{2}$$

We combine the result:

$$\int_{-\infty}^{\infty} x^2 \cdot exp\left(-\frac{(x-x_0)^2}{a}\right) dx =$$

$$a\sqrt{a} \cdot \frac{\sqrt{\pi}}{2} + {x_0}^2\sqrt{a\pi} =$$

$$\sqrt{a\pi}\left(\frac{a}{2} + {x_0}^2\right)$$