This paper inspects the Gaussian integral; it is based on:

Conceptual Basis of Quantum Mechanics, Jan-Marcus Schwindt, Springer, ISBN 978-3-319-24524-9

Hope I can help you learning quantum mechanics.

We will examine the Gaussian integral. The basic one is:

$$\int_{-\infty}^{\infty} e^{-y^2} dy$$

Part one

As the integral is finite, we can square it:

$$\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)^2$$

We get:

$$\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)^2 = \int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy\, dx$$

Renaming the variables aims to view the line integral as an integral in the plane. We then can switch to polar coordinates r and angle φ:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy\, dx = \int_{0}^{\infty} \rho \int_{0}^{2\pi} d\varphi\, d\rho$$

We use Pythagoras:

$$x^2 + y^2 = \rho^2, \qquad e^{-(x^2+y^2)} = e^{-\rho^2}$$

We get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy\, dx \to \int_{0}^{\infty}\int_{0}^{2\pi} \rho e^{-\rho^2} d\varphi\, d\rho =$$

$$2\pi \int_{0}^{\infty} \rho e^{-\rho^2} d\rho$$

We substitute:

$$u \coloneqq \rho^2, \frac{du}{d\rho} = 2\rho \to d\rho = \frac{du}{2\rho}$$

We get:

$$2\pi \int_{0}^{\infty} \rho e^{-\rho^2} d\rho = 2\pi \int_{0}^{\infty} \rho e^{-u} \frac{du}{2\rho} = \pi \int_{0}^{\infty} e^{-u} du =$$

$$-\pi |e^{-u}|_{0}^{\infty} = \pi(0-1) = \pi$$

As we calculated the square, we can now say:

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

Part two

We expand the argument:

$$\int_{-\infty}^{\infty} e^{-y^2} dy \to \int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{a}} dy$$

We substitute:

$$u = \frac{(y-y_0)}{\sqrt{a}}$$

$$\frac{du}{dy} = \frac{1}{\sqrt{a}} \to dy = \sqrt{a}du$$

We get:

$$\int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{a}} dy \to \sqrt{a}\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{a\pi}$$

Remark (without proof)

If a is complex, after substitution the path of integration is a sloped line in the complex plane.

We can connect this line to the real axis with two vertical lines at $u = \pm R$.

The integral along the resulting contour vanishes thanks to the Residue theorem.

For $R \to \infty$ the integral along the vertical pieces vanishes.

The integral along the sloped line therefore is equal to the integral along the real axis.

The only precondition needed is that the real part of a is positive to have the function vanish at infinity.

Part three

We multiply the integrand by y:

$$\int_{-\infty}^{\infty} y \cdot e^{-\frac{(y-y_0)^2}{a}} dy$$

We use the substitution:

$$u = y - y_0, \qquad \frac{du}{dy} = 1 \to dy = du, \qquad y = u + y_0$$

We get:

$$\int_{-\infty}^{\infty} y \cdot e^{-\frac{(y-y_0)^2}{a}} dy \to \int_{-\infty}^{\infty} (u + y_0) \cdot e^{-\frac{u^2}{a}} du$$

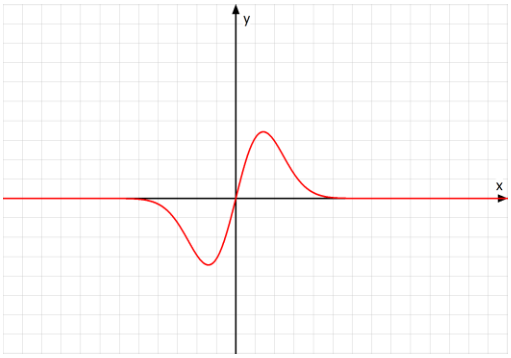
We calculate:

$$\int_{-\infty}^{\infty} (u + y_0) \cdot e^{-\frac{u^2}{a}} du = \int_{-\infty}^{\infty} u \cdot e^{-\frac{u^2}{a}} du + \int_{-\infty}^{\infty} y_0 \cdot e^{-\frac{u^2}{a}} du$$

The first integral in the sum is antisymmetric:

$$\int_{-\infty}^{\infty} u \cdot e^{-\frac{u^2}{a}} du$$

The integral of antisymmetric functions is zero.



We get:

$$\int_{-\infty}^{\infty} (u + y_0) \cdot e^{-\frac{u^2}{a}} du = y_0 \int_{-\infty}^{\infty} e^{-\frac{u^2}{a}} du = y_0 \cdot \sqrt{a\pi}$$

Part four

For this we need integration by parts.

We take the integral:

$$\int_{-\infty}^{\infty} u^2 \cdot e^{-u^2} du$$

We integrate by parts:

$$\int v \cdot w' = v \cdot w - \int v' \cdot w$$

We assign:

$$v(u) := u, \qquad w(u) = -\frac{1}{2} e^{-u^2}, \qquad w'(u) := u \cdot e^{-u^2}$$

We get:

$$\int_{-\infty}^{\infty} u \cdot u \cdot e^{-u^2} du = \left| u \cdot \left(-\frac{1}{2} e^{-u^2} \right) \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2} e^{-u^2} du$$

The antisymmetric part vanishes; we get:

$$\int_{-\infty}^{\infty} u^2 \cdot e^{-u^2} du = \int_{-\infty}^{\infty} u \cdot u \cdot e^{-u^2} du = \frac{1}{2} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

We go back and consider the integral:

$$\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{(y-y_0)^2}{a}} dy$$

Again, we substitute:

$$u = \frac{(y - y_0)}{\sqrt{a}}, \qquad \frac{du}{dy} = \frac{1}{\sqrt{a}} \rightarrow dy = \sqrt{a}du$$

$$y = u\sqrt{a} + y_0, \qquad y^2 = u^2 a + 2y_0 u\sqrt{a} + {y_0}^2$$

We get the integral:

$$\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{(y-y_0)^2}{a}} dy \rightarrow \int_{-\infty}^{\infty} \left(u^2 a + 2y_0 u\sqrt{a} + {y_0}^2\right) e^{-u^2} \sqrt{a} du =$$

$$a\sqrt{a} \int_{-\infty}^{\infty} u^2 e^{-u^2} du + 2y_0 a \int_{-\infty}^{\infty} u e^{-u^2} du + {y_0}^2 \sqrt{a} \int_{-\infty}^{\infty} e^{-u^2} du$$

Again, the antisymmetric integral vanishes:

$$2y_0 a \int_{-\infty}^{\infty} u e^{-u^2} du = 0$$

We get:

$$a\sqrt{a} \int_{-\infty}^{\infty} u^2 e^{-u^2} du + {y_0}^2 \sqrt{a} \int_{-\infty}^{\infty} e^{-u^2} du$$

We use our previous results:

$$a\sqrt{a} \int_{-\infty}^{\infty} u^2 e^{-u^2} du = a\sqrt{a} \frac{\sqrt{\pi}}{2}$$

$${y_0}^2 \sqrt{a} \int_{-\infty}^{\infty} e^{-u^2} du = {y_0}^2 \sqrt{a} \sqrt{\pi}$$

We get:

$$a\sqrt{a} \int_{-\infty}^{\infty} u^2 e^{-u^2} du + {y_0}^2 \sqrt{a} \int_{-\infty}^{\infty} e^{-u^2} du = a\sqrt{a} \frac{\sqrt{\pi}}{2} + {y_0}^2 \sqrt{a} \sqrt{\pi} =$$

$$\sqrt{\pi a} \left(\frac{a}{2} + {y_0}^2\right)$$

The result then is:

$$\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{(y-y_0)^2}{a}} dy = \sqrt{\pi a} \left(\frac{a}{2} + {y_0}^2\right)$$