The Hermite polynomials build an orthogonal system. This paper checks the orthogonality. It is based on:

“Hort-Thoma, Die Differentialgleichungen der Technik und Physik”, Leipzig 1950,

Griffiths, Introduction to quantum mechanics, third edition, page 54.

https://www.ruf.rice.edu/~baring/phys516/phys516_2021_lec_041321.pdf

For the proof of orthogonality:

https://math.stackexchange.com/questions/2701650/orthogonality-and-norm-of-hermite-polynomials

Hope I can help you with learning quantum mechanics.

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Hermite polynomials

We begin with Griffiths Problem 2.16 to become acquainted with Hermite polynomials.

We use Rodrigues formula:

$$H_n(\xi) = (-1)^n \cdot e^{\xi^2} \cdot \left(\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}\right) \tag{1}$$

We can invert this:

$$\left(\frac{d}{d\xi}\right)^n e^{-\xi^2} = H_n(\xi) \cdot (-1)^n \cdot e^{-\xi^2} \tag{1a}$$

Note: Instead of x we use $\xi = \sqrt{\frac{m \cdot \omega}{\hbar}} \cdot x$, a dimensionless variable.

We derive $H_0(\xi), H_1(\xi), H_2(\xi), H_3(\xi), H_4(\xi)$

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2 \cdot \xi$$

$$H_2(\xi) = (-1)^2 e^{\xi^2} \left(\frac{d}{d\xi}\right)^2 e^{-\xi^2} = e^{\xi^2} \frac{d}{d\xi}\left(\frac{d}{d\xi} e^{-\xi^2}\right) =$$

$$e^{\xi^2} \frac{d}{d\xi}\left(-2 \cdot \xi \cdot e^{-\xi^2}\right) = e^{\xi^2}\left(-2 \cdot e^{-\xi^2} + 4 \cdot \xi^2 \cdot e^{-\xi^2}\right) =$$

$$e^{\xi^2}\left(-2 \cdot e^{-\xi^2} + 4 \cdot \xi^2 \cdot e^{-\xi^2}\right) = -2 + 4 \cdot \xi^2$$

$$H_3(\xi) = (-1)^3 e^{\xi^2} \left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = -e^{\xi^2} \frac{d}{d\xi}\left(\frac{d}{d\xi}\left(\frac{d}{d\xi} e^{-\xi^2}\right)\right) =$$

$$-e^{\xi^2} \frac{d}{d\xi}\left(-2 \cdot e^{-\xi^2} + 4 \cdot \xi^2 \cdot e^{-\xi^2}\right) = -e^{\xi^2}\left(4 \cdot \xi \cdot e^{-\xi^2} + 8 \cdot \xi \cdot e^{-\xi^2} - 8 \cdot \xi^3 \cdot e^{-\xi^2}\right) =$$

$$8 \cdot \xi^3 - 12 \cdot \xi$$

$$H_4(\xi) = (-1)^4 e^{\xi^2} \left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = e^{\xi^2} \frac{d}{d\xi}\left(\frac{d}{d\xi}\left(\frac{d}{d\xi}\left(\frac{d}{d\xi} e^{-\xi^2}\right)\right)\right) =$$

$$e^{\xi^2} \frac{d}{d\xi}\left(12 \cdot \xi \cdot e^{-\xi^2} - 8 \cdot \xi^3 \cdot e^{-\xi^2}\right) =$$

$$e^{\xi^2}\left(12 \cdot e^{-\xi^2} - 24 \cdot \xi^2 \cdot e^{-\xi^2} - 24 \cdot \xi^2 \cdot e^{-\xi^2} + 16 \cdot \xi^4 \cdot e^{-\xi^2}\right) =$$

$$16 \cdot \xi^4 - 48 \cdot \xi^2 + 12$$

This should be enough to trust in the Rodrigues formula.

We have the recursion relation:

$$H_{n+2}(\xi) = 2 \cdot \xi \cdot H_{n+1}(\xi) - 2 \cdot (n+1) \cdot H_n(\xi)$$

With the recursion relation we build $H_5(\xi)$ and $H_6(\xi)$:

$$H_5(\xi) = 2 \cdot \xi \cdot H_4(\xi) - 2 \cdot 4 \cdot H_3(\xi) =$$

$$2 \cdot \xi \cdot (16 \cdot \xi^4 - 48 \cdot \xi^2 + 12) - 8 \cdot (8 \cdot \xi^3 - 12 \cdot \xi) =$$

$$32 \cdot \xi^5 - 96 \cdot \xi^3 + 24 \cdot \xi - 64 \cdot \xi^3 + 96 \cdot \xi =$$

$$32 \cdot \xi^5 - 160 \cdot \xi^3 + 120 \cdot \xi$$

$$H_6(\xi) = 2 \cdot \xi \cdot H_5(\xi) - 2 \cdot 5 \cdot H_4(\xi) =$$

$$2 \cdot \xi \cdot (32 \cdot \xi^5 - 160 \cdot \xi^3 + 120 \cdot \xi) - 10 \cdot (16 \cdot \xi^4 - 48 \cdot \xi^2 + 12) =$$

$$64 \cdot \xi^6 - 320 \cdot \xi^4 + 240 \cdot \xi^2 - 160 \cdot \xi^4 + 480 \cdot \xi^2 - 120 =$$

$$64 \cdot \xi^6 - 480 \cdot \xi^4 + 720 \cdot \xi^2 - 120$$

For Hermite polynomials holds:

$$\frac{d}{d\xi} H_n(\xi) = 2 \cdot n \cdot H_{n-1}(\xi)$$

We check this by differentiating $H_6(\xi)$ and $H_5(\xi)$

$$\frac{d}{d\xi} H_6(\xi) = \frac{d}{d\xi}(64 \cdot \xi^6 - 480 \cdot \xi^4 + 720 \cdot \xi^2 - 120) =$$

$$384 \cdot \xi^5 - 1920 \cdot \xi^3 + 1440 \cdot \xi =$$

$$2 \cdot 6 \cdot (32 \cdot \xi^5 - 160 \cdot \xi^3 + 120 \cdot \xi) =$$

$$12 \cdot H_5(\xi)$$

$$\frac{d}{d\xi} H_5(\xi) = \frac{d}{d\xi}(32 \cdot \xi^5 - 160 \cdot \xi^3 + 120 \cdot \xi) =$$

$$160 \cdot \xi^4 - 480 \cdot \xi^2 + 120 =$$

$$2 \cdot 5 \cdot (16 \cdot \xi^4 - 48 \cdot \xi^2 + 12) =$$

$$10 \cdot H_4(\xi)$$

The generating function for the Hermite polynomials:

$$e^{(-z^2 + 2 \cdot z \cdot \xi)}$$

We expand into Taylor power series at $z = 0$:

$$e^{(-z^2+2\cdot z\cdot\xi)} = \frac{\frac{d^0}{dz^0} e^{(-z^2+2\cdot z\cdot\xi)}}{0!} \cdot z^0 + \frac{\frac{d^1}{dz^1} e^{(-z^2+2\cdot z\cdot\xi)}}{1!} \cdot z^1 + \frac{\frac{d^2}{dz^2} e^{(-z^2+2\cdot z\cdot\xi)}}{2!} \cdot z^2 + \frac{\frac{d^3}{dz^3} e^{(-z^2+2\cdot z\cdot\xi)}}{3!} \cdot z^3 + \cdots =$$

$$\sum_{n=0}^{\infty} \frac{\frac{d^n}{dz^n} e^{(-z^2+2\cdot z\cdot\xi)}}{n!} \cdot z^n$$

We build $H_1(\xi), H_2(\xi), H_3(\xi)$.

$$H_1(\xi) = \frac{\frac{d^1}{dz^1} e^{(-z^2+2\cdot z\cdot\xi)}}{1!} =$$

$$(-2\cdot z+2\cdot\xi)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=(note\colon z=0)=2\cdot\xi$$

$$H_2(\xi)=\frac{\frac{d^2}{dz^2}e^{(-z^2+2\cdot z\cdot\xi)}}{2!}=$$

$$\frac{d}{dz}\Big((-2\cdot z+2\cdot\xi)\cdot e^{(-z^2+2\cdot z\cdot\xi)}\Big)=$$

$$(-2)\cdot e^{(-z^2+2\cdot z\cdot\xi)}+(-2\cdot z+2\cdot\xi)\cdot(-2\cdot z+2\cdot\xi)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=$$

$$(-2+4\cdot z^2-8\cdot z\cdot\xi+4\cdot\xi^2)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=(note\colon z=0)=-2+4\cdot\xi^2$$

$$H_3(\xi)=\frac{\frac{d^3}{dz^3}e^{(-z^2+2\cdot z\cdot\xi)}}{3!}=$$

$$\frac{d}{dz}\Big((-2+4\cdot z^2-8\cdot z\cdot\xi+4\cdot\xi^2)\cdot e^{(-z^2+2\cdot z\cdot\xi)}\Big)=$$

$$\Big((8\cdot z-8\cdot\xi)\cdot e^{(-z^2+2\cdot z\cdot\xi)}+(-2+4\cdot z^2-8\cdot z\cdot\xi+4\cdot\xi^2)\cdot(-2\cdot z+2\cdot\xi)\cdot e^{(-z^2+2\cdot z\cdot\xi)}\Big)=$$

$$\Big((8\cdot z-8\cdot\xi)+(-2+4\cdot z^2-8\cdot z\cdot\xi+4\cdot\xi^2)\cdot(-2\cdot z+2\cdot\xi)\Big)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=$$

$$(8\cdot z-8\cdot\xi+4\cdot z-4\cdot\xi-8\cdot z^3+8\cdot z^2\cdot\xi+16\cdot z^2\cdot\xi-16\cdot z\cdot\xi^2-8\cdot z\cdot\xi^2+8\cdot\xi^3)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=$$

$$(12\cdot z-12\cdot\xi-8\cdot z^3+24\cdot z^2\cdot\xi-24\cdot z\cdot\xi^2+8\cdot\xi^3)\cdot e^{(-z^2+2\cdot z\cdot\xi)}=(note\colon z=0)=(-12\cdot\xi+8\cdot\xi^3)$$

This might be enough to let us trust in the generating function.

We summarize:

$$H_0(\xi)=1$$

$$H_1(\xi)=2\cdot\xi$$

$$H_2(\xi)=4\cdot\xi^2-2$$

$$H_3(\xi)=8\cdot\xi^3-12\cdot\xi$$

$$H_4(\xi)=16\cdot\xi^4-48\cdot\xi^2+12$$

$$H_5(\xi)=32\cdot\xi^5-160\cdot\xi^3+120\cdot\xi$$

$$H_6(\xi)=64\cdot\xi^6-480\cdot\xi^4+720\cdot\xi^2-120$$

We note that the leading coefficient of each polynomial is 2^n. (2)

Orthogonality

The wave function for the quantum harmonic oscillator:

$$\psi_n(\xi) = e^{-\frac{\xi^2}{2}} \cdot H_n(\xi)$$

We have to proof for $m \neq n, m < n$:

$$\int_{-\infty}^{\infty} \psi_n(\xi) \cdot \psi_m(\xi)\, d\xi = 0$$

We use Rodriguez formula (1):

$$H_m(\xi) = (-1)^m e^{\xi^2} \left(\frac{d}{d\xi}\right)^m e^{-\xi^2}$$

We resolve the wave functions:

$$\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} \cdot H_m(\xi) \cdot e^{-\frac{\xi^2}{2}} \cdot H_n(\xi)\, d\xi = \int_{-\infty}^{\infty} e^{-\xi^2} \cdot H_m(\xi) \cdot H_n(\xi)\, d\xi =$$

$$\int_{-\infty}^{\infty} e^{-\xi^2} \cdot H_m(\xi) \cdot \left((-1)^n \cdot e^{\xi^2} \cdot \left(\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}\right)\right) d\xi =$$

$$(-1)^n \int_{-\infty}^{\infty} H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}\right) d\xi =;$$

We omit the factor $(-1)^n$:

$$\int_{-\infty}^{\infty} H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}\right) d\xi$$

We use partial integration:

$$\int f \cdot g' = [f \cdot g] - \int f' \cdot g$$

We assign:

$$f \coloneqq H_m(\xi) \;\rightarrow f^{(1)} = \frac{d}{d\xi}\left(H_m(\xi)\right)$$

$$g^{(1)} \coloneqq \left(\frac{d}{d\xi}\right)^n e^{-\xi^2} \rightarrow g = \left(\frac{d}{d\xi}\right)^{(n-1)} e^{-\xi^2}$$

We get for one step:

$$\int_{-\infty}^{\infty} H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n)} e^{-\xi^2}\right) d\xi \rightarrow$$

$$\left[H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-1)} e^{-\xi^2}\right)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left(\frac{d}{d\xi}\right)^{(1)} \left(H_m(\xi)\right) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-1)} e^{-\xi^2}\right) d\xi$$

We repeat this n-times and get:

$$\left[H_m(\xi)\cdot\left(\left(\frac{d}{d\xi}\right)^{(n-n)} e^{-\xi^2}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_m(\xi)\right)\right)\cdot\left(\left(\frac{d}{d\xi}\right)^{(n-n)} e^{-\xi^2}\right)d\xi$$

$$\left[H_m(\xi)\cdot\left(e^{-\xi^2}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_m(\xi)\right)\right)\cdot\left(e^{-\xi^2}\right)d\xi$$

We examine: Due to $e^{-\xi^2}$ we have:

$$\left[H_m(\xi)\cdot\left(e^{-\xi^2}\right)\right]_{-\infty}^{\infty}=0$$

The integral:

$$\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_m(\xi)\right)\right)\cdot\left(e^{-\xi^2}\right)$$

This integral is zero because $\left(H_m(\xi)\right)$ is a polynomial of grade $m<n$, so the derivatives becomes zero beginning with $n=m+1$.

Case $m=n$. We have the same process as above but this time ending with:

$$\left[H_n(\xi)\cdot\left(\left(\frac{d}{d\xi}\right)^{(n-n)} e^{-\xi^2}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\left(\frac{d}{d\xi}\right)^{n}\left(H_n(\xi)\right)\cdot\left(\left(\frac{d}{d\xi}\right)^{(n-n)} e^{-\xi^2}\right)d\xi=$$

$$\left[H_n(\xi)\cdot\left(e^{-\xi^2}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_n(\xi)\right)\right)\cdot\left(e^{-\xi^2}\right)d\xi=;$$

As before we have:

$$\left[H_n(\xi)\cdot\left(e^{-\xi^2}\right)\right]_{-\infty}^{\infty}=0$$

We look at:

$$\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_n(\xi)\right)\right)\cdot e^{-\xi^2}\,d\xi$$

We take a closer look at:

$$\left(\frac{d}{d\xi}\right)^{n}\left(H_n(\xi)\right)\tag{3}$$

$H_n(\xi)$ is a polynomial in ξ of grade ξ^n. Deriving this polynomial n-times gives the factor $n!$

The leading coefficient of $H_n(\xi)$ is 2^n as we saw above (2).

With this we get the value of the integral:

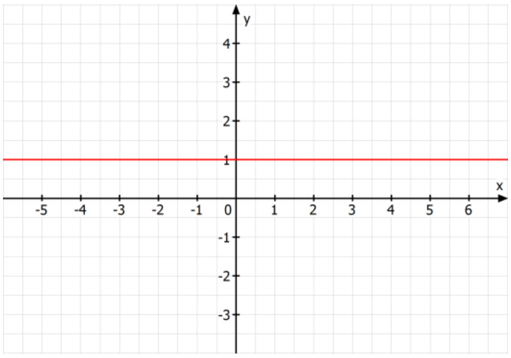
$$\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{n}\left(H_n(\xi)\right)\right)\cdot e^{-\xi^2}\,d\xi=n!\cdot 2^n\cdot\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi=$$

$$n!\cdot 2^n\cdot\sqrt{\pi}$$

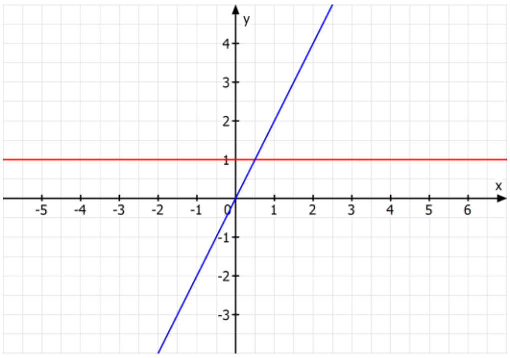
Note: If $m=n$ the Hermite polynomial disappears from the integral

Visualization

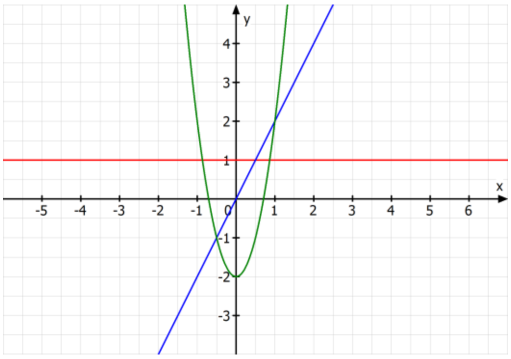
We plot the Hermite polynomials from $H_0(\xi)$ to $H_6(\xi)$.



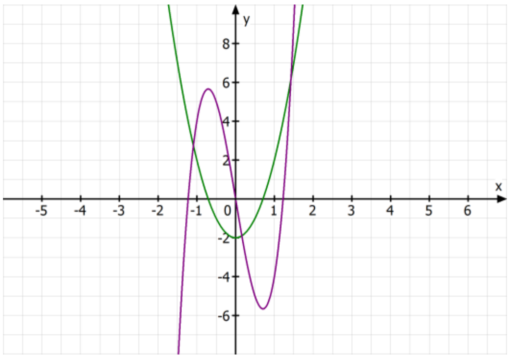
$H_0(\xi)$



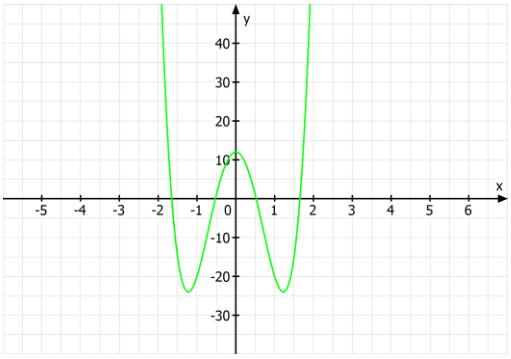
$H_1(\xi)$



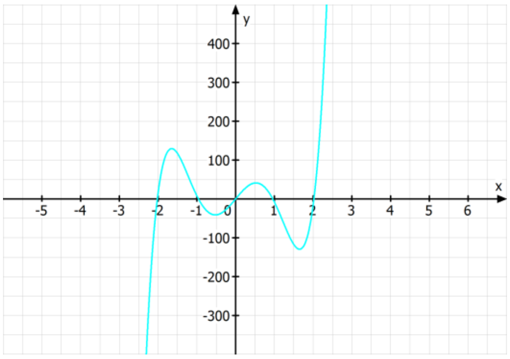
$H_0(\xi)$ $H_1(\xi)$ $H_2(\xi)$



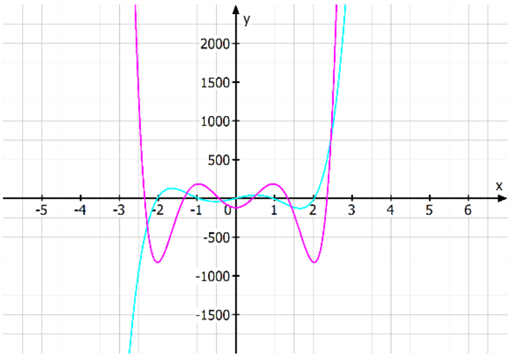
$H_2(\xi)$ $H_3(\xi)$



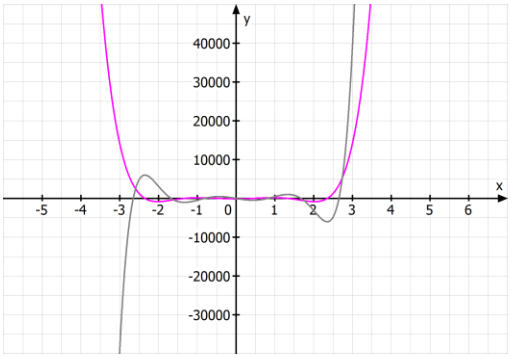
$H_4(\xi)$



$H_5(\xi)$



$H_5(\xi)$ $H_6(\xi)$



$H_6(\xi)$ $H_7(\xi)$

What we learn out of these plots is that the inner structure of the Hermite polynomials is interesting for small ξ only. If the value of ξ exceeds ± 3, the Hermite polynomials reduce to the effect of the leading power of ξ.

We want to check what dimensions we are working with and use the definition of ξ:

$$\xi \coloneqq \sqrt{\frac{m \cdot \omega}{\hbar}} \cdot x \to \sqrt{\frac{m \cdot 4 \cdot \pi^2 \cdot \nu}{h}} \cdot x$$

We need an example and choose a nonrelativistic electron.

Planck constant:	$h = 6.626 \cdot 10^{-34} J \cdot s$
Mass of an electron:	$m_0 = 9.109 \cdot 10^{-31} kg$
Speed of light:	$2.998 \cdot 10^8 \frac{m}{s}$
de Broglie relation for frequency:	$\nu = \frac{E}{h}$
From special relativity:	$E = \frac{m_0 \cdot c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$

We assume $v \ll c$ and combine:

$$\xi \coloneqq \sqrt{\frac{m_0 \cdot 4 \cdot \pi^2 \cdot \nu}{h}} \cdot x \to \sqrt{\frac{m_0 \cdot 4 \cdot \pi^2 \cdot E}{h^2}} \cdot x \to \sqrt{\frac{m_0 \cdot 4 \cdot \pi^2 \cdot m_0 \cdot c^2}{h^2}} \to \frac{m_0 \cdot 2 \cdot \pi \cdot c}{h}$$

$$\xi = \frac{9.109 \cdot 10^{-31} kg \cdot 2 \cdot \pi \cdot 2.998 \cdot 10^8 \frac{m}{s}}{6.626 \cdot 10^{-34} J \cdot s} \cdot x$$

We check the dimensions:

$$\left[\frac{kg \cdot m}{J \cdot s^2} \cdot m\right] = \left[\frac{kg \cdot m \cdot s^2}{kg \cdot m^2 \cdot s^2} \cdot m\right] = \left[\frac{1}{m} \cdot m\right] = 1$$

ξ is a dimensionless variable.

We get ξ for a slow-moving electron, $v \ll c$:

$$\xi \approx 2.590 \cdot 10^{12} \cdot x$$

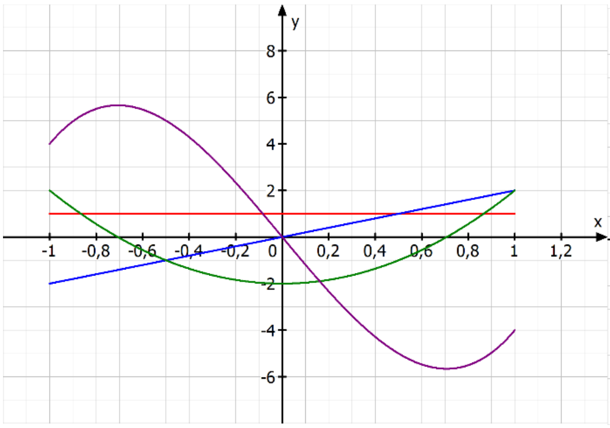
We multiply with the classical electron radius:

$$r_e \approx 2.818 \cdot 10^{-15} m$$

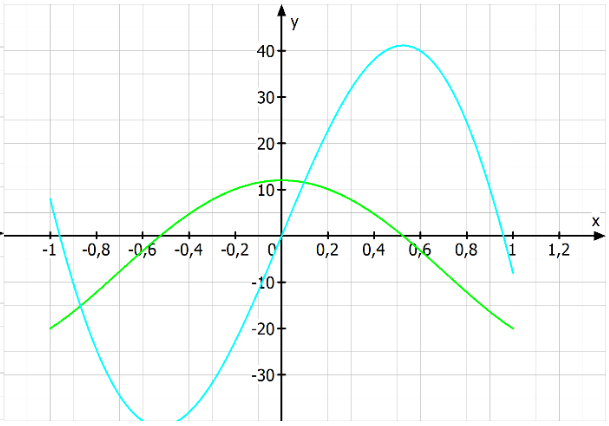
We get:

$$\xi \approx 2.6 \cdot 10^{12} \cdot x = 7.299 \cdot 10^{-3}$$

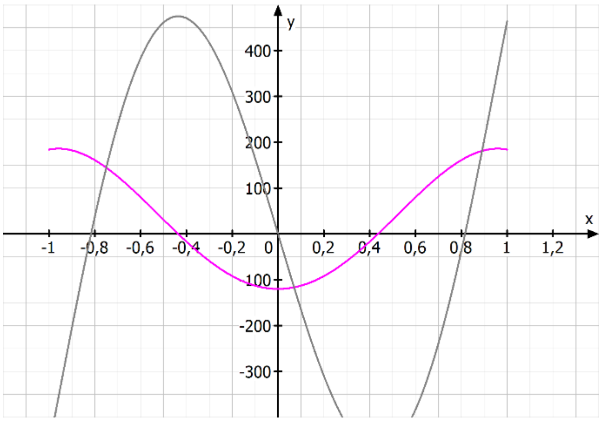
We plot ξ within the range $-1 \leq \xi \leq 1$



$H_0(\xi)$ $H_1(\xi)$ $H_2(\xi)$ $H_3(\xi)$



$H_4(\xi)$ $H_5(\xi)$



$H_6(\xi)$ $H_7(\xi)$

Orthogonality revisited

We got that the Hermite polynomials are orthogonal by a proof working with partial integration.

We proofed for $m \neq n, m < n$:

$$\int_{-\infty}^{\infty} \psi_n(\xi) \cdot \psi_m(\xi)\, d\xi = 0$$

We did this by deriving $H_m(\xi)$ n-times with the effect:

$$\left(\frac{d}{d\xi}\right)^n H_m(\xi) = 0$$

What we got was:

We examine: Due to $e^{-\xi^2}$ we have:

$$\left[H_m(\xi) \cdot \left(e^{-\xi^2}\right)\right]_{-\infty}^{\infty} = 0$$

The integral:

$$\int_{-\infty}^{\infty} \left(\left(\frac{d}{d\xi}\right)^n \left(H_m(\xi)\right)\right) \cdot \left(e^{-\xi^2}\right)$$

But what would happen if we derived m-times instead of n-times?

We would get:

$$\left[H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-m)} e^{-\xi^2}\right)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left(\left(\frac{d}{d\xi}\right)^m \left(H_m(\xi)\right)\right) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-m)} e^{-\xi^2}\right) d\xi$$

The first part goes to zero due to $e^{-\xi^2}$:

$$\left[H_m(\xi) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-m)} e^{-\xi^2}\right)\right]_{-\infty}^{\infty} = 0$$

The second part:

$$\int_{-\infty}^{\infty} \left(\left(\frac{d}{d\xi}\right)^m \left(H_m(\xi)\right)\right) \cdot \left(\left(\frac{d}{d\xi}\right)^{(n-m)} e^{-\xi^2}\right) d\xi$$

First multiplicand according to (3) gives:

$$\left(\frac{d}{d\xi}\right)^m \left(H_m(\xi)\right) = m! \cdot 2^m$$

With $d = n - m$ and $(1a)$ the second multiplicand gives:

$$\left(\frac{d}{d\xi}\right)^d e^{-\xi^2} = H_d(\xi) \cdot (-1)^d \cdot e^{-\xi^2}$$

We combine:

$$\int_{-\infty}^{\infty}\left(\left(\frac{d}{d\xi}\right)^{m}\left(H_m(\xi)\right)\right)\cdot\left(\left(\frac{d}{d\xi}\right)^{(n-m)}e^{-\xi^2}\right)d\xi \rightarrow$$

$$(-1)^d\cdot m!\cdot 2^m\cdot\int_{-\infty}^{\infty}H_d(\xi)\cdot e^{-\xi^2}\,d\xi$$

We need to calculate $\int_{-\infty}^{\infty}H_d(\xi)\cdot e^{-\xi^2}\,d\xi$.

For odd Hermite polynomials the integral will give zero because:

$$\int_{-\infty}^{\infty}\xi\cdot e^{-\xi^2}d\xi=0,\int_{-\infty}^{\infty}\xi^3\cdot e^{-\xi^2}d\xi=0,\int_{-\infty}^{\infty}\xi^{2n+1}\cdot e^{-\xi^2}d\xi=0$$

We take even Hermite polynomials:

$$H_0(\xi)=1$$

$$H_2(\xi)=4\cdot\xi^2-2$$

$$H_4(\xi)=16\cdot\xi^4-48\cdot\xi^2+12$$

$$H_6(\xi)=64\cdot\xi^6-480\cdot\xi^4+720\cdot\xi^2-120$$

$$H_8(\xi)=256\cdot\xi^8-3584\cdot\xi^6+13440\cdot\xi^4-13440\cdot\xi^2+1680$$

$$H_{10}(\xi)=1024\cdot\xi^{10}-23040\cdot\xi^8+161280\cdot\xi^6-403200\cdot\xi^4+302400\cdot\xi^2-30240$$

We use:

$$\int_{-\infty}^{\infty}\xi^{2k}\cdot e^{-\xi^2}d\xi=\frac{(2k-1)!!}{2^k}\cdot\sqrt{\pi}$$

We get for $H_2(\xi)$:

$$\int_{-\infty}^{\infty}(4\cdot\xi^2-2)\cdot e^{-\xi^2}d\xi=4\cdot\int_{-\infty}^{\infty}\xi^2d\xi-2\cdot\int_{-\infty}^{\infty}e^{-\xi^2}d\xi=$$

$$2\cdot\sqrt{\pi}-2\cdot\sqrt{\pi}=0$$

We get for $H_4(\xi)$:

$$\int_{-\infty}^{\infty}(16\cdot\xi^4-48\cdot\xi^2+12)\cdot e^{-\xi^2}d\xi=$$

$$16\cdot\int_{-\infty}^{\infty}\xi^4\cdot e^{-\xi^2}d\xi-48\cdot\int_{-\infty}^{\infty}\xi^2\cdot e^{-\xi^2}d\xi+12\cdot\int_{-\infty}^{\infty}e^{-\xi^2}d\xi=$$

$$16\cdot\frac{3}{4}\sqrt{\pi}-24\cdot\frac{1}{2}\cdot\sqrt{\pi}+12\cdot\sqrt{\pi}=0$$

This cannot be by chance; we check this by help of excel up to $H_{12}(\xi)$:

k	n	((2k-1)!!)/2^k	H2	H4	H6	H8	H10	H12	H14	H16	H18	H20	
0	0	1	-2	12	-120	1680	-30240	665280	-17297280	518918400	-17643225600	6,70443E+11	x^0
1	2	0,5	4	-48	720	-13440	302400	-7983360	242161920	-8302694400	3,17578E+11	-1,34089E+13	x^2
2	4	0,75		16	-480	13440	-403200	13305600	-484323840	19372953600	-8,46875E+11	4,02266E+13	x^4
3	6	1,875			64	-3584	161280	-7096320	322882560	-15498362880	7,90417E+11	-4,29083E+13	x^6
4	8	6,5625				256	-23040	1520640	-92252160	5535129600	-3,3875E+11	2,14542E+13	x^8
5	10	29,53125					1024	-135168	12300288	-984023040	75277762560	-5,72111E+12	x^10
6	12	162,421875						4096	-745472	89456640	-9124577280	8,66835E+11	x^12
7	14	1055,742188							16384	-3932160	601620480	-76205260800	x^14
8	16	7918,066406								65536	-20054016	3810263040	x^16
9	18	67303,56445									262144	-99614720	x^18
10	20	639383,8623	We build the sum "-2*1+4*0.5" etc.									1048576	x^20
11	22	6713530,554	0	0	0	0	0	0	0	0	0	0	

What we can assume (and should be proofed) is that $\int_{-\infty}^{\infty} H_d(\xi) \cdot e^{-\xi^2}\, d\xi$ is zero for every Hermite polynomial with exception of $H_0(\xi)$.