This short paper deals with the Lagrangian equation of motion, especially with generalized coordinates. We use cartesian coordinates and polar coordinates and check that $\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$ and $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j}$ holds for both cases: r_i cartesian and q_i polar and vice versa.

It follows:

https://phys.libretexts.org/Bookshelves/Classical_Mechanics/Classical_Mechanics_(Tatum)/13%3A_Lagrangian_Mechanics/13.04%3A_The_Lagrangian_Equations_of_Motion

Hope I can help you with learning quantum mechanics.

Prerequisite

We describe a point in space with the Cartesian coordinates x, y, z, corresponding to q_1, q_2, q_3.

We could also use spherical coordinates r, θ, ϕ corresponding to q_1, q_2, q_3.

We express:

$$x = r \cdot sin(\phi) \cdot cos(\theta) \tag{P1}$$

$$y = r \cdot cos(\phi) \cdot cos(\theta) \tag{P2}$$

$$z = r \cdot sin(\theta) \tag{P3}$$

We take a look at the partial derivatives.

$\frac{\partial x}{\partial r} = cos(\phi) \cdot cos(\theta)$	$\frac{\partial x}{\partial \theta} = -r \cdot sin(\phi) \cdot cos(\theta)$	$\frac{\partial x}{\partial \phi} = -r \cdot cos(\phi) \cdot sin(\theta)$
$\frac{\partial y}{\partial r} = sin(\phi) \cdot cos(\theta)$	$\frac{\partial y}{\partial \theta} = r \cdot cos(\phi) \cdot cos(\theta)$	$\frac{\partial y}{\partial \phi} = -r \cdot sin(\phi) \cdot sin(\theta)$
$\frac{\partial z}{\partial r} = sin(\theta)$	$\frac{\partial z}{\partial \phi} = 0$	$\frac{\partial z}{\partial \theta} = r \cdot cos(\theta)$

We need $\dot{x}, \dot{y}, \dot{z}$. We differentiate:

$$\dot{x} = \dot{r} \cdot cos(\phi) \cdot cos(\theta) - r \cdot \dot{\phi} \cdot sin(\phi) \cdot cos(\theta) - r \cdot \dot{\theta} \cdot cos(\phi) \cdot sin(\theta)$$

$$\dot{y} = \dot{r} \cdot sin(\phi) \cdot cos(\theta) + r \cdot \dot{\phi} \cdot cos(\phi) \cdot cos(\theta) - r \cdot \dot{\theta} \cdot sin(\phi) \cdot sin(\theta)$$

$$\dot{z} = \dot{r} \cdot sin(\theta) + r \cdot \dot{\theta} \cdot cos(\theta)$$

We differentiate partially with respect to $\dot{r}, \dot{\theta}, \dot{\phi}$:

$\frac{\partial \dot{x}}{\partial \dot{r}} = cos(\phi) \cdot cos(\theta)$	$\frac{\partial \dot{x}}{\partial \dot{\theta}} = -r \cdot sin(\phi) \cdot cos(\theta)$	$\frac{\partial \dot{x}}{\partial \dot{\phi}} = -r \cdot cos(\phi) \cdot sin(\theta)$
$\frac{\partial \dot{y}}{\partial \dot{r}} = sin(\phi) \cdot cos(\theta)$	$\frac{\partial \dot{y}}{\partial \dot{\theta}} = r \cdot cos(\phi) \cdot cos(\theta)$	$\frac{\partial \dot{y}}{\partial \dot{\phi}} = -r \cdot sin(\phi) \cdot sin(\theta)$
$\frac{\partial \dot{z}}{\partial \dot{r}} = sin(\theta)$	$\frac{\partial \dot{z}}{\partial \dot{\phi}} = 0$	$\frac{\partial \dot{z}}{\partial \dot{\theta}} = r \cdot cos(\theta)$

We compare:

$\frac{\partial \dot{x}}{\partial \dot{r}} = \frac{\partial x}{\partial r}$	$\frac{\partial \dot{x}}{\partial \dot{\phi}} = \frac{\partial x}{\partial \phi}$	$\frac{\partial \dot{x}}{\partial \dot{\theta}} = \frac{\partial x}{\partial \theta}$
$\frac{\partial \dot{y}}{\partial \dot{r}} = \frac{\partial y}{\partial r}$	$\frac{\partial \dot{y}}{\partial \dot{\phi}} = \frac{\partial y}{\partial \phi}$	$\frac{\partial \dot{y}}{\partial \dot{\theta}} = \frac{\partial y}{\partial \theta}$
$\frac{\partial \dot{z}}{\partial \dot{r}} = \frac{\partial z}{\partial r}$	$\frac{\partial \dot{z}}{\partial \dot{\phi}} = \frac{\partial z}{\partial \phi}$	$\frac{\partial \dot{z}}{\partial \dot{\theta}} = \frac{\partial z}{\partial \theta}$

We generalize:

$$\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \tag{P4}$$

We differentiate partially with respect to r, θ, ϕ:

<table>
<tr><td colspan="3">$$\frac{\partial \dot{x}}{\partial r} = -\dot{\phi} \cdot sin(\phi) \cdot cos(\theta) - \dot{\theta} \cdot cos(\phi) \cdot sin(\theta)$$
$$\frac{\partial \dot{x}}{\partial \theta} = -\dot{r} \cdot sin(\phi) \cdot cos(\theta) - r \cdot \dot{\phi} \cdot cos(\phi) \cdot cos(\theta) + r \cdot \dot{\theta} \cdot sin(\phi) \cdot sin(\theta)$$
$$\frac{\partial \dot{x}}{\partial \phi} = -\dot{r} \cdot cos(\phi) \cdot sin(\theta) + r \cdot \dot{\phi} \cdot sin(\phi) \cdot sin(\theta) - r \cdot \dot{\theta} \cdot cos(\phi) \cdot cos(\theta)$$</td></tr>
<tr><td colspan="3">$$\frac{\partial \dot{y}}{\partial r} = \dot{\phi} \cdot cos(\phi) \cdot cos(\theta) - \dot{\theta} \cdot sin(\phi) \cdot sin(\theta)$$
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</table>

We take a look at the time derivatives.

<table>
<tr><td colspan="3">$$\frac{d}{dt}\left(\frac{\partial x}{\partial r}\right) = -\dot{\phi} \cdot sin(\phi) \cdot cos(\theta) - \dot{\theta} \cdot cos(\phi) \cdot sin(\theta)$$
$$\frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right) = -\dot{r} \cdot sin(\phi) \cdot cos(\theta) - r \cdot \dot{\phi} \cdot cos(\phi) \cdot cos(\theta) + r \cdot \dot{\theta} \cdot sin(\phi) \cdot sin(\theta)$$
$$\frac{d}{dt}\left(\frac{\partial x}{\partial \phi}\right) = -\dot{r} \cdot cos(\phi) \cdot sin(\theta) + r \cdot \dot{\phi} \cdot sin(\phi) \cdot sin(\theta) - r \cdot \dot{\theta} \cdot cos(\phi) \cdot cos(\theta)$$</td></tr>
<tr><td colspan="3">$$\frac{d}{dt}(y) = \dot{\phi} \cdot cos(\phi) \cdot cos(\theta) - \dot{\theta} \cdot sin(\phi) \cdot sin(\theta)$$
$$\frac{d}{dt}\left(\frac{\partial y}{\partial \theta}\right) = \dot{r} \cdot cos(\phi) \cdot cos(\theta) - r \cdot \dot{\phi} \cdot sin(\phi) \cdot cos(\theta) - r \cdot \dot{\theta} \cdot cos(\phi) \cdot sin(\theta)$$
$$\frac{d}{dt}\left(\frac{\partial y}{\partial \phi}\right) = -\dot{r} \cdot sin(\phi) \cdot sin(\theta) - r \cdot \dot{\phi} \cdot sin(\phi) \cdot sin(\theta) - r \cdot \dot{\theta} \cdot sin(\phi) \cdot cos(\theta)$$</td></tr>
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</table>

We generalize:

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j} \tag{P5}$$

End prerequisite

We use generalized coordinates q_i. With each generalized coordinate we have an associated generalized force P_i.

If the work required to change the coordinate q_i by δq_i is $P_i \delta q_i$, then P_i is the generalized force associated with the coordinate q_i:

$$P_i = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_i} \tag{1}$$

We have a system of n particles.

The force on the i-th particle is F_i.

If the i-th particle moves, the total work done on the system is:

$$\sum_i F_i \cdot \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j \rightarrow \cdot \sum_j \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \tag{2}$$

With the definition of the generalized force P_j we rewrite this:

$$\sum_j P_j \cdot \delta q_j \tag{3}$$

We use:

$$P_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \tag{4}$$

From Newton we use:

$$F_i = m_i \cdot \ddot{r}_i \tag{5}$$

We get:

$$P_j = \sum_i m_i \cdot \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \tag{6}$$

From the product rule we get:

$$\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) = \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} + \dot{r}_i \cdot \frac{d}{dt}\frac{\partial r_i}{\partial q_j} \rightarrow$$

$$\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \cdot \frac{d}{dt}\frac{\partial r_i}{\partial q_j} \tag{7}$$

We replace in (6):

$$P_j = \sum_i m_i \cdot \left(\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \cdot \frac{d}{dt}\frac{\partial r_i}{\partial q_j}\right) \tag{8}$$

We use (P4) and (P5):

$$\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j}$$

We get:

$$P_j = \sum_i m_i \cdot \left(\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) - \dot{r}_i \cdot \frac{d}{dt}\frac{\partial r_i}{\partial q_j} \right) =$$

$$\sum_i m_i \cdot \left(\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \right) - \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \right) \quad (9)$$

We write the kinetic energy:

$$T = \frac{1}{2} \cdot \sum_i m_i \cdot \dot{r}_i^{\,2} \quad (10)$$

We build the partial derivations:

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{1}{2} \cdot \sum_i m_i \cdot \dot{r}_i^{\,2} \right) = \frac{1}{2} \cdot \sum_i m_i \cdot 2 \cdot \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} =$$

$$\sum_i m_i \cdot \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \quad (11)$$

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} \cdot \sum_i m_i \cdot \dot{r}_i^{\,2} \right) = \frac{1}{2} \cdot \sum_i m_i \cdot 2 \cdot \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} =$$

$$\sum_i m_i \cdot \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \quad (12)$$

We plug (11) and (12) into (9):

$$P_j = \sum_i m_i \cdot \left(\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \right) - \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} \right) =$$

$$\sum_i m_i \cdot \left(\frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \right) \right) - \sum_i m_i \cdot \dot{r}_i \cdot \frac{\partial \dot{r}_i}{\partial q_j} =$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \quad (13)$$

This is the generalized force associated with a given generalized coordinate.

We reverse the coordinates.

We have:

$$x = r \cdot sin(\theta) \cdot cos(\phi)$$

$$y = r \cdot sin(\theta) \cdot sin(\phi)$$

$$z = r \cdot cos(\theta)$$

We express the coordinates the other way around:

$r = \sqrt{x^2 + y^2 + z^2}$	$\dot{r} = \frac{z(t)\,(\frac{d}{dt}z(t) + 2\,y(t)\,(\frac{d}{dt}y(t)) + 2\,x(t)\,(\frac{d}{dt}x(t))}{\sqrt{z(t)^2 + y(t)^2 + x(t)^2}}$
$\phi = arctan\left(\frac{y}{x}\right)$	$\dot{\phi} = \frac{x(t)\frac{d}{dt}y(t) - y(t)\,(\frac{d}{dt}x(t))}{y(t)^2 + x(t)^2}$
$\theta = arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$	$\dot{\theta} = \frac{\frac{\frac{d}{dt}z(t)}{\sqrt{z(t)^2 + y(t)^2 + x(t)^2}} - \frac{z(t)\,(z(t)\,(\frac{d}{dt}z(t)) + y(t)\,(\frac{d}{dt}y(t)) + x(t)\,(\frac{d}{dt}x(t)))}{(z(t)^2 + y(t)^2 + x(t)^2)^{\frac{3}{2}}}}{\sqrt{1 - \frac{z(t)^2}{z(t)^2 + y(t)^2 + x(t)^2}}}$

Note: ϕ simplified for $x > 0$.

Let us take a look at the partial derivatives of r, ϕ, θ with respect to x, y, z:

$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$	$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2}$	$\frac{\partial \theta}{\partial x} = \frac{x \cdot z}{(x^2 + y^2 + z^2) \cdot \sqrt{x^2 + y^2}}$
$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$	$\frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}$	$\frac{\partial \theta}{\partial y} = \frac{y \cdot z}{(x^2 + y^2 + z^2) \cdot \sqrt{x^2 + y^2}}$
$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$	$\frac{\partial \phi}{\partial z} = 0$	$\frac{\partial \theta}{\partial z} = -\frac{x^2 + y^2}{(x^2 + y^2 + z^2) \cdot \sqrt{x^2 + y^2}}$

We get the identities:

$\frac{d}{dt}\frac{\partial r}{\partial x} = \frac{\dot{x}}{\sqrt{x^2 + y^2 + z^2}} - \frac{x(z\dot{z} + y\dot{y} + x\dot{x})}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{\partial \dot{r}}{\partial x}$
$\frac{d}{dt}\frac{\partial r}{\partial y} = \frac{\dot{y}}{\sqrt{x^2 + y^2 + z^2}} - \frac{y(z\dot{z} + y\dot{y} + x\dot{x})}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{\partial \dot{r}}{\partial y}$
$\frac{d}{dt}\frac{\partial r}{\partial z} = \frac{\dot{z}}{\sqrt{x^2 + y^2 + z^2}} - \frac{z(z\dot{z} + y\dot{y} + x\dot{x})}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{\partial \dot{r}}{\partial z}$

$\frac{d}{dt}\frac{\partial \phi}{\partial x} = \frac{2xy^2\dot{y} - 2\dot{x}y^3}{x^5\left(\frac{y^2}{x^2} + 1\right)^2} - \frac{x\dot{y} - 2\dot{x}y}{x^3\left(\frac{y^2}{x^2} + 1\right)} = \frac{\partial \dot{\phi}}{\partial x}$
$\frac{d}{dt}\frac{\partial \phi}{\partial y} = -\frac{2xy\dot{y} - 2y^2\dot{x}}{x^4\left(\frac{y^2}{x^2} + 1\right)^2} - \frac{\dot{x}\dot{y}}{x^2 + y^2} = \frac{\partial \dot{\phi}}{\partial y}$
$\frac{\partial \dot{\phi}}{\partial z} = 0 = \frac{\partial \dot{\phi}}{\partial z}$

$$\frac{d}{dt}\frac{\partial\theta}{\partial x} = -\frac{xz^3(x\dot{x}+y\dot{y}+z\dot{z}) - xz^2\dot{z}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{3}{2}}} - \frac{3xz(x\dot{x}+y\dot{y}+z\dot{z})}{(x^2+y^2+z^2)^{\frac{5}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{1}{2}}} + \frac{x\dot{z}+z\dot{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{1}{2}}} = \frac{\partial\dot{\theta}}{\partial x}$$

$$\frac{d}{dt}\frac{\partial\theta}{\partial y} = \frac{yz^3(z\dot{z}+y\dot{y}+x\dot{x}) - yz^2\dot{z}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{7}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{3}{2}}} - \frac{3yz(z\dot{z}+y\dot{y}+x\dot{x})}{(x^2+y^2+z^2)^{\frac{5}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{1}{2}}} + \frac{y\dot{z}+z\dot{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{1}{2}}} = \frac{\partial\dot{\theta}}{\partial y}$$

$$\frac{d}{dt}\frac{\partial\theta}{\partial z} = \frac{-\frac{z\dot{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{z^2(z\dot{z}+y\dot{y}+x\dot{x})}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{z^3\dot{z}}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{z^4(z\dot{z}+y\dot{y}+x\dot{x})}{(x^2+y^2+z^2)^{\frac{7}{2}}}}{\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{3}{2}}} + \frac{(3z\dot{z}+y\dot{y}+x\dot{x})(x^2+y^2+z^2) - 3z^2(z\dot{z}+y\dot{y}+x\dot{x})}{(x^2+y^2+z^2)^{\frac{5}{2}}\left(1-\frac{z^2}{x^2+y^2+z^2}\right)^{\frac{1}{2}}} = \frac{\partial\dot{\theta}}{\partial z}$$

Note: The calculations were made by wxmaxima. I tried to summarize the results and put them into a readable form. For your convenience I added the maxima-file.

Remark

There is an apparent asymmetry in complexity. While the derivations from cartesian coordinates to polar coordinates $\frac{\partial x}{\partial r}$ etc. are relatively simple, the derivations from polar coordinates to cartesian coordinates $\frac{\partial r}{\partial x}$ etc. are very elaborated.