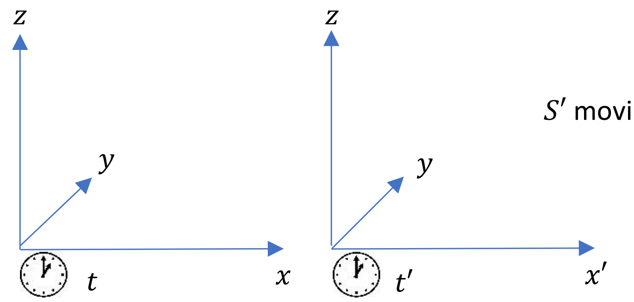
The model

The Lorentz transformation assumes two inertial systems S and S' moving with a relative speed of v. We can consider the system S as resting, system S' as moving.



Note: The position of origin of S' has the distance $v \cdot t$ when referenced in S.

Note: In case that the particle is moving in both Systems we replace them by a pair of systems in such way that one system moves with the speed of the particle and thus the particle is resting in this system.

The Lorentz transformation:

$$t' = \gamma\left(t - \frac{v}{c^2} \cdot x\right)$$

$$x' = \gamma(x - v \cdot t)$$

$$y' = y$$

$$z' = z$$

Most papers express the velocity v relative to the speed of light:

$$\beta = \frac{v}{c}$$

In this case γ becomes:

$$\gamma := \frac{1}{\sqrt{1 - \beta^2}}$$

Note: We work in 1D only, so we need no vector notation.

We can switch between the two systems and write:

$$t = \gamma\left(t' + \frac{v}{c^2} \cdot x'\right)$$

$$x = \gamma(x' + v \cdot t)$$

$$y = y'$$

$$z = z'$$

Two systems

In system S we have a particle resting, $v = 0$ with mass m_0.	In system S' we have a particle moving with speed $-v$ with mass m_0.
The total Energy E of the particle, viewed in System S: $$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$	The total Energy E of the particle, viewed in System S': $$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$
The particle is at rest, $\beta = 0$, we get in system S: $$E = \frac{m_0 c^2}{\sqrt{1-0}} = m_0 c^2$$	The particle is moving with speed $-v$, $\beta = -\frac{v}{c}$, we get in system S': $$E' = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$ The energy increases.
The momentum p of the particle, viewed in system S: $$p = mv = \frac{m_0 v_0}{\sqrt{1-\beta^2}} = \frac{0}{\sqrt{1-0}} = 0$$ Note: v_0=0 is the speed of the particle in frame S.	The momentum p of the particle, viewed in system S': $$p' = -mv = -\frac{m_0 v}{\sqrt{1-\beta^2}}$$ Note: The momentum increases. Note: v is the speed of system S' relative to S.

The impact of momentum

The formula for the total energy does not show the impact of momentum. We can make this impact visible by transforming the expression for the total energy.

We begin with:

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$

$$(mc^2)^2 = \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right)^2$$

$$m^2 c^4 = \frac{{m_0}^2 c^4}{1-\beta^2}$$

$$m^2 c^4 (1-\beta^2) = {m_0}^2 c^4$$

$$m^2 c^4 - \beta^2 m^2 c^4 = {m_0}^2 c^4$$

$$m^2 c^4 = {m_0}^2 c^4 + \beta^2 m^2 c^4$$

$$m^2 c^4 = {m_0}^2 c^4 + \frac{v^2}{c^2} m^2 c^4$$

$$m^2 c^4 = {m_0}^2 c^4 + p^2 c^2$$

$$E^2 = {m_0}^2 c^4 + p^2 c^2$$

$$E = \sqrt{{m_0}^2 c^4 + p^2 c^2}$$

We get $E(p) = ({m_0}^2c^4 + p^2c^2)^{\frac{1}{2}}$. We expand this with Taylor:

$$E(p)_{|p=0} = E(0) + E'(0) \cdot p + \frac{E''(0)}{2} \cdot p^2 + \cdots$$

From Taylor we get.

$$E(p)_{|p_0=0} = m_0c^2 + \left(\frac{1}{2}({m_0}^2c^4 + {p_0}^2c^2)^{-\frac{1}{2}} \cdot 2p_0c^2\right) \cdot p$$
$$+ \left(-\frac{1}{4}({m_0}^2c^4 + {p_0}^2c^2)^{-\frac{3}{2}} \cdot 2p_0c^2 \cdot 2p_0c^2 + \frac{1}{2}({m_0}^2c^4 + {p_0}^2c^2)^{-\frac{1}{2}} \cdot 2c^2\right)\frac{p^2}{2} + \cdots$$
$$=$$

$$m_0c^2 + \left(\frac{1}{2}({m_0}^2c^4 + {p_0}^2c^2)^{-\frac{1}{2}} \cdot 2c^2\right)\frac{p^2}{2} + \cdots =$$

$$m_0c^2 + \left(({m_0}^2c^4)^{-\frac{1}{2}}\right)\frac{c^2p^2}{2} + \cdots \approx$$

$$m_0c^2 + \frac{c^2p^2}{2m_0c^2} =$$

$$m_0c^2 + \frac{p^2}{2m_0}$$

Note: For small p this is the classical formula:

$$E = m_0c^2 + \frac{p^2}{2m_0}$$

Result: The total energy E can be expressed as a sum of rest energy and momentum energy.

If p^2c^2 becomes large, we get:

$$E = \sqrt{{m_0}^2c^4 + p^2c^2} \to E = cp$$

Independence of $E^2 - p^2c^2 = {m_0}^2c^4$

We take the equation:

$$E = \sqrt{{m_0}^2c^4 + p^2c^2}$$

We square it:

$$E^2 = {m_0}^2c^4 + p^2c^2$$

We rearrange:

$$E^2 - p^2c^2 = {m_0}^2c^4$$

We notice that ${m_0}^2c^4$ is independent of any frame. The difference $E^2 - p^2c^2$ should be independent too.

We express E in terms of E' and p'.

We begin with E:

$$E = \frac{1}{\sqrt{1-\beta^2}}(E' + v \cdot p')$$

We check this (using that v is positive:

$$\frac{1}{\sqrt{1-\beta^2}}(E' - v \cdot p') = \frac{1}{\sqrt{1-\beta^2}}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}} - v \cdot \frac{m_0 |v|}{\sqrt{1-\beta^2}}\right) =$$

$$\frac{1}{\sqrt{1-\beta^2}}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}} - \frac{m_0 v^2}{\sqrt{1-\beta^2}}\right) =$$

$$\left(\frac{m_0 c^2 - m_0 v^2}{1-\beta^2}\right) = \left(\frac{m_0 c^2 - m_0 v^2}{1 - \frac{v^2}{c^2}}\right) =$$

$$\left(\frac{m_0 c^2 - m_0 v^2}{\frac{c^2 - v^2}{c^2}}\right) = \left(\frac{c^2 (m_0 c^2 - m_0 v^2)}{c^2 - v^2}\right) =$$

$$\left(\frac{m_0 c^2 (c^2 - v^2)}{c^2 - v^2}\right) = m_0 c^2$$

We continue with p:

$$p = \frac{p' + \frac{v}{c^2} E'}{\sqrt{1-\beta^2}}$$

We check this:

$$p = \frac{p' + \frac{v}{c^2} E'}{\sqrt{1-\beta^2}} =$$

$$\frac{-\frac{m_0 v}{\sqrt{1-\beta^2}} + \frac{v}{c^2}\frac{m_0 c^2}{\sqrt{1-\beta^2}}}{\sqrt{1-\beta^2}} =$$

$$\frac{-m_0 v + \frac{v}{c^2} m_0 c^2}{1-\beta^2} =$$

$$\frac{-m_0 v + v m_0}{1-\beta^2} = 0$$

These are the Lorentz transformations for energy and momentum of a particle.

We check the independency of $E^2 - p^2 c^2$:

$$E^2 - p^2 c^2 = {m_0}^2 c^4 = E'^2 - p'^2 c^2$$

Left:

$$E^2 - p^2 c^2 = {m_0}^2 c^4 - 0 = {m_0}^2 c^4$$

Right:

$$E'^2 - {p'}^2 c^2 = \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right)^2 + \left(\frac{m_0 v}{\sqrt{1-\beta^2}}\right)^2 c^2 =$$

$$\frac{{m_0}^2c^4 - {m_0}^2v^2c^2}{1-\beta^2} = \frac{{m_0}^2c^4 - {m_0}^2v^2c^2}{1-\frac{v^2}{c^2}} =$$

$$\frac{{m_0}^2c^4 - {m_0}^2v^2c^2}{\frac{c^2-v^2}{c^2}} = \frac{c^2({m_0}^2c^4 - {m_0}^2v^2c^2)}{c^2-v^2} =$$

$$\frac{{m_0}^2c^4(c^2-v^2)}{c^2-v^2} = {m_0}^2c^4$$

The difference $E^2 - p^2c^2$ is independent of the frame it is calculated in.