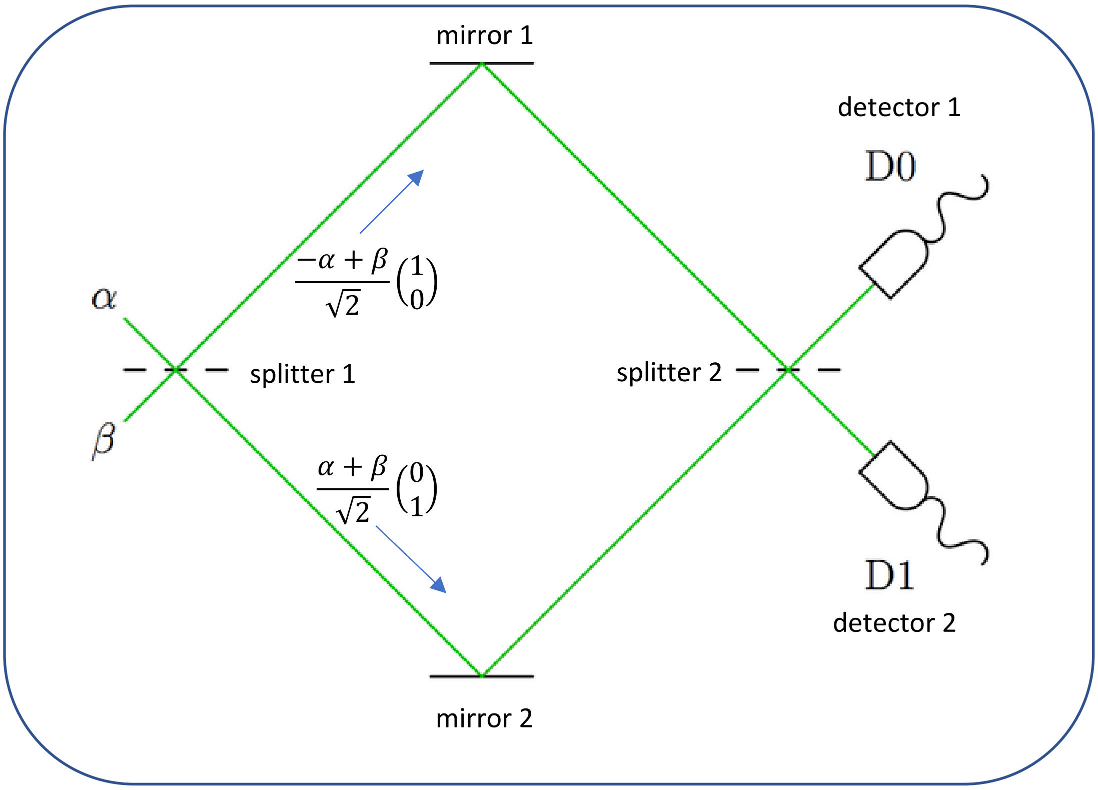
This paper deals with the Mach Zehnder interferometer. If you are interested in results only maybe you better search other available information on the internet.

We will work through this problem carefully and do a lot of arithmetic with complex numbers.

Hope I can help you with learning quantum mechanics.

Content

Mach-Zehnder Interferometer without phase shift



Input beam α: $\alpha\begin{pmatrix}0\\1\end{pmatrix}$

Input beam β: $\beta\begin{pmatrix}1\\0\end{pmatrix}$

Note: $\alpha^2+\beta^2=1$

Matrix splitter 1: $\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$	Matrix splitter 2: $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

Input into splitter 1: $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

Effect of splitter 1:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha+\beta\\ \alpha+\beta\end{pmatrix}$$

Input into splitter 2:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha+\beta\\ \alpha+\beta\end{pmatrix}$$

Effect of splitter 2:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha+\beta\\ \alpha+\beta\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\alpha+\beta+\alpha+\beta\\ -\alpha+\beta-\alpha-\beta\end{pmatrix}=\begin{pmatrix}\beta\\ -\alpha\end{pmatrix}$$

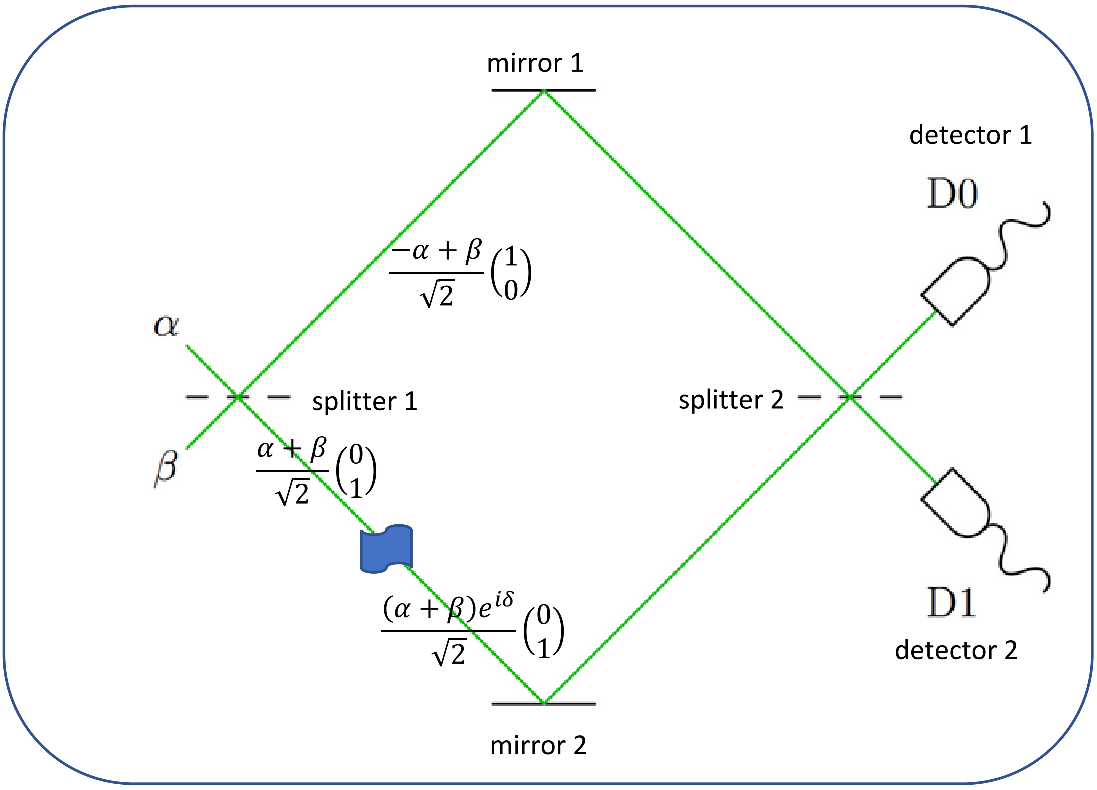
We calculate the probabilities for the photon to arrive at $D0$ or $D1$:

$$P(D0) = \beta\beta^* = |\beta|^2$$

$$P(D1) = (-\alpha)(-\alpha^*) = |\alpha|^2$$

$$P(D0) + P(D1) = |\alpha|^2 + |\beta|^2 = 1$$

We insert a phase shifter into the lower path



The phase shifter shifts the phase of the lower beam by $e^{i\delta}$.

We get the new input into splitter 2:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha+\beta\\(\alpha+\beta)e^{i\delta}\end{pmatrix}$$

The effect of splitter 2 onto the new input:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha+\beta\\(\alpha+\beta)e^{i\delta}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\alpha+\beta+(\alpha+\beta)e^{i\delta}\\-\alpha+\beta-(\alpha+\beta)e^{i\delta}\end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix}\alpha\left(e^{i\delta}-1\right)+\beta\left(e^{i\delta}+1\right)\\-\alpha\left(e^{i\delta}+1\right)+\beta\left(1-e^{i\delta}\right)\end{pmatrix}$$

We calculate the probabilities:

$$P(D0) = \frac{1}{4}\left(\alpha\left(e^{i\delta}-1\right)+\beta\left(e^{i\delta}+1\right)\right)\left(\alpha^*\left(e^{-i\delta}-1\right)+\beta^*\left(e^{-i\delta}+1\right)\right) =$$

$$\frac{1}{4}\Big(\alpha\alpha^*\left(e^{i\delta}-1\right)\left(e^{-i\delta}-1\right) + \alpha\beta^*\left(e^{i\delta}-1\right)\left(e^{-i\delta}+1\right) + \beta\alpha^*\left(e^{i\delta}+1\right)\left(e^{-i\delta}-1\right)$$

$$+\beta\beta^*\left(e^{i\delta}+1\right)\left(e^{-i\delta}+1\right)\Big) =;$$

We do this in parts.

$$* \qquad \frac{1}{4}\alpha\alpha^*(e^{i\delta}-1)(e^{-i\delta}-1)=\frac{1}{4}\alpha\alpha^*(2-e^{-i\delta}-e^{i\delta})$$

$$\frac{1}{4}\alpha\beta^*(e^{i\delta}-1)(e^{-i\delta}+1)=\frac{1}{4}\alpha\beta^*(e^{i\delta}-e^{-i\delta})$$

$$\frac{1}{4}\beta\alpha^*(e^{i\delta}+1)(e^{-i\delta}-1)=\frac{1}{4}\beta\alpha^*(-e^{i\delta}+e^{-i\delta})$$

$$\frac{1}{4}\beta\beta^*(e^{i\delta}+1)(e^{-i\delta}+1)=\frac{1}{4}\beta\beta^*\left(2+\left(e^{i\delta}+e^{-i\delta}\right)\right)$$

Note: these are the four parts of the sum.

The sum of a complex number with its complex conjugated gives two times its real part: $(x+iy)+(x-iy)=2x$	The difference of a complex number with its complex conjugate gives two time its imaginary part: $(x+iy)-(x-iy)=2\cdot i\cdot y$
For exponentials we get:	
$e^{i\delta}+e^{-i\delta}=2\cdot re(\delta)=2\cdot cos(\delta)$	$e^{i\delta}-e^{-i\delta}=2\cdot i\cdot im(\delta)=2\cdot i\cdot sin(\delta)$

We work with the middle part:

$$\frac{1}{4}\alpha\beta^*(e^{i\delta}-e^{-i\delta})+\frac{1}{4}\beta\alpha^*(-e^{i\delta}+e^{-i\delta})=\frac{1}{4}\alpha\beta^*(e^{i\delta}-e^{-i\delta})-\frac{1}{4}\beta\alpha^*(e^{i\delta}-e^{-i\delta})=$$

$$\frac{1}{4}(e^{i\delta}-e^{-i\delta})(\alpha\beta^*-\alpha^*\beta)=;$$

Note: $\alpha\beta^*-\alpha^*\beta$ is of kind $(x+iy)-(x-iy)$.

$$\frac{1}{4}\left(2\cdot i\cdot im\left(e^{i\delta}\right)\right)\left(2\cdot i\cdot im(\alpha\beta^*)\right)=-\left(im\left(e^{i\delta}\right)\right)\left(im(\alpha\beta^*)\right)=$$

$$-sin(\delta)\left(im(\alpha\beta^*)\right)$$

We treat first and fourth part parallel:

First part	Fourth part
$\frac{1}{4}\alpha\alpha^*(2-e^{-i\delta}-e^{i\delta})=$ $\frac{1}{4}\alpha\alpha^*\left(2-\left(e^{-i\delta}+e^{i\delta}\right)\right)=$ $\frac{1}{4}\alpha\alpha^*\left(2-2\cdot re\left(e^{i\delta}\right)\right)=$ $\frac{1}{2}\alpha\alpha^*\left(1-re\left(e^{i\delta}\right)\right)=$ $\alpha\alpha^*\left(\frac{1-cos(\delta)}{2}\right)=$ $\alpha\alpha^*\left(sin\left(\frac{\delta}{2}\right)\right)^2$	$\frac{1}{4}\beta\beta^*\left(2+\left(e^{i\delta}+e^{-i\delta}\right)\right)=$ $\frac{1}{4}\beta\beta^*\left(2+2\cdot re\left(e^{i\delta}\right)\right)=$ $\frac{1}{2}\beta\beta^*\left(1+re\left(e^{i\delta}\right)\right)=$ $\beta\beta^*\left(\frac{1+cos(\delta)}{2}\right)=$ $\beta\beta^*\left(cos\left(\frac{\delta}{2}\right)\right)^2$

Note: we used $\left(\frac{1-cos(\delta)}{2}\right)=\left(sin\left(\frac{\delta}{2}\right)\right)^2$ and $\left(\frac{1+cos(\delta)}{2}\right)=\left(cos\left(\frac{\delta}{2}\right)\right)^2$.

We combine the result:

$$P(D0)=\alpha\alpha^*\left(sin\left(\frac{\delta}{2}\right)\right)^2-sin(\delta)\big(im(\alpha\beta^*)\big)+\beta\beta^*\left(cos\left(\frac{\delta}{2}\right)\right)^2$$

We calculate $P(D1)$:

$$P(D1)=\frac{1}{4}\Big(-\alpha\big(e^{i\delta}+1\big)+\beta\big(1-e^{i\delta}\big)\Big)\Big(-\alpha^*\big(e^{-i\delta}+1\big)+\beta^*\big(1-e^{-i\delta}\big)\Big)=$$

$$\frac{1}{4}\Big(\alpha\alpha^*\big(e^{i\delta}+1\big)\big(e^{-i\delta}+1\big)-\alpha\beta^*\big(e^{i\delta}+1\big)\big(1-e^{-i\delta}\big)-\alpha^*\beta\big(1-e^{i\delta}\big)\big(e^{-i\delta}+1\big)+\beta\beta^*\big(1-e^{i\delta}\big)\big(1-e^{-i\delta}\big)\Big)=;$$

We do this in parts.

As above part one and four parallel:

Part one	Part four
$\frac{1}{4}\alpha\alpha^*\big(e^{i\delta}+1\big)\big(e^{-i\delta}+1\big)=$ $\frac{1}{4}\alpha\alpha^*\left(2+\big(e^{i\delta}+e^{-i\delta}\big)\right)=\cdots$ This is the same expression as above for $\beta\beta^*$. We get: $\alpha\alpha^*\left(cos\left(\frac{\delta}{2}\right)\right)^2$	$\frac{1}{4}\beta\beta^*\big(1-e^{i\delta}\big)\big(1-e^{-i\delta}\big)=$ $\frac{1}{4}\beta\beta^*\big(2-e^{-i\delta}-e^{i\delta}\big)=\cdots$ This is the same expression as above for $\alpha\alpha^*$. We get: $\beta\beta^*\left(sin\left(\frac{\delta}{2}\right)\right)^2$

Middle part:

$$\frac{1}{4}\Big(-\alpha\beta^*\big(e^{i\delta}+1\big)\big(1-e^{-i\delta}\big)-\alpha^*\beta\big(1-e^{i\delta}\big)\big(e^{-i\delta}+1\big)\Big)=$$

$$\frac{1}{4}\Big(-\alpha\beta^*\big(e^{i\delta}-e^{-i\delta}\big)-\alpha^*\beta\big(e^{-i\delta}-e^{i\delta}\big)\Big)=\frac{1}{4}\Big(-\alpha\beta^*\big(e^{i\delta}-e^{-i\delta}\big)+\alpha^*\beta\big(e^{i\delta}-e^{-i\delta}\big)\Big)=$$

$$\frac{1}{4}\big(e^{i\delta}-e^{-i\delta}\big)\big(\alpha^*\beta-\alpha\beta^*\big)=\frac{1}{4}\cdot\Big(2\cdot i\cdot im\big(e^{i\delta}\big)\Big)\cdot\big(-2\cdot i\cdot im(\alpha\beta^*)\big)=$$

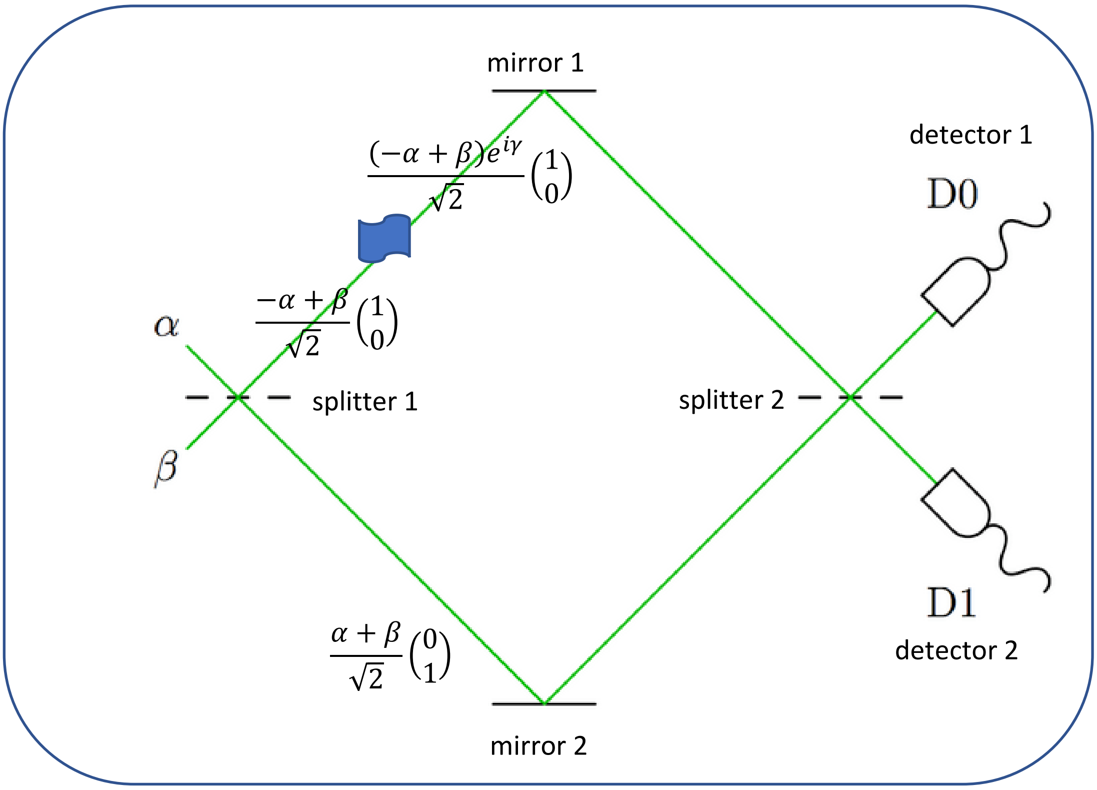
$$im\big(e^{i\delta}\big)\cdot im(\alpha\beta^*)=sin(\delta)\cdot im(\alpha\beta^*)$$

We assemble the complete solution:

$$P(D0)=\alpha\alpha^*\left(sin\left(\frac{\delta}{2}\right)\right)^2-sin(\delta)im(\alpha\beta^*)+\beta\beta^*\left(cos\left(\frac{\delta}{2}\right)\right)^2$$

$$P(D1)=\alpha\alpha^*\left(cos\left(\frac{\delta}{2}\right)\right)^2+sin(\delta)im(\alpha\beta^*)+\beta\beta^*\left(sin\left(\frac{\delta}{2}\right)\right)^2$$

We insert a phase shifter into the upper path



The phase shifter shifts the phase of the upper beam by $e^{i\gamma}$.

We get the new input into splitter 2:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\ \alpha+\beta\end{pmatrix}$$

The effect of splitter 2 onto the new input:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\ \alpha+\beta\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}+\alpha+\beta\\ (-\alpha+\beta)e^{i\gamma}-\alpha-\beta\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}\alpha\left(1-e^{i\gamma}\right)+\beta\left(1+e^{i\gamma}\right)\\ -\alpha\left(1+e^{i\gamma}\right)-\beta\left(1-e^{i\gamma}\right)\end{pmatrix}$$

We calculate the probabilities:

$$P(D0)=\frac{1}{4}\left(\alpha\left(1-e^{i\gamma}\right)+\beta\left(1+e^{i\gamma}\right)\right)\left(\alpha^*\left(1-e^{-i\gamma}\right)+\beta^*\left(1+e^{-i\gamma}\right)\right)=$$

$$\frac{1}{4}\Big(\alpha\alpha^*\left(1-e^{i\gamma}\right)\left(1-e^{-i\gamma}\right)+\alpha\beta^*\left(1-e^{i\gamma}\right)\left(1+e^{-i\gamma}\right)+\beta\alpha^*\left(1+e^{i\gamma}\right)\left(1-e^{-i\gamma}\right)$$
$$+\beta\beta^*\left(1+e^{i\gamma}\right)\left(1+e^{-i\gamma}\right)\Big)=;$$

We do this in parts and compare it with the results we already have for the phase shifter in the lower path:

upper path	lower path (* see page 5)
$\frac{1}{4}\alpha\alpha^*(1-e^{i\gamma})(1-e^{-i\gamma}) = \frac{1}{4}\alpha\alpha^*(2-e^{-i\gamma}-e^{i\gamma})$	$\frac{1}{4}\alpha\alpha^*(e^{i\delta}-1)(e^{-i\delta}-1) = \frac{1}{4}\alpha\alpha^*(2-e^{-i\delta}-e^{i\delta})$
$\frac{1}{4}\alpha\beta^*(1-e^{i\gamma})(1+e^{-i\gamma}) = \frac{1}{4}\alpha\beta^*(-e^{i\gamma}+e^{-i\gamma})$	$\frac{1}{4}\alpha\beta^*(e^{i\delta}-1)(e^{-i\delta}+1) = \frac{1}{4}\alpha\beta^*(e^{i\delta}-e^{-i\delta})$
$\frac{1}{4}\beta\alpha^*(1+e^{i\gamma})(1-e^{-i\gamma}) = \frac{1}{4}\beta\alpha^*(e^{i\gamma}-e^{-i\gamma})$	$\frac{1}{4}\beta\alpha^*(e^{i\delta}+1)(e^{-i\delta}-1) = \frac{1}{4}\beta\alpha^*(-e^{i\delta}+e^{-i\delta})$
$\frac{1}{4}\beta\beta^*(1+e^{i\gamma})(1+e^{-i\gamma}) = \frac{1}{4}\beta\beta^*\left(2+\left(e^{i\gamma}+e^{-i\gamma}\right)\right)$	$\frac{1}{4}\beta\beta^*(e^{i\delta}+1)(e^{-i\delta}+1) = \frac{1}{4}\beta\beta^*\left(2+\left(e^{i\delta}+e^{-i\delta}\right)\right)$

Result:

The results for $\alpha\alpha^*$ and $\beta\beta^*$ are the same.

The results for the mixed terms $\alpha\beta^*$ and $\beta\alpha^*$ switch the sign:

The result $\frac{1}{4}\alpha\beta^*(-e^{i\delta}+e^{-i\delta})$ for the mixed term in the upper path is minus the result $\frac{1}{4}\alpha\beta^*(e^{i\gamma}-e^{-i\gamma})$ for the mixed term in the lower path.

The result $\frac{1}{4}\beta\alpha^*(e^{i\delta}-e^{-i\delta})$ for the mixed term in the upper path is minus the result $\frac{1}{4}\beta\alpha^*(-e^{i\gamma}+e^{-i\gamma})$ for the mixed term in the lower path.

We expect that the sign of the result will change too.

We check this:

$$\frac{1}{4}\alpha\beta^*(-e^{i\gamma}+e^{-i\gamma})+\frac{1}{4}\beta\alpha^*(e^{i\gamma}-e^{-i\gamma}) = -\frac{1}{4}\alpha\beta^*(e^{i\gamma}-e^{-i\gamma})+\frac{1}{4}\beta\alpha^*(e^{i\gamma}-e^{-i\gamma}) =$$

$$(e^{i\gamma}-e^{-i\gamma})\left(-\frac{1}{4}\alpha\beta^*+\frac{1}{4}\beta\alpha^*\right) = \frac{1}{4}(e^{i\gamma}-e^{-i\gamma})(-\alpha\beta^*+\alpha^*\beta) =$$

$$-\frac{1}{4}(e^{i\gamma}-e^{-i\gamma})(\alpha\beta^*-\alpha^*\beta) = -\frac{1}{4}(e^{i\gamma}-e^{-i\gamma})(2\cdot i\cdot im(\alpha\beta^*)) =$$

$$-\frac{1}{4}\left(2\cdot i\cdot im(e^{i\gamma})\right)(2\cdot i\cdot im(\alpha\beta^*)) = \left(im(e^{i\gamma})\right)(im(\alpha\beta^*)) =$$

$$sin(\gamma)\left(im(\alpha\beta^*)\right)$$

We get the result:

$$P(D0) = \alpha\alpha^*\left(sin\left(\frac{\gamma}{2}\right)\right)^2 + sin(\gamma)\left(im(\alpha\beta^*)\right) + \beta\beta^*\left(cos\left(\frac{\gamma}{2}\right)\right)^2$$

We calculate $P(D1)$:

$$P(D1) = \frac{1}{4}\left(-\alpha(1+e^{i\gamma}) - \beta(1-e^{i\gamma})\right)\left(-\alpha^*(1+e^{-i\gamma}) - \beta^*(1-e^{-i\gamma})\right) =$$

$$\frac{1}{4}\Big(\alpha\alpha^*(1+e^{i\gamma})(1+e^{-i\gamma}) + \alpha\beta^*(1+e^{i\gamma})(1-e^{-i\gamma}) + \alpha^*\beta(1-e^{i\gamma})(1+e^{-i\gamma}) + \beta\beta^*(1-e^{i\gamma})(1-e^{-i\gamma})\Big) =;$$

We do this in parts.

Part one and part four parallel:

Part one	Part four
$\frac{1}{4}\alpha\alpha^*(1+e^{i\gamma})(1+e^{-i\gamma}) =$ $\frac{1}{4}\alpha\alpha^*\left(2+(e^{i\gamma}+e^{-i\gamma})\right) = \cdots$ This is the same expression as above for $\beta\beta^*$. We get: $\alpha\alpha^*\left(cos\left(\frac{\gamma}{2}\right)\right)^2$	$\frac{1}{4}\beta\beta^*(1-e^{i\gamma})(1-e^{-i\gamma}) =$ $\frac{1}{4}\beta\beta^*(2-e^{-i\gamma}-e^{i\gamma}) = \cdots$ This is the same expression as above for $\alpha\alpha^*$. We get: $\beta\beta^*\left(sin\left(\frac{\gamma}{2}\right)\right)^2$

The middle part:

$$\frac{1}{4}\left(\alpha\beta^*(1+e^{i\gamma})(1-e^{-i\gamma}) + \alpha^*\beta(1-e^{i\gamma})(1+e^{-i\gamma})\right) =$$

$$\frac{1}{4}\left(\alpha\beta^*(e^{i\gamma}-e^{-i\gamma}) + \alpha^*\beta(e^{-i\gamma}-e^{i\gamma})\right) = \frac{1}{4}\left(\alpha\beta^*(e^{i\gamma}-e^{-i\gamma}) - \alpha^*\beta(e^{i\gamma}-e^{-i\gamma})\right) =$$

$$\frac{1}{4}(e^{i\gamma}-e^{-i\gamma})(\alpha\beta^*-\alpha^*\beta) = \frac{1}{4}\cdot 2\cdot i\cdot im(e^{i\gamma})\cdot 2\cdot i\cdot im(\alpha\beta^*) =$$

$$i\cdot im(e^{i\gamma})\cdot i\cdot im(\alpha\beta^*) = -im(e^{i\gamma})\cdot im(\alpha\beta^*) =$$

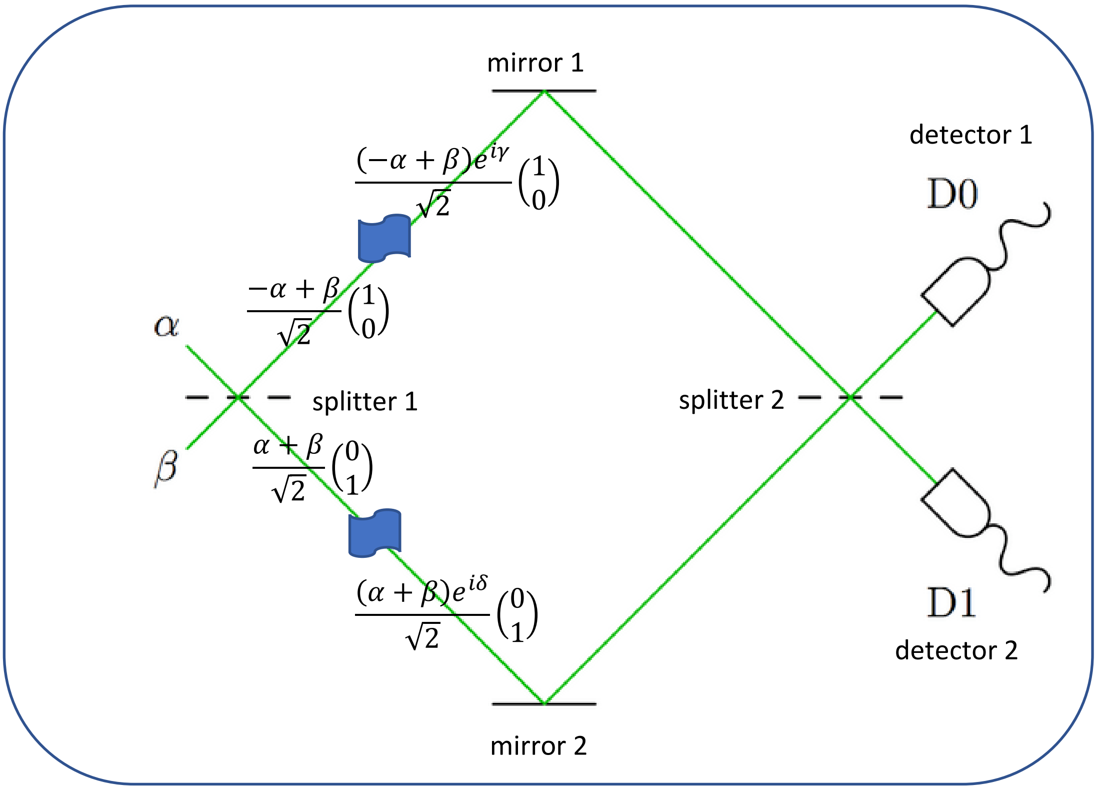
$$-sin(\gamma)\cdot im(\alpha\beta^*)$$

We assemble the complete solution:

$$P(D0) = \alpha\alpha^*\left(sin\left(\frac{\gamma}{2}\right)\right)^2 + sin(\gamma)im(\alpha\beta^*) + \beta\beta^*\left(cos\left(\frac{\gamma}{2}\right)\right)^2$$

$$P(D1) = \alpha\alpha^*\left(cos\left(\frac{\gamma}{2}\right)\right)^2 - sin(\gamma)im(\alpha\beta^*) + \beta\beta^*\left(sin\left(\frac{\gamma}{2}\right)\right)^2$$

We insert phase shifters in both paths



We get the new input into splitter 2:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\(\alpha+\beta)e^{i\delta}\end{pmatrix}$$

The effect of splitter 2 onto the new input:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\(\alpha+\beta)e^{i\delta}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}+(\alpha+\beta)e^{i\delta}\\(-\alpha+\beta)e^{i\gamma}-(\alpha+\beta)e^{i\delta}\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}-\alpha e^{i\gamma}+\beta e^{i\gamma}+\alpha e^{i\delta}+\beta e^{i\delta}\\-\alpha e^{i\gamma}+\beta e^{i\gamma}-\alpha e^{i\delta}-\beta e^{i\delta}\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}\alpha\left(e^{i\delta}-e^{i\gamma}\right)+\beta\left(e^{i\delta}+e^{i\gamma}\right)\\-\alpha\left(e^{i\delta}+e^{i\gamma}\right)-\beta\left(e^{i\delta}-e^{i\gamma}\right)\end{pmatrix}$$

We calculate the probability $P(D0)$:

$$P(D0)=\frac{1}{4}\left(\left(\alpha\left(e^{i\delta}-e^{i\gamma}\right)+\beta\left(e^{i\delta}+e^{i\gamma}\right)\right)\left(\alpha^*\left(e^{-i\delta}-e^{-i\gamma}\right)+\beta^*\left(e^{-i\delta}+e^{-i\gamma}\right)\right)\right)=$$

$$\frac{1}{4}\Big(\big(\alpha\alpha^*\left(e^{i\delta}-e^{i\gamma}\right)\left(e^{-i\delta}-e^{-i\gamma}\right)+\alpha\beta^*\left(e^{i\delta}-e^{i\gamma}\right)\left(e^{-i\delta}+e^{-i\gamma}\right)+\alpha^*\beta\left(e^{i\delta}+e^{i\gamma}\right)\left(e^{-i\delta}-e^{-i\gamma}\right)$$
$$+\beta\beta^*\left(e^{i\delta}+e^{i\gamma}\right)\left(e^{-i\delta}+e^{-i\gamma}\right)\big)\Big)=$$

$$\frac{1}{4}\Big(\big(\alpha\alpha^*\left(2-e^{i\delta}e^{-i\gamma}-e^{-i\delta}e^{i\gamma}\right)+\alpha\beta^*\left(e^{i\delta}e^{-i\gamma}-e^{-i\delta}e^{i\gamma}\right)+\alpha^*\beta\left(-e^{i\delta}e^{-i\gamma}+e^{-i\delta}e^{i\gamma}\right)$$
$$+\beta\beta^*\left(2+e^{i\delta}e^{-i\gamma}+e^{-i\delta}e^{i\gamma}\right)\big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2-e^{i(\delta-\gamma)}-e^{i(\gamma-\delta)}\big)+\alpha\beta^*\big(e^{i(\delta-\gamma)}-e^{i(\gamma-\delta)}\big)+\alpha^*\beta\big(-e^{i(\delta-\gamma)}+e^{i(\gamma-\delta)}\big) + \beta\beta^*\big(2+e^{i(\delta-\gamma)}+e^{i(\gamma-\delta)}\big)\Big)\Big) =;$$

We rename $\gamma-\delta \coloneqq \Delta$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\left(2-\left(e^{-i(\Delta)}+e^{i(\Delta)}\right)\right)+\alpha\beta^*\big(e^{-i(\Delta)}-e^{i(\Delta)}\big)-\alpha^*\beta\big(e^{-i(\Delta)}-e^{i(\Delta)}\big)+\beta\beta^*\big(2+e^{-i(\Delta)}+e^{i(\Delta)}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\left(2-\left(e^{-i(\Delta)}+e^{i(\Delta)}\right)\right)+\big(e^{-i(\Delta)}-e^{i(\Delta)}\big)(\alpha\beta^*-\alpha^*\beta)+\beta\beta^*\big(2+e^{-i(\Delta)}+e^{i(\Delta)}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2-2\cdot cos(\Delta)\big)+\big(-2\cdot i\cdot sin(\Delta)\big)(\alpha\beta^*-\alpha^*\beta)+\beta\beta^*\big(2+2\cdot cos(\Delta)\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2-2\cdot cos(\Delta)\big)+\big(-2\cdot i\cdot sin(\Delta)\big)\big(2\cdot i\cdot im(\alpha\beta^*)\big)+\beta\beta^*\big(2+2\cdot cos(\Delta)\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2-2\cdot cos(\Delta)\big)+\big(2\cdot sin(\Delta)\big)\big(2\cdot im(\alpha\beta^*)\big)+\beta\beta^*\big(2+2\cdot cos(\Delta)\big)\Big)\Big)=$$

$$\frac{1}{2}\left(\left(\alpha\alpha^*\left(\frac{1-cos(\Delta)}{2}\right)+sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(\frac{1+cos(\Delta)}{2}\right)\right)\right)=$$

$$\alpha\alpha^*\left(sin\left(\frac{\Delta}{2}\right)\right)^2+sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(cos\left(\frac{\Delta}{2}\right)\right)^2$$

We calculate the probability $P(D1)$:

$$P(D1)=\frac{1}{4}\Big(\Big(-\alpha\big(e^{i\delta}+e^{i\gamma}\big)-\beta\big(e^{i\delta}-e^{i\gamma}\big)\Big)\Big(-\alpha^*\big(e^{-i\delta}+e^{-i\gamma}\big)-\beta^*\big(e^{-i\delta}-e^{-i\gamma}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(e^{i\delta}+e^{i\gamma}\big)\big(e^{-i\delta}+e^{-i\gamma}\big)+\alpha\beta^*\big(e^{i\delta}+e^{i\gamma}\big)\big(e^{-i\delta}-e^{-i\gamma}\big)+\alpha^*\beta\big(e^{i\delta}-e^{i\gamma}\big)\big(e^{-i\delta}+e^{-i\gamma}\big) + \beta\beta^*\big(e^{i\delta}-e^{i\gamma}\big)\big(e^{-i\delta}-e^{-i\gamma}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2+e^{i\delta}e^{-i\gamma}+e^{-i\delta}e^{i\gamma}\big)+\alpha\beta^*\big(e^{-i\delta}e^{i\gamma}-e^{i\delta}e^{-i\gamma}\big)+\alpha^*\beta\big(e^{i\delta}e^{-i\gamma}-e^{-i\delta}e^{i\gamma}\big) + \beta\beta^*\big(2-e^{i\delta}e^{-i\gamma}-e^{-i\delta}e^{i\gamma}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2+e^{i(\delta-\gamma)}+e^{i(\gamma-\delta)}\big)+\alpha\beta^*\big(e^{i(\gamma-\delta)}-e^{i(\delta-\gamma)}\big)+\alpha^*\beta\big(e^{i(\delta-\gamma)}-e^{i(\gamma-\delta)}\big) + \beta\beta^*\big(2-e^{i(\delta-\gamma)}-e^{i(\gamma-\delta)}\big)\Big)\Big)=;$$

We rename $\gamma-\delta \coloneqq \Delta$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2+e^{-i(\Delta)}+e^{i\Delta}\big)+\alpha\beta^*\big(e^{i\Delta}-e^{-i(\Delta)}\big)+\alpha^*\beta\big(e^{-i(\Delta)}-e^{i\Delta}\big)+\beta\beta^*\big(2-e^{-i(\Delta)}-e^{i\Delta}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2+e^{i\Delta}+e^{-i(\Delta)}\big)+\alpha\beta^*\big(e^{i\Delta}-e^{-i(\Delta)}\big)-\alpha^*\beta\big(e^{i\Delta}-e^{-i(\Delta)}\big)+\beta\beta^*\big(2-e^{i\Delta}-e^{-i(\Delta)}\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\Big(\alpha\alpha^*\big(2+2\cdot cos(\Delta)\big)+\big(e^{i\Delta}-e^{-i(\Delta)}\big)(\alpha\beta^*-\alpha^*\beta)+\beta\beta^*\big(2-2\cdot cos(\Delta)\big)\Big)\Big)=$$

$$\frac{1}{4}\Big(\big(\alpha\alpha^*(2+2\cdot cos(\Delta)\big)+\big(2\cdot i\cdot sin(\Delta)\big)\big(2\cdot i\cdot im(\alpha\beta^*)\big)+\beta\beta^*\big(2-2\cdot cos(\Delta)\big)\big)\Big)=$$

$$\alpha\alpha^*\left(\frac{1+cos(\Delta)}{2}\right)-sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(\frac{1-cos(\Delta)}{2}\right)=$$

$$\alpha\alpha^*\left(cos\left(\frac{\Delta}{2}\right)\right)^2-sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(sin\left(\frac{\Delta}{2}\right)\right)^2$$

The complete solution:

$$P(D0)=\alpha\alpha^*\left(sin\left(\frac{\Delta}{2}\right)\right)^2+sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(cos\left(\frac{\Delta}{2}\right)\right)^2$$

$$P(D1)=\alpha\alpha^*\left(cos\left(\frac{\Delta}{2}\right)\right)^2-sin(\Delta)im(\alpha\beta^*)+\beta\beta^*\left(sin\left(\frac{\Delta}{2}\right)\right)^2$$

Remarks

a) The Δ-solution seems identic with the upper path solution. If γ shifts more than δ, $\gamma-\delta$ becomes positive and we have a net phase shift in the upper branch. If γ shifts less than δ, $\gamma-\delta$ becomes negative and we have a net phase shift in the lower branch. The sin-function is antisymmetric and shows this behavior too: $sin(-x)=-sin(x)$.

b) We invested a lot of work in the Δ-case. We can avoid this by manipulating the vector.

The vector with both phase shifters:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\(\alpha+\beta)e^{i\delta}\end{pmatrix}$$

We draw out the phase of the upper path:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}(-\alpha+\beta)e^{i\gamma}\\(\alpha+\beta)e^{i\delta}\end{pmatrix}\rightarrow\frac{1}{\sqrt{2}}\cdot e^{i\gamma}\cdot\begin{pmatrix}(-\alpha+\beta)\\(\alpha+\beta)e^{i(\delta-\gamma)}\end{pmatrix}\rightarrow$$

$$\frac{1}{\sqrt{2}}\cdot e^{i\gamma}\cdot\begin{pmatrix}(-\alpha+\beta)\\(\alpha+\beta)e^{-i\Delta}\end{pmatrix}$$

Note: We get $-\Delta$ because of $\delta-\gamma$ compared to the Δ used above.

The overall phase $e^{i\gamma}$ has no effect, the interferometer reacts to the value of Δ in the lower path only.

c) A mirror adds a phase of 180° or $e^{i\pi}$ to the beam. The two mirrors will add this effect to 360° or $e^{2i\pi}$ thus cancelling out.

We use splitter matrices:

Matrix splitter 1:	Matrix splitter 2:
$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$

The -1 belongs to the part of the beam that is reflected by the splitter. The $+1$ belongs to the part transmitted. Remember that $e^{i\pi}=-1$.

Matrix and splitter

We work with an input beam represented as $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ where α belongs to the upper beam, β to the lower beam, α and β are complex numbers. Quantum mechanics treat this as probability amplitudes requiring $|\alpha|^2 + |\beta|^2 = 1$, so $\alpha\alpha^* + \beta\beta^* = 1$.

Any linear optical element used in the path of the interferometer can be represented by a 2×2 matrix (at least I hope so …).

The conservation of the probability requires the matrices M to be unitary:

$$M^\dagger M = 1$$

Note: † stands for transposition and complex conjugation of the matrix, † is called "dagger".

Unitary matrices preserve the probability.

We check with all numbers being complex:

$$M \coloneqq \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$M^\dagger = \begin{pmatrix}a^* & c^*\\ b^* & d^*\end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix}a^* & c^*\\ b^* & d^*\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^*a + c^*c & a^*b + c^*d\\ b^*a + d^*c & b^*b + d^*d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a^* & c^*\\ b^* & d^*\end{pmatrix} = \begin{pmatrix}aa^* + bb^* & ac^* + bd^*\\ ca^* + db^* & cc^* + dd^*\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

We get the requirements:

$$a^*a + c^*c = 1 = aa^* + bb^*$$

$$b^*b + d^*d = 1 = cc^* + dd^*$$

$$a^*b + c^*d = 0 = ac^* + bd^*$$

$$b^*a + d^*c = 0 = ca^* + db^*$$

For complex numbers holds:

$$a^*a = aa^*\ etc.$$

$$a^*a \in \mathbb{R}, a^*a \geq 0$$

From the requirements we get dependencies for the coefficients:

$$b^*b = c^*c$$

∗∗

$$a^*b = -c^*d$$

$$c^*a = -d^*b$$

$$b^*a = -d^*c$$

$$a^*c = -b^*d$$

We check:

$$M^\dagger M = \begin{pmatrix}a^* & c^*\\ b^* & d^*\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^*a + b^*b & -c^*d + c^*d\\ -d^*c + d^*c & b^*b + d^*d\end{pmatrix} = \begin{pmatrix}a^*a + b^*b & 0\\ 0 & b^*b + d^*d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & 0 \\ 0 & b^*b + dd^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We get back the unity matrix.

The Matrix M must conserve the probability:

$$(\alpha'^* \beta'^*)\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \alpha'^*\alpha' + \beta'^*\beta' = 1$$

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix}$$

We use the dependencies (**) and calculate:

$$\big((a^*\alpha^* + b^*\beta^*)\ (c^*\alpha^* + d^*\beta^*)\big)\begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix} =$$

$$(a^*\alpha^* + b^*\beta^*)(a\alpha + b\beta) + (c^*\alpha^* + d^*\beta^*)(c\alpha + d\beta) =$$

$$(a^*a)\alpha^*\alpha + (a^*b)\alpha^*\beta + (b^*a)\beta^*\alpha + (b^*b)\beta^*\beta + (c^*c)\alpha^*\alpha + (c^*d)\alpha^*\beta + (d^*c)\beta^*\alpha + (d^*d)\beta^*\beta =$$

$$(a^*a)\alpha^*\alpha + (b^*b)\beta^*\beta + (c^*c)\alpha^*\alpha + (d^*d)\beta^*\beta =$$

$$(a^*a + c^*c)\alpha^*\alpha + (b^*b + d^*d)\beta^*\beta =$$

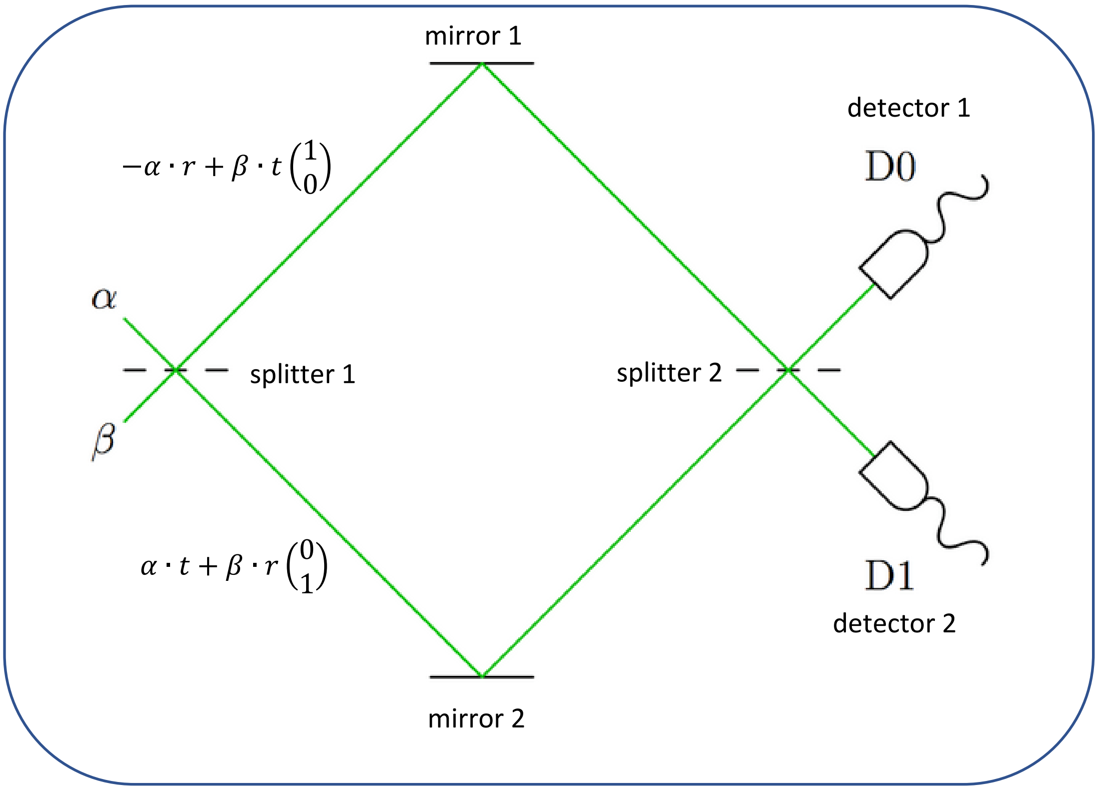
$$\alpha^*\alpha + \beta^*\beta = 1$$

Matrix M preserves the probability.

Variable reflection and transmission ratio

We modify the splitters and allow a ratio in transmission T and reflection R, $T + R = 1$. We use square roots $\sqrt{T} \coloneqq t$ and $\sqrt{R} \coloneqq r$.

The picture changes:



Input beam α: $\alpha\begin{pmatrix}0\\1\end{pmatrix}$

Input beam β: $\beta\begin{pmatrix}1\\0\end{pmatrix}$

Note: $\alpha^2+\beta^2=1$

Matrix splitter 1, M_{S1}: $M_{S1}:=\begin{pmatrix}-r & t\\ t & r\end{pmatrix}$	Matrix splitter 2, M_{S2}: $M_{S2}:=\begin{pmatrix}r & t\\ t & -r\end{pmatrix}$

Note: we could either swap the input to splitter two or swap the matrix of splitter two by its antidiagonal. This is done by sandwiching the first matrix with the swap-matrix X:

$$X:=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

We check:

$$XM_{S1}X=$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-r & t\\ t & r\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}t & r\\ -r & t\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}r & t\\ t & -r\end{pmatrix}$$

Result:

$$XM_{S1}X=M_{S2}$$

Input into splitter 1:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Effect of splitter 1:

$$\begin{pmatrix}-r & t\\ t & r\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}-\alpha\cdot r+\beta\cdot t\\ \alpha\cdot t+\beta\cdot r\end{pmatrix}$$

Input into splitter 2:

$$\begin{pmatrix}-\alpha\cdot r+\beta\cdot t\\ \alpha\cdot t+\beta\cdot r\end{pmatrix}$$

Effect of splitter 2:

$$\begin{pmatrix}r & t\\ t & -r\end{pmatrix}\begin{pmatrix}-\alpha\cdot r+\beta\cdot t\\ \alpha\cdot t+\beta\cdot r\end{pmatrix}=\begin{pmatrix}r\cdot(-\alpha\cdot r+\beta\cdot t)+t\cdot(\alpha\cdot t+\beta\cdot r)\\ t\cdot(-\alpha\cdot r+\beta\cdot t)-r\cdot(\alpha\cdot t+\beta\cdot r)\end{pmatrix}=$$

$$\begin{pmatrix}-\alpha\cdot r^2+\beta\cdot rt+\alpha\cdot t^2+\beta\cdot rt\\ -\alpha\cdot rt+\beta\cdot t^2-\alpha\cdot rt-\beta\cdot r^2\end{pmatrix}=\begin{pmatrix}\alpha(t^2-r^2)+2\beta rt\\ -2\alpha rt+\beta(t^2-r^2)\end{pmatrix}$$

We can compact this process by building the combined matrix M_S for M_{S1} and splitter M_{S2}:

$$M_S=M_{S2}M_{S1}=$$

$$\begin{pmatrix}r & t\\ t & -r\end{pmatrix}\begin{pmatrix}-r & t\\ t & r\end{pmatrix}=\begin{pmatrix}-r^2+t^2 & 2rt\\ -2rt & t^2-r^2\end{pmatrix}$$

Note: The order of matrices is important.

We check the effect of the combined matrix to the input beam:

$$\begin{pmatrix} -r^2+t^2 & 2rt \\ -2rt & t^2-r^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha r^2+\alpha t^2+2\beta rt \\ -2\alpha rt+\beta t^2-\beta r^2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha(t^2-r^2)+2\beta rt \\ -2\alpha rt+\beta(t^2-r^2) \end{pmatrix}$$

Result: The combination of the two matrices gives the same result.

We can treat reflection and transmission as real and imaginary part of a complex constant $z \coloneqq e^{i\varphi}$:

$$r = re(z) = cos(\varphi)$$

$$t = im(z) = sin(\varphi)$$

We get the matrix M_S:

$$\begin{pmatrix} -r^2+t^2 & 2rt \\ -2rt & t^2-r^2 \end{pmatrix} \to \begin{pmatrix} -cos^2(\varphi)+sin^2(\varphi) & 2cos(\varphi)sin(\varphi) \\ -2cos(\varphi)sin(\varphi) & sin^2(\varphi)-cos^2(\varphi) \end{pmatrix} =;$$

We use trigonometric formulas:

$$2cos(\varphi)sin(\varphi) = sin(2\varphi)$$

$$sin^2(\varphi)-cos^2(\varphi) = -cos(2\varphi)$$

We proceed:

$$\begin{pmatrix} -cos^2(\varphi)+sin^2(\varphi) & 2cos(\varphi)sin(\varphi) \\ -2cos(\varphi)sin(\varphi) & sin^2(\varphi)-cos^2(\varphi) \end{pmatrix} \to \begin{pmatrix} -cos(2\varphi) & sin(2\varphi) \\ -sin(2\varphi) & -cos(2\varphi) \end{pmatrix}$$

Note: we write this with exponentials only:

$$\frac{1}{2}\begin{pmatrix} -(z^2+(z^*)^2) & i(z^2-(z^*)^2) \\ -i(z^2-(z^*)^2) & -(z^2+(z^*)^2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -(e^{i2\varphi}+e^{-i2\varphi}) & i(e^{i2\varphi}-e^{-i2\varphi}) \\ -i(e^{i2\varphi}-e^{-i2\varphi}) & -(e^{i2\varphi}+e^{-i2\varphi}) \end{pmatrix}$$

Result:

$$M_S = \frac{1}{2}\begin{pmatrix} -(e^{i2\varphi}+e^{-i2\varphi}) & i(e^{i2\varphi}-e^{-i2\varphi}) \\ -i(e^{i2\varphi}-e^{-i2\varphi}) & -(e^{i2\varphi}+e^{-i2\varphi}) \end{pmatrix}$$

Note: $(z^2-(z^*)^2)$ is pure imaginary, so $i(z^2-(z^*)^2)$ is real.

Note: $(z^2+(z^*)^2)$ is a real number.

Note: The matrix M_S has real values only.

Concatenating several interferometers

By using the matrices we can concatenate interferometers with different reflection and transmission parameters.

It is helpful to use the four Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices build a basis for the space of Hermitian 2×2-matrices.

The matrix M_S depends on the argument φ only.

We express the matrix M_S by help of the Pauli matrices I and Y:

$$M_S = -\frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2}\left(e^{i2\varphi} - e^{-i2\varphi}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

We use:

$$e^{i\theta\sigma_i} = id \cdot \frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right) + i\sigma_i \cdot \frac{1}{2}\left(e^{i2\varphi} - e^{-i2\varphi}\right)$$

Note: σ_i are the Pauli matrices.

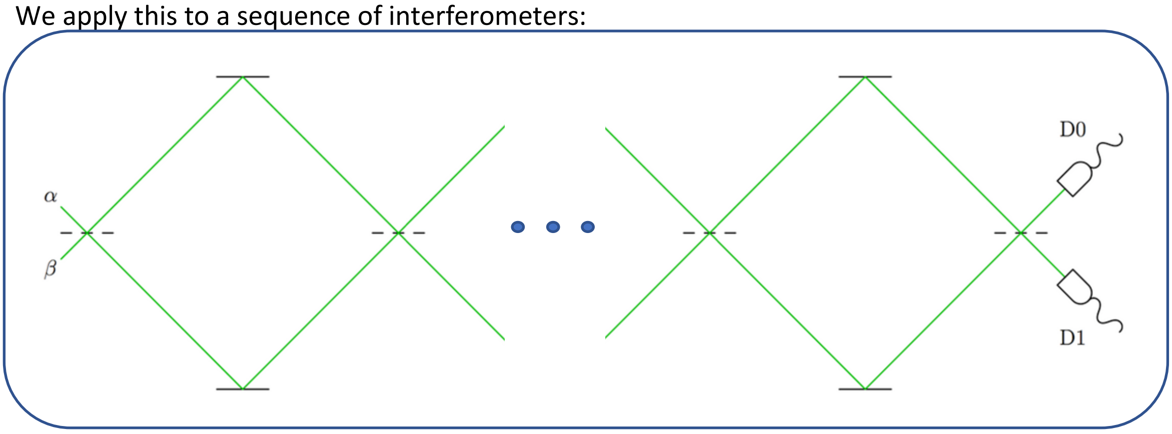
We apply:

$$e^{i2\varphi Y} = id \cdot \left(\frac{e^{i2\varphi} + e^{-i2\varphi}}{2}\right) + Y \cdot \left(\frac{e^{i2\varphi} - e^{-i2\varphi}}{2}\right)$$

We get:

$$M_S = -e^{i2\varphi Y}$$

We apply this to a sequence of interferometers:



The transformation matrix M_n is a sequence of n matrices M_S:

$$M_n = (M_S)^n = \left(-e^{i2\varphi Y}\right)^n = (-1)^n \cdot e^{i2n\varphi Y}$$

This result is valid if the ratios of reflection and transmission are the same for each interferometer. If we concatenate interferometers with different ratios, φ changes. We set the ratio φ_i for interferometer i and get:

$$M_n = M_1 \cdot \ldots \cdot M_n = -e^{i2(\varphi_1 + \cdots + \varphi_n)Y}$$

Note that in any case $i2n\varphi$ resp. $i2(\varphi_1 + \cdots + \varphi_n)$ is modulo 2π.

Experimental settings

We will work with a set of n interferometers having the same reflection and transmission ratio.

Beams staying on their side

How have we to adjust reflection and transmission ratio to hold the output on the same side as the input? The input beam α to go completely through $D0$?

In this case reflection must be 100%. It is sufficient to take a look at a single transfer matrix:

$$M_S = -\frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2}\left(e^{i2\varphi} - e^{-i2\varphi}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

The transmission part:

$$-\frac{1}{2}\left(e^{i2\varphi}+e^{-i2\varphi}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

The reflection part:

$$-\frac{1}{2}\left(e^{i2\varphi}-e^{-i2\varphi}\right)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

The transmission part must be zero:

$$-\frac{1}{2}\left(e^{i2\varphi}+e^{-i2\varphi}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=0\rightarrow$$

$$e^{i2\varphi}+e^{-i2\varphi}=0\rightarrow$$

$$e^{i2\varphi}=-e^{-i2\varphi}\rightarrow 2\varphi=\frac{\pi}{2}\rightarrow$$

$$\varphi=\frac{\pi}{4}$$

We check the transmission part:

$$-\frac{1}{2}\left(e^{i2\varphi}+e^{-i2\varphi}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\rightarrow$$

$$-\frac{1}{2}\left(e^{i2\frac{\pi}{4}}+e^{-i2\frac{\pi}{4}}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=$$

$$-\frac{1}{2}\left(e^{i\frac{\pi}{2}}+e^{-i\frac{\pi}{2}}\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=$$

$$-\frac{1}{2}\left(i+(-i)\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=0$$

We check the reflection part:

$$-\frac{1}{2}\left(e^{i2\varphi}-e^{-i2\varphi}\right)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\rightarrow$$

$$-\frac{1}{2}\left(e^{i2\frac{\pi}{4}}-e^{-i2\frac{\pi}{4}}\right)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}=$$

$$-\frac{1}{2}\left(i-(-i)\right)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}=$$

$$-i\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}=$$

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

The beam is reflected 100% at any splitter, any beam remains on its side.

Beams passing through

How have we to adjust reflection and transmission ratio to swap the output onto the other side? The input beam α to go completely through $D1$?

In this case we must work with the whole transfer matrix M_n:

$$M_n=(-1)^n\cdot e^{i2n\varphi Y}$$

We omit the factor $(-1)^n$:

$$M_n = e^{i2n\varphi Y} = \left(id \cdot \frac{1}{2}\left(e^{i2n\varphi} + e^{-i2n\varphi}\right) + Y \cdot \frac{1}{2}\left(e^{i2n\varphi} - e^{-i2n\varphi}\right)\right)$$

The (reflection) swapping part:

$$Y \cdot \frac{1}{2}\left(e^{i2n\varphi} - e^{-i2n\varphi}\right)$$

The transmission part:

$$id \cdot \frac{1}{2}\left(e^{i2n\varphi} + e^{-i2n\varphi}\right)$$

The transmission part must become zero:

$$e^{i2n\varphi} + e^{-i2n\varphi} = 0 \rightarrow$$

$$e^{i2n\varphi} = -e^{-i2n\varphi} \rightarrow$$

$$2n\varphi = \frac{\pi}{2} \rightarrow$$

$$\varphi = \frac{\pi}{4n}$$

We check the transmission part:

$$id \cdot \frac{1}{2}\left(e^{i2n\varphi} + e^{-i2n\varphi}\right) \rightarrow$$

$$id \cdot \frac{1}{2}\left(e^{i2n\frac{\pi}{4n}} + e^{-i2n\frac{\pi}{4n}}\right) =$$

$$id \cdot \frac{1}{2}\left(e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}}\right) =$$

$$id \cdot \frac{1}{2}(i - i) = 0$$

We check the reflection part:

$$Y \cdot \frac{1}{2}\left(e^{i2n\varphi} - e^{-i2n\varphi}\right) \rightarrow$$

$$Y \cdot \frac{1}{2}\left(e^{i2n\frac{\pi}{4n}} - e^{-i2n\frac{\pi}{4n}}\right) =$$

$$Y \cdot \frac{1}{2}\left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right) =$$

$$Y \cdot \frac{1}{2}\left(i - (-i)\right) =$$

$$Y \cdot \frac{1}{2} 2i =$$

$$Y \cdot i =$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot i =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In the transfer matrix we find the term $(-1)^n$.

Depending on n being odd or even this will direct the output to $D1$.