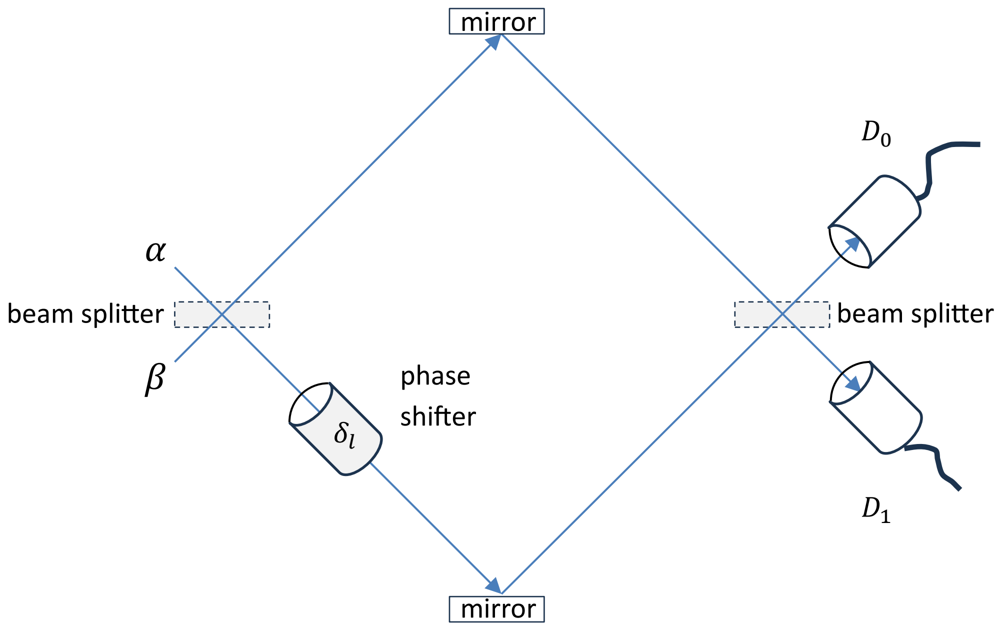
This paper deals with the Mach Zehnder Interferometer and the following scenarios: plain interferometer, phase shifter in one path and in both two paths.

A more elaborated version dealing with transmission and reflection you find in the paper "Mach Zehnder".

Hope I can help you with learning quantum mechanics.

Mach-Zehnder Interferometer



We have an input beam:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Note that the photon is in superposition with α and β complex numbers, normalized to the probability $|\alpha|^2 + |\beta|^2 = 1$.

The left beam splitter can be expressed as a matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$

The right beam splitter:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

The phase shifter can be expressed as a matrix too:

$$\begin{pmatrix}1 & 0\\ 0 & e^{i\delta_l}\end{pmatrix}$$

All matrices act on the input photon:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\delta_l}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\delta_l}\end{pmatrix}\begin{pmatrix}-\alpha+\beta\\ \alpha+\beta\end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}-\alpha+\beta\\ e^{i\delta_l}(\alpha+\beta)\end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} -\alpha+\beta+e^{i\delta_l}(\alpha+\beta) \\ -\alpha+\beta-e^{i\delta_l}(\alpha+\beta) \end{pmatrix} =;$$

As we know we will get a solution of double angle. We rewrite:

$$\frac{1}{2}\begin{pmatrix} \alpha\left(e^{i\delta_l}-1\right)+\beta\left(e^{i\delta_l}+1\right) \\ -\alpha\left(e^{i\delta_l}+1\right)+\beta\left(1-e^{i\delta_l}\right) \end{pmatrix} =;$$

We isolate the double angle:

$$\frac{e^{i\frac{\delta_l}{2}}}{2}\begin{pmatrix} \alpha\left(e^{i\frac{\delta_l}{2}}-e^{-i\frac{\delta_l}{2}}\right)+\beta\left(e^{i\frac{\delta_l}{2}}+e^{-i\frac{\delta_l}{2}}\right) \\ -\alpha\left(e^{i\frac{\delta_l}{2}}+e^{-i\frac{\delta_l}{2}}\right)-\beta\left(e^{i\frac{\delta_l}{2}}-e^{-i\frac{\delta_l}{2}}\right) \end{pmatrix}$$

We use complex identities:

$$e^{i\frac{\delta_l}{2}}-e^{-i\frac{\delta_l}{2}}=2isin\frac{\delta_l}{2}, \qquad e^{i\frac{\delta_l}{2}}+e^{-i\frac{\delta_l}{2}}=2cos\frac{\delta_l}{2}$$

We get:

$$e^{i\frac{\delta_l}{2}}\begin{pmatrix} \alpha\cdot i\cdot sin\frac{\delta_l}{2}+\beta\cdot cos\frac{\delta_l}{2} \\ -\alpha\cdot cos\frac{\delta_l}{2}-\beta\cdot i\cdot sin\frac{\delta_l}{2} \end{pmatrix}$$

We check by calculating the probabilities on the detectors, the sum must give one.

$$|P_0|^2=\left(\alpha\cdot i\cdot sin\frac{\delta_l}{2}+\beta\cdot cos\frac{\delta_l}{2}\right)\left(\overline{\alpha}\cdot -i\cdot sin\frac{\delta_l}{2}+\overline{\beta}\cdot cos\frac{\delta_l}{2}\right)=$$

$$\left(\alpha\cdot i\cdot sin\frac{\delta_l}{2}\right)\left(\overline{\alpha}\cdot -i\cdot sin\frac{\delta_l}{2}\right)+\left(\alpha\cdot i\cdot sin\frac{\delta_l}{2}\right)\left(\overline{\beta}\cdot cos\frac{\delta_l}{2}\right)+\left(\beta\cdot cos\frac{\delta_l}{2}\right)\left(\overline{\alpha}\cdot -i\cdot sin\frac{\delta_l}{2}\right)$$
$$+\left(\beta\cdot cos\frac{\delta_l}{2}\right)\left(\overline{\beta}\cdot cos\frac{\delta_l}{2}\right)=$$

$$|\alpha|^2sin^2\frac{\delta_l}{2}+\alpha\overline{\beta}isin\frac{\delta_l}{2}cos\frac{\delta_l}{2}-\overline{\alpha}\beta icos\frac{\delta_l}{2}sin\frac{\delta_l}{2}+|\beta|^2cos^2\frac{\delta_l}{2}=$$

$$|\alpha|^2sin^2\frac{\delta_l}{2}+isin\frac{\delta_l}{2}cos\frac{\delta_l}{2}\left(\alpha\overline{\beta}-\overline{\alpha}\beta\right)+|\beta|^2cos^2\frac{\delta_l}{2}=;$$

We use trigonometric identities:

$$sin\frac{\delta_l}{2}cos\frac{\delta_l}{2}=\frac{1}{2}sin\delta_l$$

We get:

$$|\alpha|^2sin^2\frac{\delta_l}{2}+i\frac{1}{2}sin\delta_l\left(\alpha\overline{\beta}-\overline{\alpha}\beta\right)+|\beta|^2cos^2\frac{\delta_l}{2}=;$$

We calculate the complex number using:

$$\alpha\overline{\beta} = x + iy, \qquad i\alpha\overline{\beta} = -y + ix$$

$$\overline{\alpha}\beta = x - iy, \qquad i\overline{\alpha}\beta = y + ix$$

$$i\alpha\overline{\beta} - i\overline{\alpha}\beta = -2y = -2 \cdot Im\left(\alpha\overline{\beta}\right)$$

We get the probability for the upper path:

$$|\alpha|^2 sin^2\frac{\delta_l}{2} - sin\delta_l \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 cos^2\frac{\delta_l}{2}$$

With the same calculation we find the probability for the lower path:

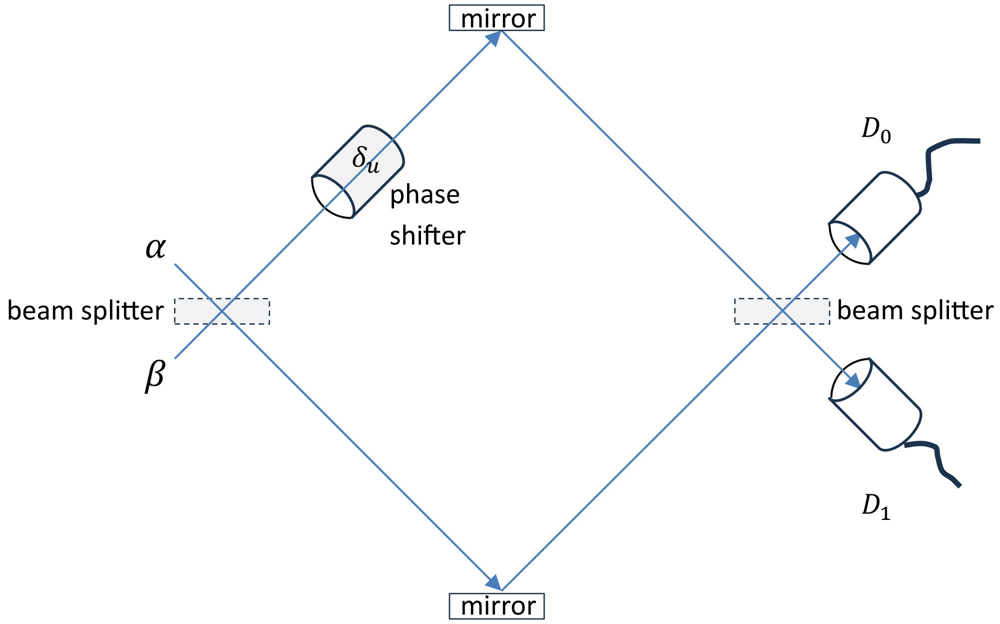
$$|\alpha|^2 cos^2\frac{\delta_l}{2} + sin\delta_l \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 sin^2\frac{\delta_l}{2}$$

Both probabilities must give 1:

$$|\alpha|^2 sin^2\frac{\delta_l}{2} - sin\delta_l \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 cos^2\frac{\delta_l}{2} + |\alpha|^2 cos^2\frac{\delta_l}{2} + sin\delta_l \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 sin^2\frac{\delta_l}{2} =$$

$$|\alpha|^2\left(sin^2\frac{\delta_l}{2} + cos^2\frac{\delta_l}{2}\right) + |\beta|^2\left(cos^2\frac{\delta_l}{2} + sin^2\frac{\delta_l}{2}\right) = |\alpha|^2 + |\beta|^2 = 1$$

Mach-Zehnder Interferometer with the phase shifter in the upper path



We have an input beam:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Note that the photon is in superposition with α and β complex numbers, normalized to the probability $|\alpha|^2 + |\beta|^2 = 1$.

The left beam splitter can be expressed as a matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$

The right beam splitter:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

The phase shifter can be expressed as a matrix too:

$$\begin{pmatrix}e^{i\delta_u} & 0\\ 0 & 1\end{pmatrix}$$

All matrices act on the input photon:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{i\delta_u} & 0\\ 0 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Instead of running through the same calculation we use:

$$\begin{pmatrix}e^{i\delta_u} & 0\\ 0 & 1\end{pmatrix} = e^{i\delta_u}\begin{pmatrix}1 & 0\\ 0 & e^{-i\delta_u}\end{pmatrix}$$

The overall phase will not affect the probability because $\left|e^{i\delta_u}\right|^2 = 1$.

We take our result from above and replace δ_l by $-\delta_u$:

The probability for the upper path:

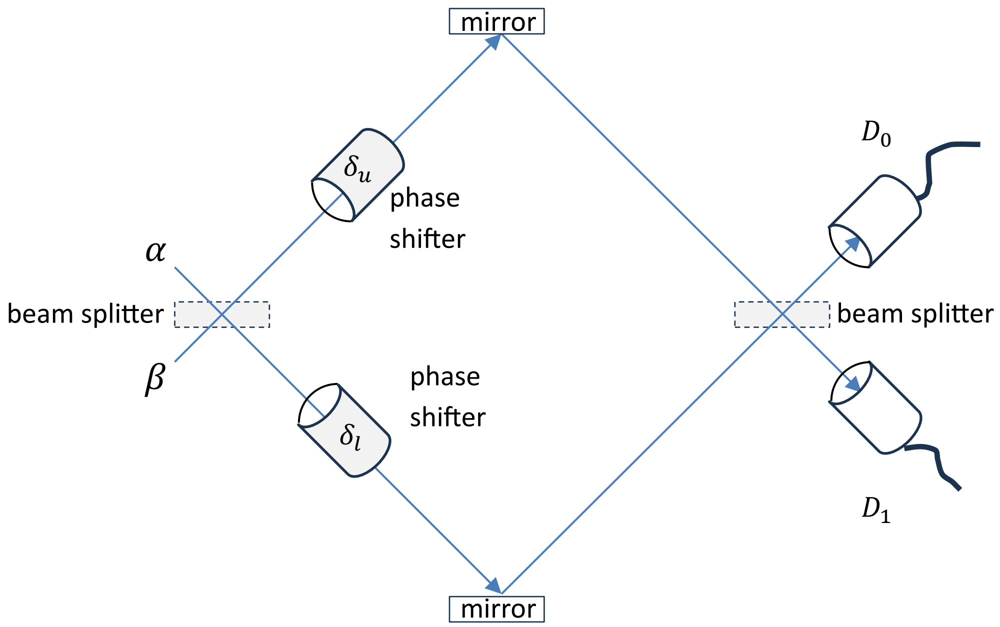
$$|\alpha|^2 sin^2\frac{-\delta_u}{2} - sin(-\delta_u) \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 cos^2\frac{-\delta_u}{2}$$

The probability of the lower path:

$$|\alpha|^2 cos^2\frac{-\delta_u}{2} + sin(-\delta_u) \cdot Im\left(\alpha\overline{\beta}\right) + |\beta|^2 sin^2\frac{-\delta_u}{2}$$

The probabilities sum up to one.

Mach-Zehnder Interferometer with two phase shifters in each path



We have an input beam:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Note that the photon is in superposition with α and β complex numbers, normalized to the probability $|\alpha|^2 + |\beta|^2 = 1$.

The pair of beam splitters can be expressed as a matrix:

$$\begin{pmatrix} e^{i\delta_u} & 0 \\ 0 & e^{i\delta_l} \end{pmatrix}$$

We use:

$$\begin{pmatrix} e^{i\delta_u} & 0 \\ 0 & e^{i\delta_l} \end{pmatrix} = e^{i\delta_u} \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\delta_l - \delta_u)} \end{pmatrix} = e^{i\delta_u} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta} \end{pmatrix}$$

Note $\Delta = \delta_l - \delta_u$

We take our result from above and replace δ_l by Δ:

The probability for the upper path:

$$|\alpha|^2 sin^2 \frac{-\Delta}{2} - sin(-\Delta) \cdot Im(\alpha\overline{\beta}) + |\beta|^2 cos^2 \frac{-\Delta}{2}$$

The probability of the lower path:

$$|\alpha|^2 cos^2 \frac{-\Delta}{2} + sin(-\Delta) \cdot Im(\alpha\overline{\beta}) + |\beta|^2 sin^2 \frac{-\Delta}{2}$$

The probabilities sum up to one.