A matrix (an operator) mathematically is a map from one vector space to another. This paper describes how the change of basis vectors alters a given matrix.

We will calculate this in two ways, a graphical representation might help.

Related information you may find at:

https://www.math.tamu.edu/~fnarc/psfiles/change_basis_m311.pdf

Hope I can help you with learning quantum mechanics.

Method 1

$f: V \to V$ is a linear map of a vector space into itself.

B and C are a complete set of basis vectors of V.

M_{BC} maps the basis B onto the basis C.

M_{CB} maps the basis C onto the basis B.

Then:

$$M_{BC} \cdot M_{CB} = id$$

If we have a matrix A_B representing the linear map f with respect to the basis B,
then we can calculate the matrix A_C representing the linear map f with respect to the basis C:

$$A_C = M_{CB} \cdot A_B \cdot M_{BC}$$

Example

We use the vector space $\mathbb{R}^2$.

The linear map f:

$$f(x_1, x_2) \coloneqq (x_1 + 2x_2, x_1 - 3x_2)$$

Basis B is the standard basis:

$$B \coloneqq \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

f is represented by the matrix A_B with respect to the standard basis B:

$$A_B \coloneqq \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$$

Basis C is a second basis (not orthogonal):

$$C \coloneqq \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$$

We calculate the matrix M_{BC} that maps the basis B onto the basis C. For this matrix must hold:

$$M_{BC} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$M_{BC} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We express the basis vectors C with the basis B:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We get the matrix M_{BC}:

$$M_{BC} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Check:

$$M_{BC} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$M_{BC} \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

To calculate the reverse matrix M_{CB} we calculate the inverse to M_{BC}:

$$M_{CB} = \begin{pmatrix}-2 & \frac{3}{2}\\1 & -\frac{1}{2}\end{pmatrix}$$

We check:

$$M_{CB} \cdot \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2 & \frac{3}{2}\\1 & -\frac{1}{2}\end{pmatrix} \cdot \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$M_{CB} \cdot \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}-2 & \frac{3}{2}\\1 & -\frac{1}{2}\end{pmatrix} \cdot \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

Note: for higher dimensions, the use of a computer algebra system recommended.

We check $M_{CB} \cdot M_{BC}$:

$$M_{CB} \cdot M_{BC} = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}\begin{pmatrix}-2 & \frac{3}{2}\\1 & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

The linear map A_B with respect to the basis B:

$$A_B \coloneqq \begin{pmatrix}1 & 2\\1 & -3\end{pmatrix}$$

The linear map A_C with respect to the basis C:

$$A_C = M_{CB} \cdot A_B \cdot M_{BC}$$

$$A_C = \begin{pmatrix}-2 & \frac{3}{2}\\1 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 2\\1 & -3\end{pmatrix}\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} =$$

$$\begin{pmatrix}-\frac{35}{2} & -\frac{71}{2}\\ \frac{15}{2} & \frac{31}{2}\end{pmatrix}$$

Method 2

f is represented by the matrix A_B, the matrix with respect to the standard basis:

$$A_B := \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$$

Basis B is the standard basis:

$$B := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Basis C is a second basis (not orthogonal):

$$C := \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$$

We calculate the effect of A_B on the basis vectors C:

$$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \end{pmatrix}$$

We express the result in the basis C:

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ -9 \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \end{pmatrix} + d \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We get (after some calculation):

$$a = -\frac{35}{2}, b = \frac{15}{2}$$

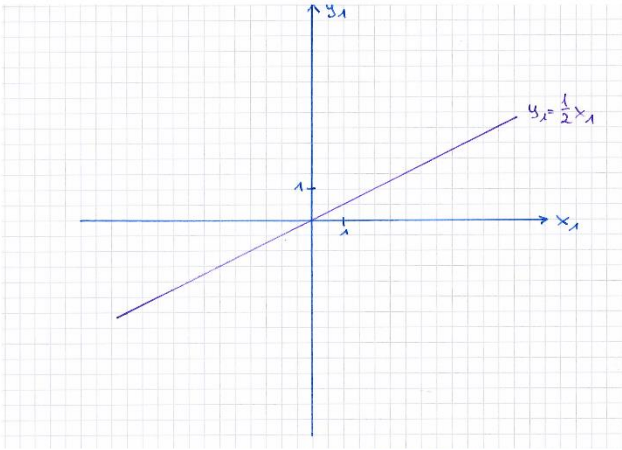
$$c = -\frac{71}{2}, b = \frac{31}{2}$$

We assemble the linear map A_C with respect to the basis C:

$$A_C = \begin{pmatrix} -\frac{35}{2} & -\frac{71}{2} \\ \frac{15}{2} & \frac{31}{2} \end{pmatrix}$$

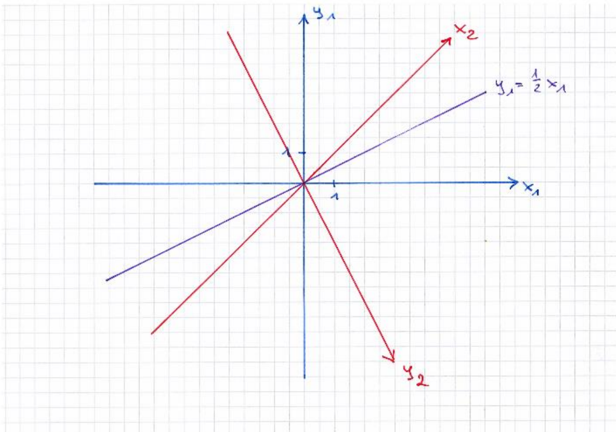
Graphical representation

The traditional cartesian coordinates with function $f(x_1) = \frac{1}{2}x_1$:



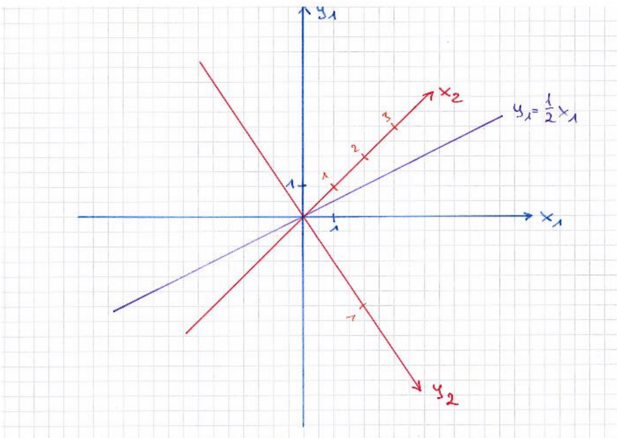
The effect of the linear map f, represented by the matrix A_B onto the (blue) axes x_1 and y_1 gives the new (red) axes x_2 and y_2:

$$A_B := \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$$

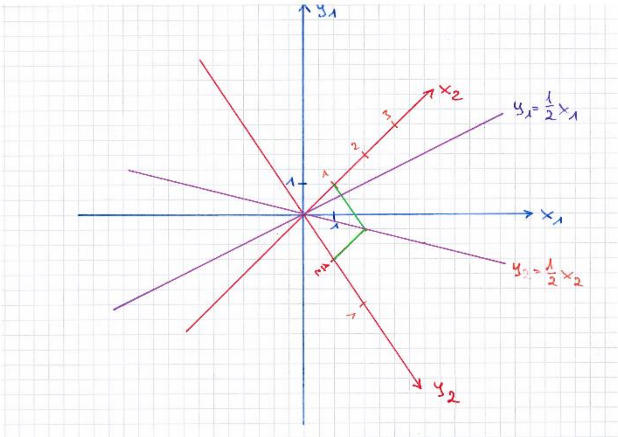


Note: the linear map changes the orientation, the determinant of A_B is negative.

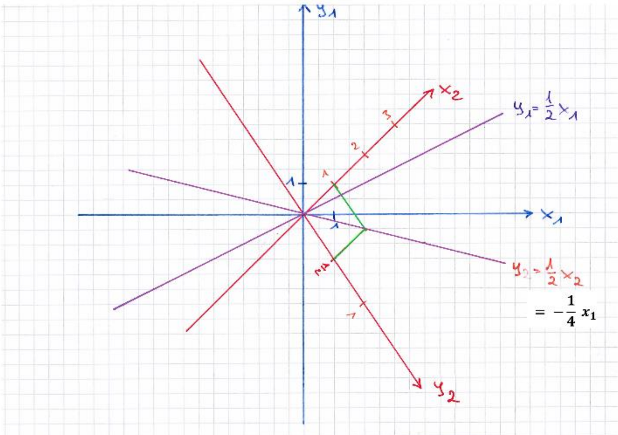
The scaling of the new coordinate system is altered:



The effect of the linear map on the function $f(x_1) = \frac{1}{2}x_1$: $f(x_2) = \frac{1}{2}x_2$, the expression remains the same.



The new function expressed in the "old" coordinate system:



Changing the basis could help to make dependencies easier to handle.