This paper checks that the number operator, applied to ground/first/second excited state of the quantum harmonic oscillator gives back these states with the appropriate factor in front.

Hope I can help you with learning quantum mechanics.

Contents

Ground state

The ground state:

$$\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$$

Note: the normalized ground state for a quantum harmonic oscillator complete is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

but we work with the core function only to minimize the calculation effort.

Note: the number operator N is the product of the creation operator $\hat{a}^+$ and the annihilation operator $\hat{a}^-$:

$\hat{a}^+ = (P + i\omega X)$	$\hat{a}^- = (P - i\omega X)$

$$N = \hat{a}^+\hat{a}^-$$

The number operator N:

$$(P + i\omega X)(P - i\omega X) =$$

$$PP - i\omega PX + i\omega XP + \omega^2 XX =$$

$$PP + i\omega(XP - PX) + \omega^2 XX =$$

$$PP + i\omega[X, P] + \omega^2 XX$$

Note: the commutator $[X, P] = i\hbar$.

$$PP - \hbar\omega + \omega^2 XX$$

We need the details:

$P = -i\hbar\frac{\partial}{\partial x}$	$X = x\,\cdot$
$PP = (-i\hbar)(-i\hbar)\frac{\partial^2}{\partial x^2} = -\hbar^2\frac{\partial^2}{\partial x^2}$	$XX = x^2$

We apply the number operator to the wave function of the ground state:

$$\left(-\hbar^2\frac{\partial^2}{\partial x^2} - \hbar\omega + \omega^2 x^2\right) e^{-\frac{\omega}{2\hbar}x^2}$$

We do this in parts:

First the derivation $-\hbar^2\frac{\partial^2}{\partial x^2} e^{-\frac{\omega}{2\hbar}x^2}$:

$$-\hbar^2\frac{\partial^2}{\partial x^2} e^{-\frac{\omega}{2\hbar}x^2} = -\hbar^2\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$\hbar^2\frac{\partial}{\partial x}\left(\frac{\omega x}{\hbar} e^{-\frac{\omega}{2\hbar}x^2}\right) = \hbar^2\left(\frac{\omega}{\hbar} e^{-\frac{\omega}{2\hbar}x^2} - \frac{\omega x}{\hbar}\frac{\omega x}{\hbar} e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$\hbar^2\left(\frac{\omega}{\hbar} e^{-\frac{\omega}{2\hbar}x^2} - \frac{\omega^2 x^2}{\hbar^2} e^{-\frac{\omega}{2\hbar}x^2}\right) = (\hbar\omega - \omega^2 x^2) e^{-\frac{\omega}{2\hbar}x^2}$$

$e^{-\frac{\omega}{2\hbar}x^2}$ is the wave function so we write in operator style:

$$-\hbar^2\frac{\partial^2}{\partial x^2} = (\hbar\omega - \omega^2 x^2)$$

We complete the number operator:

$$(\hbar\omega - \omega^2 x^2 - \hbar\omega + \omega^2 x^2)e^{-\frac{\omega}{2\hbar}x^2} = 0$$

The number operator N applied to the ground state gives (correctly) 0.

First excited state

The ground state:

$$\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$$

The first excited state:

$$\psi_1(x) = 2i\omega x e^{-\frac{\omega}{2\hbar}x^2} = 2i\omega x\psi_0(x)$$

The number operator N remains the same:

$$PP - \hbar\omega + \omega^2 XX$$

We apply the number operator to the wave function of the first excited state:

$$\left(-\hbar^2\frac{\partial^2}{\partial x^2} - \hbar\omega + \omega^2 x^2\right)2i\omega x e^{-\frac{\omega}{2\hbar}x^2} =$$

$$-2i\omega\hbar^2\frac{\partial^2}{\partial x^2}\left(xe^{-\frac{\omega}{2\hbar}x^2}\right) - 2i\hbar\omega^2 x e^{-\frac{\omega}{2\hbar}x^2} + 2i\omega^3 x^3 e^{-\frac{\omega}{2\hbar}x^2} =$$

We do this in parts:

First the derivation $-2i\omega\hbar^2\frac{\partial^2}{\partial x^2}\left(xe^{-\frac{\omega}{2\hbar}x^2}\right)$:

$$-2i\omega\hbar^2\frac{\partial^2}{\partial x^2}\left(xe^{-\frac{\omega}{2\hbar}x^2}\right) = -2i\omega\hbar^2\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(xe^{-\frac{\omega}{2\hbar}x^2}\right) =;$$

Deriving first time:

$$\frac{\partial}{\partial x}\left(xe^{-\frac{\omega}{2\hbar}x^2}\right) = \left(e^{-\frac{\omega}{2\hbar}x^2} - \frac{\omega x^2}{\hbar}e^{-\frac{\omega}{2\hbar}x^2}\right) = \left(1 - \frac{\omega x^2}{\hbar}\right)e^{-\frac{\omega}{2\hbar}x^2}$$

Deriving second time:

$$\frac{\partial}{\partial x}\left(\left(1 - \frac{\omega x^2}{\hbar}\right)e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$\left(-\frac{2\omega x}{\hbar}e^{-\frac{\omega}{2\hbar}x^2} + \left(1 - \frac{\omega x^2}{\hbar}\right)\left(-\frac{\omega x}{\hbar}\right)e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$\left(-\frac{2\omega x}{\hbar} - \frac{\omega x}{\hbar} + \frac{\omega^2 x^3}{\hbar^2}\right)e^{-\frac{\omega}{2\hbar}x^2}$$

We multiply by the factor $-2i\omega\hbar^2$:

$$-2i\omega\hbar^2\left(-\frac{2\omega x}{\hbar}-\frac{\omega x}{\hbar}+\frac{\omega^2x^3}{\hbar^2}\right)e^{-\frac{\omega}{2\hbar}x^2} =$$

$$(4i\omega^2x\hbar+2i\omega^2x\hbar-2i\omega^3x^3)e^{-\frac{\omega}{2\hbar}x^2} =$$

$$(6i\omega^2x\hbar-2i\omega^3x^3)e^{-\frac{\omega}{2\hbar}x^2}$$

We complete the number operator:

$$(6i\omega^2x\hbar-2i\omega^3x^3-2i\omega^2\hbar x+2i\omega^3x^3)e^{-\frac{\omega}{2\hbar}x^2} =$$

$$4i\omega^2x\hbar e^{-\frac{\omega}{2\hbar}x^2} =;$$

We simplified little bit and must correct that. The raising operator and the lowering operator need a normalizing factor, the correct values are:

$$a^- = \frac{i}{\sqrt{2\omega\hbar}}(P-i\omega X)$$

$$a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P+i\omega X)$$

This gives the correct number operator:

$$N = \frac{1}{2\omega\hbar}(P-i\omega X)(P+i\omega X)$$

We take the result above and divide it by $2\omega\hbar$:

$$\frac{4i\omega^2x\hbar}{2\omega\hbar}e^{-\frac{\omega}{2\hbar}x^2} = 2i\omega xe^{-\frac{\omega}{2\hbar}x^2} = 1\cdot\psi_1(x)$$

The number operator N applied to the first excited state $2i\omega xe^{-\frac{\omega}{2\hbar}x^2}$ gives correct one time the first excited state.

Second excited state

We check the number operator N on the second excited state.

The ground state:

$$\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$$

The second excited state:

$$\psi_2(x) = (-\hbar+2\omega x^2)\psi_0(x) = (-\hbar+2\omega x^2)e^{-\frac{\omega}{2\hbar}x^2}$$

The number operator N:

$$(P+i\omega X)(P-i\omega X) =$$

$$PP-i\omega PX+i\omega XP+\omega^2XX =$$

$$PP+i\omega(XP-PX)+\omega^2XX =$$

$$PP + i\omega[X,P] + \omega^2 XX =$$

Note: the commutator $[X,P] = i\hbar$.

$$PP - \hbar\omega + \omega^2 XX$$

We need the details:

$P = -i\hbar \frac{\partial}{\partial x}$	$X = x\cdot$
$PP = (-i\hbar)(-i\hbar)\frac{\partial^2}{\partial x^2} = -\hbar^2 \frac{\partial^2}{\partial x^2}$	$XX = x^2$

We apply the number operator to the wave function of the second excited state:

$$\left(-\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar\omega + \omega^2 x^2\right)\psi_2(x) = \left(-\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar\omega + \omega^2 x^2\right)(-\hbar + 2\omega x^2)e^{-\frac{\omega}{2\hbar}x^2}$$

We do this in parts:

First $-\hbar^2 \frac{\partial^2}{\partial x^2}\psi_2(x)$.

Inner derivative:

$$-\hbar^2 \frac{\partial^2}{\partial x^2}\psi_2(x) = -\hbar^2 \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left((-\hbar + 2\omega x^2)e^{-\frac{\omega}{2\hbar}x^2}\right)\right) =$$

$$-\hbar^2 \frac{\partial}{\partial x}\left(4\omega x e^{-\frac{\omega}{2\hbar}x^2} - \frac{\omega x}{\hbar}(-\hbar + 2\omega x^2)e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$-\hbar^2 \frac{\partial}{\partial x}\left(4\omega x e^{-\frac{\omega}{2\hbar}x^2} + \omega x e^{-\frac{\omega}{2\hbar}x^2} - \frac{2\omega^2 x^3}{\hbar} e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$-\hbar^2 \frac{\partial}{\partial x}\left(\left(4\omega x + \omega x - \frac{2\omega^2 x^3}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$-\hbar^2 \frac{\partial}{\partial x}\left(\left(5\omega x - \frac{2\omega^2 x^3}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2}\right)$$

Outer derivative:

$$-\hbar^2 \frac{\partial}{\partial x}\left(\left(5\omega x - \frac{2\omega^2 x^3}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$-\hbar^2 \left(\left(5\omega - \frac{6\omega^2 x^2}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2} - \frac{\omega x}{\hbar}\left(5\omega x - \frac{2\omega^2 x^3}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2}\right) =$$

$$-\hbar^2 \left(\left(5\omega - \frac{6\omega^2 x^2}{\hbar}\right) - \frac{\omega x}{\hbar}\left(5\omega x - \frac{2\omega^2 x^3}{\hbar}\right)\right) e^{-\frac{\omega}{2\hbar}x^2} =$$

$$-\hbar^2 \left(5\omega - \frac{6\omega^2 x^2}{\hbar} - \frac{\omega x}{\hbar}5\omega x + \frac{\omega x}{\hbar}\frac{2\omega^2 x^3}{\hbar}\right) e^{-\frac{\omega}{2\hbar}x^2} =$$

$$-\hbar^2\left(5\omega-\frac{6\omega^2x^2}{\hbar}-\frac{5\omega^2x^2}{\hbar}+\frac{2\omega^3x^4}{\hbar^2}\right)e^{-\frac{\omega}{2\hbar}x^2}=$$

$$-\hbar^2\left(5\omega-\frac{11\omega^2x^2}{\hbar}+\frac{2\omega^3x^4}{\hbar^2}\right)e^{-\frac{\omega}{2\hbar}x^2}=$$

$$-5\omega\hbar^2+11\omega^2\hbar x^2-2\omega^3x^4\psi_0(x)$$

Second $-\hbar\omega\psi_2(x)$:

$$-\hbar\omega\psi_2(x)=-\hbar\omega(-\hbar+2\omega x^2)\psi_0(x)=$$

$$(\omega\hbar^2-2\omega^2\hbar x^2)\psi_0(x)$$

Third $\omega^2x^2\psi_2(x)$:

$$\omega^2x^2\psi_2(x)=$$

$$\omega^2x^2(-\hbar+2\omega x^2)\psi_0(x)=$$

$$(-\omega^2\hbar x^2+2\omega^3x^4)\psi_0(x)$$

We add:

$$\psi_0(x)(-5\omega\hbar^2+11\omega^2\hbar x^2-2\omega^3x^4+\omega\hbar^2-2\omega^2\hbar x^2-\omega^2\hbar x^2+2\omega^3x^4)\psi_0(x)=$$

$$\psi_0(x)(-4\omega\hbar^2+8\omega^2\hbar x^2)=$$

$$2\cdot 2\cdot(-\omega\hbar^2+2\omega^2\hbar x^2)\psi_0(x)=$$

$$2\hbar\omega\cdot 2\cdot(-\hbar+2\omega x^2)\psi_0(x)$$

We simplified. The raising operator and the lowering operator need a normalizing factor, the correct values are:

$$a^-=\frac{i}{\sqrt{2\omega\hbar}}(P-i\omega X)$$

$$a^+=\frac{-i}{\sqrt{2\omega\hbar}}(P+i\omega X)$$

This gives the correct number operator:

$$N=\frac{1}{2\omega\hbar}(P-i\omega X)(P+i\omega X)$$

We take the result above and divide it by $2\omega\hbar$:

$$2\cdot(-\hbar+2\omega x^2)\psi_0(x)=2\cdot\psi_2(x)$$

The number operator applied to the wave function of the second excited state gives back two times the second excited state.