In this paper we take the quantum mechanical Hamiltonian and factorize it into two new operators, raising and lowering operator. From this we form the number operator.

We check the effect of these operators to the ground state wave function and calculate the wave function for the first and second excited state. In the last step we apply the number operator to the second excited state and verify that it gives back the number of this state.

This paper is based on:

https://ocw.mit.edu/courses/physics/8-04-quantum-physics-i-spring-2016/lecture-notes/MIT8_04S16_LecNotes14_15.pdf

In Griffiths the operators are named "ladder operator", you find information about in "2.3 Harmonic oscillator".

Hope I can help you with learning quantum mechanics

The classic harmonic oscillator:

$$E = \frac{1}{2} m\omega^2 x^2 + \frac{p^2}{2m}$$

In quantum mechanics we use the position operator $\hat{x}$ and the momentum operator $\hat{p}$.

Analog to the classical system we define the Hamiltonian operator $\hat{H}$:

$$\hat{H} \coloneqq \frac{1}{2} m\omega^2 \hat{x}^2 + \frac{\hat{p}^2}{2m}$$

We work in the Hilbert space $\mathcal{H}$, the space of the square-integrable complex valued functions of x.

We want to factorize the Hamiltonian such that we can write it as the product of two operators $\hat{V}^\dagger$ and $\hat{V}$:

$$\hat{H} \sim V^\dagger V$$

We rewrite the Hamiltonian $\hat{H}$:

$$\hat{H} \coloneqq \frac{1}{2} m\omega^2 \left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right)$$

For complex numbers a, b we have:

$$a^2 + b^2 = (a - ib)(a + ib)$$

This might fit for our sum of operators too, but we must not forget that operators do not necessarily commute. We try and transform the sum:

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \rightarrow \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right)$$

Note: we chose the i to go with $\hat{p}$ because $\hat{x}$ should give a real (position) value.

We calculate the product:

$$\left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) =$$

$$\hat{x}\hat{x} + \hat{x}\frac{i\hat{p}}{m\omega} - \frac{i\hat{p}}{m\omega}\hat{x} - \frac{i\hat{p}}{m\omega}\frac{i\hat{p}}{m\omega} =$$

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) =$$

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[\hat{x}, \hat{p}] =;$$

Note the commutation relation $[\hat{x}, \hat{p}] = i\hbar\hat{1}$

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\hat{1}$$

$\left(\hat{x} + \frac{i\hat{p}}{m\omega} \right)$ and $\left(\hat{x} - \frac{i\hat{p}}{m\omega} \right)$ seems to be good candidates for $\hat{V}^\dagger$ and $\hat{V}$. Because $\hat{p}$ and $\hat{x}$ do not commute we got the extra factor $-\frac{\hbar}{m\omega}\hat{1}$

We define:

$$\hat{V} \coloneqq \hat{x} + \frac{i\hat{p}}{m\omega}$$

$$\hat{V}^{\dagger} \coloneqq \hat{x} - \frac{i\hat{p}}{m\omega}$$

The product $\hat{V}^{\dagger}V$:

$$\hat{V}^{\dagger}V = \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\hat{1}$$

Note: $\hat{1}$ is the identity operator.

We insert the definitions in the Hamiltonian and write:

$$\hat{H} \coloneqq \frac{1}{2}m\omega^2\left(\hat{V}^{\dagger}\hat{V} + \frac{\hbar}{m\omega}\hat{1}\right) =$$

$$\frac{1}{2}m\omega^2\hat{V}^{\dagger}\hat{V} + \frac{\hbar\omega}{2}\hat{1}$$

Note: We must add $\frac{\hbar}{m\omega}\hat{1}$ to keep the result correct.

The operators $V^{\dagger}$ and V need to be scaled so that the commutator gives a simple, unit-free constant.

We calculate:

$$\left[\hat{V}, \hat{V}^{\dagger}\right] = \left[\hat{x} + \frac{i\hat{p}}{m\omega}, \hat{x} - \frac{i\hat{p}}{m\omega}\right] =$$

$$[\hat{x}, \hat{x}] + \frac{i}{m\omega}[\hat{p}, \hat{x}] - \frac{i}{m\omega}[\hat{x}, \hat{p}] + \frac{1}{m^2\omega^2}[\hat{p}, \hat{p}] =$$

$$\frac{i}{m\omega}[\hat{p}, \hat{x}] + \frac{i}{m\omega}[\hat{p}, \hat{x}] = \frac{2i}{m\omega}\left(-i\hbar\hat{1}\right) =$$

$$\frac{2\hbar}{m\omega}\hat{1}$$

With this we scale $V^{\dagger}$ and V and give them new names:

$$\hat{a}^{\dagger} \coloneqq \sqrt{\frac{m\omega}{2\hbar}}\hat{V}^{\dagger}$$

$$\hat{a} \coloneqq \sqrt{\frac{m\omega}{2\hbar}}\hat{V}$$

$\hat{a}^{\dagger}$ is often named as $\hat{a}^{+}$, the raising operator.

$\hat{a}$ is often named as $\hat{a}^{-}$, the lowering operator.

The scaling of $V^{\dagger}$ and V leads to the commutator relation $\left[\hat{a}, \hat{a}^{\dagger}\right]$:

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$$

We can go back and express $\hat{a}$ and $\hat{a}^\dagger$ in terms of the operators $\hat{x}$ and $\hat{p}$:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

Often used are the inverse relations:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \cdot (\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} \cdot (\hat{a}^\dagger - \hat{a})$$

Note: $\hat{a}$ and $\hat{a}^\dagger$ are not Hermitian, they are Hermitian conjugates of each other.

We define the number operator $\hat{N}$:

$$\hat{N} := \hat{a}^\dagger \hat{a}$$

Note: the number operator $\hat{N}$ is Hermitian.

Number operator, raising and lowering operators form a commutator algebra, a set of operators that closes under commutation:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$[\hat{a}, \hat{N}] = \hat{a}$$

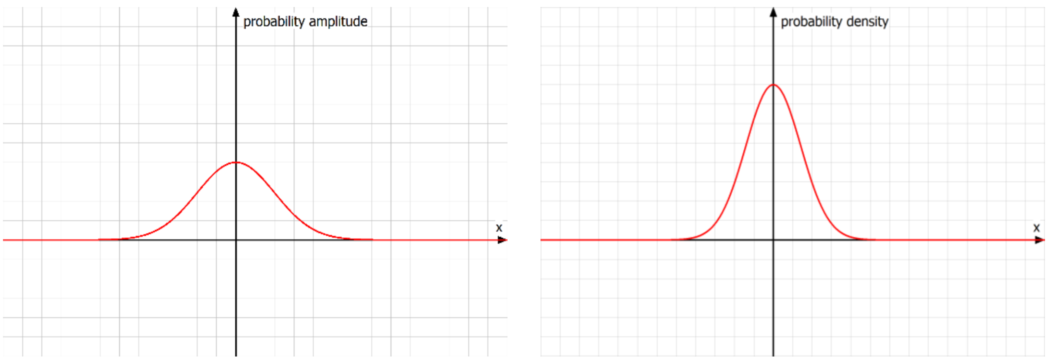
$$[\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger$$

With lowering and raising operator we rewrite the Hamiltonian:

$$\hat{H} = \omega\hbar\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$\hat{H} = \omega\hbar\left(\hat{N} + \frac{1}{2}\right)$$

We apply the raising operator to the ground-state wave function:



$$\hat{a}^\dagger\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) =;$$

We remember:

$$\hat{x}\varphi(x) = x\varphi(x)$$

$$\hat{p}\varphi(x) = -i\hbar\frac{d}{dx}\varphi(x)$$

We calculate the parts:

$$\hat{x}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) = xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$\hat{p}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) = -i\hbar\frac{d}{dx}e^{-\frac{m\omega}{2\hbar}x^2} =$$

$$i\hbar \cdot \frac{m\omega x}{\hbar} \cdot e^{-\frac{m\omega}{2\hbar}x^2} =$$

$$im\omega x \cdot e^{-\frac{m\omega}{2\hbar}x^2}$$

We do the complete calculation:

$$\hat{a}^\dagger\left(e^{-\frac{\omega}{2\hbar}x^2}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) =$$

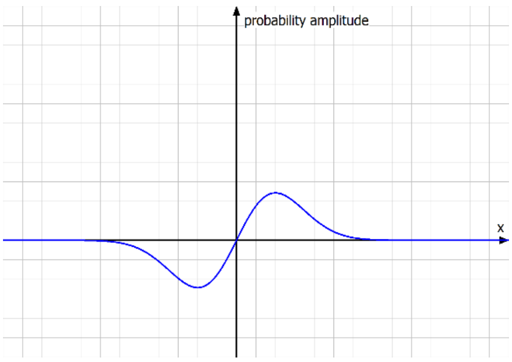
$$\sqrt{\frac{m\omega}{2\hbar}}\left(x \cdot e^{-\frac{m\omega}{2\hbar}x^2} - \frac{i}{m\omega} \cdot im\omega x \cdot e^{-\frac{m\omega}{2\hbar}x^2}\right) =$$

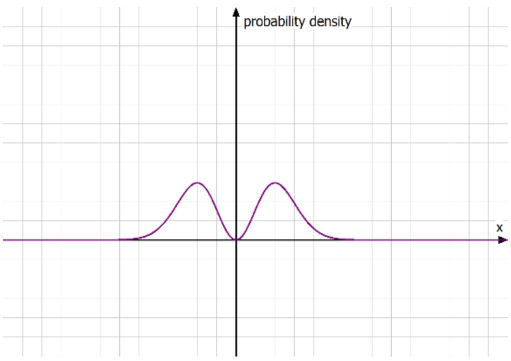
$$\sqrt{\frac{m\omega}{2\hbar}}\left(x \cdot e^{-\frac{m\omega}{2\hbar}x^2} + x \cdot e^{-\frac{m\omega}{2\hbar}x^2}\right) =$$

$$\sqrt{\frac{2m\omega}{\hbar}}\left(x \cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

The raising operator $\hat{a}^\dagger$ applied to the ground state function gives the first excited state:

$$\hat{a}^\dagger\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) = \left(\sqrt{\frac{2m\omega}{\hbar}} \cdot x\right) \cdot e^{-\frac{m\omega}{2\hbar}x^2}$$





We apply the raising operator to the first excited state again:

$$\hat{a}^\dagger\left(\sqrt{\frac{2m\omega}{\hbar}} \cdot x\right) \cdot e^{-\frac{m\omega}{2\hbar}x^2} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(\sqrt{\frac{2m\omega}{\hbar}} \cdot x\right) \cdot e^{-\frac{m\omega}{2\hbar}x^2} =$$

$$\frac{m\omega}{\hbar}\left(\hat{x}-\frac{i\hat{p}}{m\omega}\right)\left(x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\frac{m\omega}{\hbar}\left(\left(x^2\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)-\frac{i\hat{p}}{m\omega}\left(x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)\right)=$$

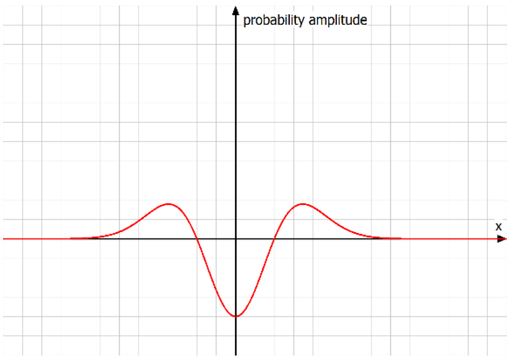
$$\frac{m\omega}{\hbar}\left(\left(x^2\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)-\frac{\hbar}{m\omega}\frac{d}{dx}\left(x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)\right)=$$

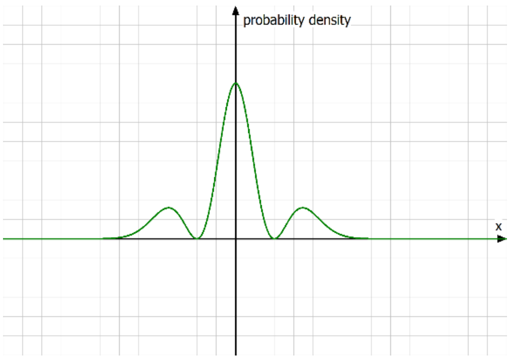
$$\frac{m\omega}{\hbar}\left(\left(x^2\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)-\frac{\hbar}{m\omega}\left(1-x^2\frac{m\omega}{\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\frac{m\omega}{\hbar}\left(2x^2e^{-\frac{m\omega}{2\hbar}x^2}-\frac{\hbar}{m\omega}e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\frac{2x^2m\omega}{\hbar}e^{-\frac{m\omega}{2\hbar}x^2}-e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega}{2\hbar}x^2}$$





Note: the graphics show the probability amplitudes that can be negative. The corresponding probability densities are always positive.

We try the other way round and apply the lowering operator $\hat{a}$ to the first excited state:

$$\hat{a}\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=;$$

We calculate $\hat{p}$:

$$\hat{p}\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=-i\hbar\frac{d}{dx}\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$-i\hbar\sqrt{\frac{2m\omega}{\hbar}}\left(e^{-\frac{m\omega}{2\hbar}x^2}-x\cdot\frac{m\omega x}{\hbar}\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$-i\hbar\sqrt{\frac{2m\omega}{\hbar}}\left(1-\frac{m\omega x^2}{\hbar}\right)\cdot e^{-\frac{m\omega}{2\hbar}x^2}$$

We do the complete calculation:

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x^2\cdot e^{-\frac{m\omega}{2\hbar}x^2}-\frac{i}{m\omega}\cdot i\hbar\sqrt{\frac{2m\omega}{\hbar}}\left(1-\frac{m\omega x^2}{\hbar}\right)\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\left(x^2\cdot e^{-\frac{m\omega}{2\hbar}x^2}+\frac{\hbar}{m\omega}\left(1-\frac{m\omega x^2}{\hbar}\right)\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\frac{m\omega}{\hbar}\left(x^2+\frac{\hbar}{m\omega}\left(1-\frac{m\omega x^2}{\hbar}\right)\right)\cdot e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$\frac{m\omega}{\hbar}\left(x^2+\frac{\hbar}{m\omega}-x^2\right)\cdot e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$e^{-\frac{m\omega}{2\hbar}x^2}$$

Result: the lowering operator does what it should. Applied to first excited state function it brings back the ground state wave function.

We check what happens if we apply the lowering operator to the ground state wave function:

$$\hat{a}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x\cdot e^{-\frac{m\omega}{2\hbar}x^2}+\frac{i}{m\omega}\cdot im\omega x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x\cdot e^{-\frac{m\omega}{2\hbar}x^2}-x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)=0$$

The lowering operator applied to the ground state annihilates the ground state. We have no wave function anymore.

To see the number operator working we apply the number operator to the second excited state wave function:

$$\hat{N}\coloneqq\hat{a}^{\dagger}\hat{a}$$

$$\hat{a}^{\dagger}\hat{a}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\hat{a}^{\dagger}\left(\hat{a}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)\right)=;$$

We calculate the lowering operator:

$$\hat{a}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=;$$

We calculate the effect of the position operator $\hat{x}$:

$$\hat{x}\left(\left(2x^2\frac{m\omega}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

We calculate the effect of the momentum operator $\hat{p}$:

$$\hat{p}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=-i\hbar\frac{d}{dx}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$-i\hbar\frac{d}{dx}\left(2\cdot\frac{m\omega x^2}{\hbar}e^{-\frac{m\omega}{2\hbar}x^2}-e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$-i\hbar\left(\frac{4m\omega x}{\hbar}e^{-\frac{m\omega}{2\hbar}x^2}-2\cdot\frac{m\omega x^2}{\hbar}\frac{m\omega x}{\hbar}e^{-\frac{m\omega}{2\hbar}x^2}+\frac{m\omega x}{\hbar}e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$-i\hbar\left(\frac{4m\omega x}{\hbar}-2\cdot\frac{m\omega x^2}{\hbar}\frac{m\omega x}{\hbar}+\frac{m\omega x}{\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$-i\hbar\left(\frac{4m\omega x}{\hbar}-\frac{2m^2\omega^2x^3}{\hbar^2}+\frac{m\omega x}{\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$-i\hbar\left(\frac{5m\omega x}{\hbar}-\frac{2m^2\omega^2x^3}{\hbar^2}\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

We combine:

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}+\frac{\hbar}{m\omega}\left(\frac{5m\omega x}{\hbar}-\frac{2m^2\omega^2x^3}{\hbar^2}\right)e^{-\frac{m\omega}{2\hbar}x^2}\right)=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)\cdot x+\frac{\hbar}{m\omega}\left(\frac{5m\omega x}{\hbar}-\frac{2m^2\omega^2x^3}{\hbar^2}\right)\right)e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{2m\omega x^3}{\hbar}-x+5x-\frac{2m\omega x^3}{\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}=$$

$$4x\sqrt{\frac{m\omega}{2\hbar}}e^{-\frac{m\omega}{2\hbar}x^2}$$

We rewrite this to show that this is twice the first excited state:

$$4x\sqrt{\frac{m\omega}{2\hbar}}e^{-\frac{m\omega}{2\hbar}x^2} = 2\left(\sqrt{\frac{2m\omega}{\hbar}}\cdot x\cdot e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

If we now apply the raising operator to this twice first excited state, we get (as shown above) twice the second excited state.

Result:

$$\widehat{N}\left(\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}\right) = 2\cdot\left(2\cdot\frac{m\omega x^2}{\hbar}-1\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

The number operator applied to the second excited state gives back the number of this state.

One remarkable result: the lowering operator gives back both the lower state and the number of the state above. In a certain sense it contains the information of the number operator.