Quantum mechanics works with cartesian and polar coordinate systems. It is useful to switch between them and especially, to know how to transform partial derivatives, e.g. work with the momentum operator.

This paper is based on http://www.math.uni-kiel.de/geometrie/klein/ingws9/mo1611.pdf

Hope I can help you with learning quantum mechanics.

A function in cartesian coordinates:	$f(x,y): R^2 \to R$
Written in polar coordinates:	$\tilde{f}(r,\varphi): \mathbb{C} \to R$
We transform polar coordinates in cartesian coordinates:	$x = r \cdot \cos\varphi$ $y = r \cdot sin\varphi$ $\tilde{f}(r,\varphi) = f(r \cdot \cos\varphi, r \cdot sin\varphi)$
The partial derivative with respect to r:	$\frac{\partial \tilde{f}}{\partial r} = \frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial y}{\partial r} =$ $\frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial}{\partial r}(r \cdot \cos\varphi) + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial}{\partial r}(r \cdot sin\varphi) =$ $\frac{\partial}{\partial x} f(x,y) \cdot \cos\varphi + \frac{\partial}{\partial y} f(x,y) \cdot sin\varphi$
The partial derivative with respect to φ:	$\frac{\partial \tilde{f}}{\partial \varphi} = \frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial y}{\partial \varphi} =$ $\frac{\partial}{\partial x} f(x,y) \cdot \frac{\partial}{\partial \varphi}(r \cdot \cos\varphi) + \frac{\partial}{\partial y} f(x,y) \cdot \frac{\partial}{\partial \varphi}(r \cdot sin\varphi) =$ $\frac{\partial}{\partial x} f(x,y) \cdot (-r \cdot sin\varphi) + \frac{\partial}{\partial y} f(x,y) \cdot (r \cdot cos\varphi)$
We use:	$cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$ $sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$
We get:	$\frac{\partial \tilde{f}}{\partial r} = \frac{\partial}{\partial x} f(x,y) \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial y} f(x,y) \cdot \frac{y}{\sqrt{x^2+y^2}}$ $\frac{\partial \tilde{f}}{\partial \varphi} = \frac{\partial}{\partial x} f(x,y) \cdot (-y) + \frac{\partial}{\partial y} f(x,y) \cdot x$
We omit the target function and write:	$\frac{\partial}{\partial r} = \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial y}$ $\frac{\partial}{\partial \varphi} = x \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial x}$

We will check this with an example:

$$f(x,y) \coloneqq x \cdot y$$

$$\tilde{f}(r,\varphi) = \frac{r^2}{2} \cdot sin(2\varphi)$$

We do a quick check whether this gives the same values.

Cartesian coordinates:

$y = 0$ $f(x,0) = x \cdot 0 = 0$	$x = y$ $f(x,x) = x \cdot x = x^2$

The same in polar coordinates:

$y = 0 \rightarrow \varphi = 0$ $\tilde{f}(r,0) = \frac{r^2}{2} \cdot sin(0) = 0$	$x = y \rightarrow \varphi = \frac{\pi}{4}$ $\tilde{f}\left(r,\frac{\pi}{4}\right) = \frac{r^2}{2} \cdot sin\left(\frac{\pi}{2}\right) = \frac{r^2}{2} =$ $\frac{x^2 + y^2}{2} = \frac{x^2 + x^2}{2} = x^2$

We derivate $\tilde{f}(r,\varphi)$:

$$\frac{\partial}{\partial r}\tilde{f}(r,\varphi) = r \cdot sin(2\varphi)$$

$$\frac{\partial}{\partial \varphi}\tilde{f}(r,\varphi) = r^2 \cdot cos(2\varphi)$$

We transform this to cartesian coordinates:

$$\frac{\partial}{\partial r}\tilde{f}(r,\varphi) = r \cdot sin(2\varphi) \rightarrow \sqrt{x^2 + y^2} \cdot 2 \cdot \frac{y}{\sqrt{x^2 + y^2}} \cdot \frac{x}{\sqrt{x^2 + y^2}} =$$

$$\frac{2xy}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial}{\partial \varphi}\tilde{f}(r,\varphi) = r^2 \cdot cos(2\varphi) \rightarrow (x^2 + y^2) \cdot \left(\frac{x^2 - y^2}{x^2 + y^2}\right) =$$

$$x^2 - y^2$$

We compare with the transformation of the partial derivation. We had:

$$\frac{\partial}{\partial r} = \frac{x}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \varphi} = x \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial x}$$

We check $\frac{\partial}{\partial r}$:

$$\frac{\partial}{\partial r} = \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{\partial f(x,y)}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{\partial f(x,y)}{\partial y} =$$

$$\frac{x}{\sqrt{x^2+y^2}} \cdot y + \frac{y}{\sqrt{x^2+y^2}} \cdot x = \frac{2xy}{\sqrt{x^2+y^2}}$$

We check $\frac{\partial}{\partial \varphi}$:

$$\frac{\partial}{\partial \varphi} = x \cdot \frac{\partial f(x,y)}{\partial y} - y \cdot \frac{\partial f(x,y)}{\partial x} = x^2 - y^2$$

Our example works.

To calculate the other way round we formalize the procedure and use symbolic vectors.

We rewrite $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \varphi}$:

$$\frac{\partial}{\partial r} = cos\varphi \cdot \frac{\partial f(x,y)}{\partial x} + sin\varphi \cdot \frac{\partial f(x,y)}{\partial y}$$

$$\frac{\partial}{\partial \varphi} = -r \cdot sin\varphi \cdot \frac{\partial f(x,y)}{\partial x} + r \cdot cos\varphi \cdot \frac{\partial f(x,y)}{\partial y}$$

We construct the transformation matrix T:

$$T = \begin{pmatrix} cos\varphi & sin\varphi \\ -r \cdot sin\varphi & r \cdot cos\varphi \end{pmatrix}$$

We need the inverse matrix T^{-1}:

$$T^{-1} = \begin{pmatrix} cos\varphi & \frac{-sin\varphi}{r} \\ sin\varphi & \frac{cos\varphi}{r} \end{pmatrix}$$

We check whether this is the correct inverse:

$$T \cdot T^{-1} = \begin{pmatrix} cos\varphi & sin\varphi \\ -r \cdot sin\varphi & r \cdot cos\varphi \end{pmatrix} \begin{pmatrix} cos\varphi & \frac{-sin\varphi}{r} \\ sin\varphi & \frac{cos\varphi}{r} \end{pmatrix} =$$

$$\begin{pmatrix} cos^2\varphi + sin^2\varphi & \frac{-sin\varphi}{r} + \frac{sin\varphi}{r} \\ -r \cdot sin\varphi cos\varphi + r \cdot sin\varphi cos\varphi & cos^2\varphi + sin^2\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We write partial derivatives as vectors:

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\widetilde{\nabla} := \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

We apply the transformation matrix:

$$T = \begin{pmatrix} cos\varphi & sin\varphi \\ -r \cdot sin\varphi & r \cdot cos\varphi \end{pmatrix}$$

We calculate:

$$\widetilde{\nabla} = T \cdot \nabla = \begin{pmatrix} cos\varphi & sin\varphi \\ -r \cdot sin\varphi & r \cdot cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} =$$

$$\begin{pmatrix} cos\varphi \cdot \frac{\partial}{\partial x} + sin\varphi \cdot \frac{\partial}{\partial y} \\ -r \cdot sin\varphi \cdot \frac{\partial}{\partial x} + r \cdot cos\varphi \cdot \frac{\partial}{\partial y} \end{pmatrix}$$

Result:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} cos\varphi \cdot \frac{\partial}{\partial x} + sin\varphi \cdot \frac{\partial}{\partial y} \\ -r \cdot sin\varphi \cdot \frac{\partial}{\partial x} + r \cdot cos\varphi \cdot \frac{\partial}{\partial y} \end{pmatrix}$$

Now we can calculate the partial derivatives of the cartesian coordinates by polar coordinates, using the symbolic description.

$$\nabla = T^{-1} \widetilde{\nabla} = \begin{pmatrix} cos\varphi & \frac{-sin\varphi}{r} \\ sin\varphi & \frac{cos\varphi}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \varphi} \end{pmatrix} =$$

$$\begin{pmatrix} cos\varphi \cdot \frac{\partial}{\partial r} - \frac{sin\varphi}{r} \cdot \frac{\partial}{\partial \varphi} \\ sin\varphi \cdot \frac{\partial}{\partial r} + \frac{cos\varphi}{r} \cdot \frac{\partial}{\partial \varphi} \end{pmatrix}$$

Result:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} cos\varphi \cdot \frac{\partial}{\partial r} - \frac{sin\varphi}{r} \cdot \frac{\partial}{\partial \varphi} \\ sin\varphi \cdot \frac{\partial}{\partial r} + \frac{cos\varphi}{r} \cdot \frac{\partial}{\partial \varphi} \end{pmatrix}$$