This paper deals with periodic functions. We express the function in terms of eigenfunctions. We then apply the momentum operator and calculate the uncertainty in momentum and energy.

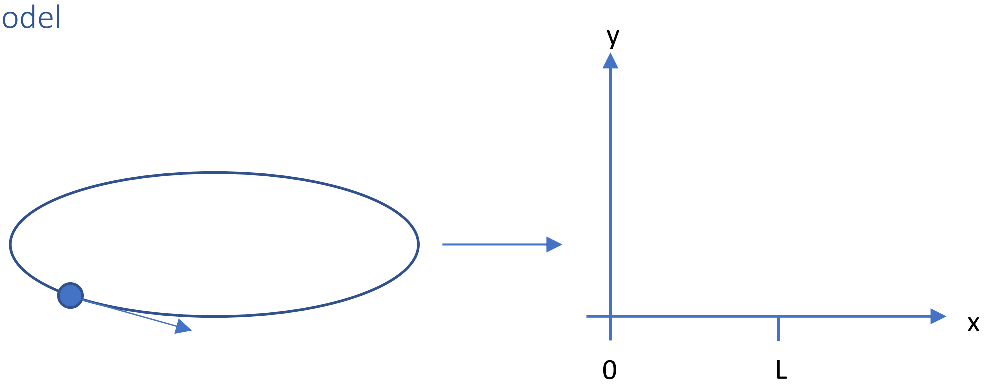
This paper follows a lecture B. Zwiebach gave at the MIT:

https://ocw.mit.edu/courses/8-04-quantum-physics-i-spring-2016/resources/solving-particle-on-a-circle/

Hope I can help you with learning quantum mechanics.

Inhalt

The model



We transform the path of the particle from a circle to a straight line, thus omitting the problems how to hold the particle on the circle. We have no potential we need to calculate.

Note: Ψ is a function of x only.

The wave functions

For the wave function we need:

$$\Psi(0) = \Psi(L)$$

$$\frac{d}{dx}\Psi(0) = \frac{d}{dx}\Psi(L)$$

We use the following wave functions:

$\Psi_{\#1}(x) = \sqrt{\frac{2}{L}} sin\left(\frac{2\pi}{L}\cdot x\right)$	$\Psi_{\#2}(x) = \sqrt{\frac{2}{3L}}\cdot sin\left(\frac{2\cdot\pi\cdot x}{L}\right) + \sqrt{\frac{4}{3L}}\cdot cos\left(\frac{6\cdot\pi\cdot x}{L}\right)$

Note: The factors in front of sin and cos needed for normalization of the integrals $\int \Psi(x)^2$.

We calculate the derivatives:

$\frac{d}{dx}\Psi_{\#1}(x) = \frac{2\pi}{L}\cdot\sqrt{\frac{2}{L}}\cdot cos\left(\frac{2\pi}{L}\cdot x\right)$	$\frac{d}{dx}\Psi_{\#2}(x) = \sqrt{\frac{2}{3L}}\cdot cos\left(\frac{2\cdot\pi\cdot x}{L}\right)\frac{2\cdot\pi}{L} - \sqrt{\frac{4}{3L}}\cdot sin\left(\frac{6\cdot\pi\cdot x}{L}\right)\frac{6\cdot\pi}{L}$
$\frac{d^2}{dx^2}\Psi_{\#1}(x) = -\left(\frac{2\pi}{L}\right)^2\cdot\sqrt{\frac{2}{L}}\cdot sin\left(\frac{2\pi}{L}\cdot x\right) = -\left(\frac{2\pi}{L}\right)^2\cdot\sqrt{\frac{2}{L}}\cdot\Psi(x)$	$\frac{d^2}{dx^2}\Psi_{\#2}(x) = -\sqrt{\frac{2}{3L}}\cdot sin\left(\frac{2\cdot\pi\cdot x}{L}\right)\left(\frac{2\cdot\pi}{L}\right)^2 - \sqrt{\frac{4}{3L}}\cdot cos\left(\frac{6\cdot\pi\cdot x}{L}\right)\left(\frac{6\cdot\pi}{L}\right)^2$

We check the condition $\Psi(0) = \Psi(L)$:

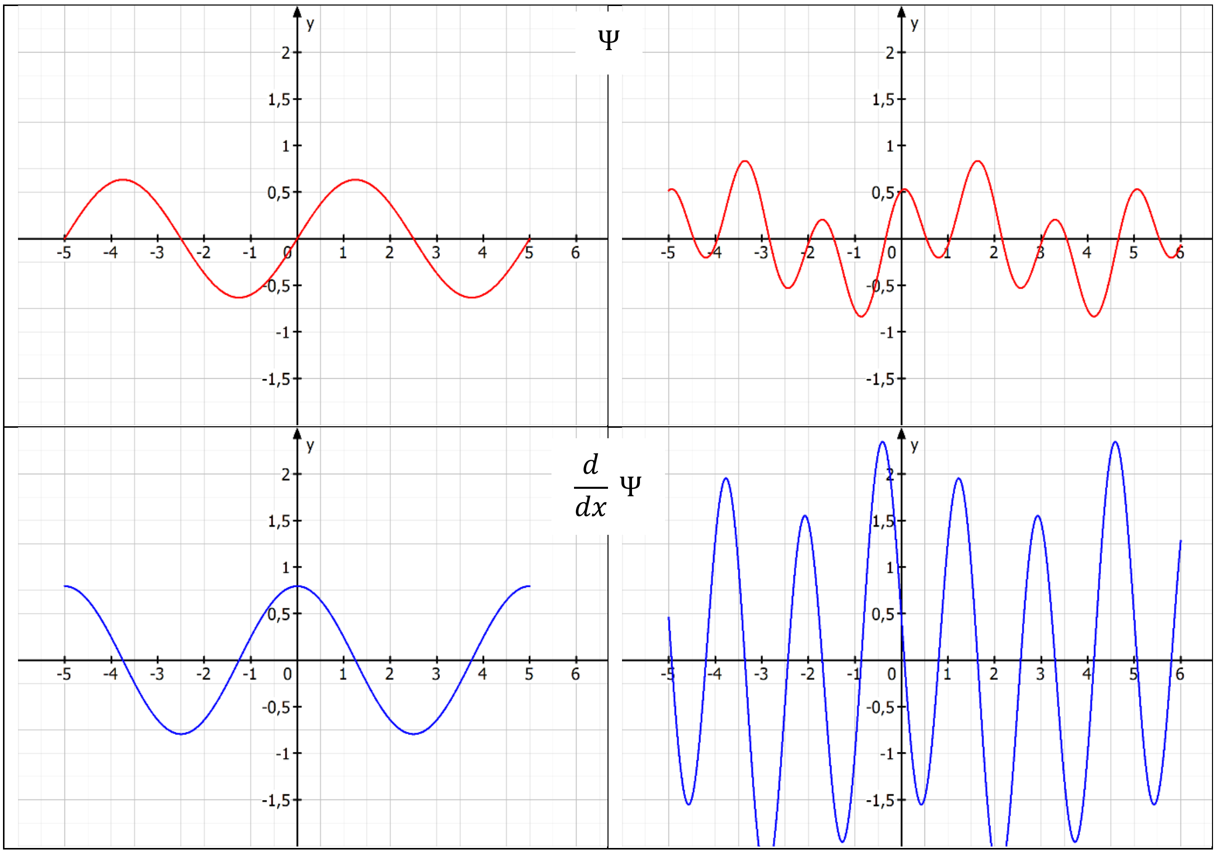
$\Psi_{\#1}(0) = 0$ $\Psi_{\#1}(L) = 0$	$\Psi_{\#2}(0) = \sqrt{\frac{4}{3L}}$ $\Psi_{\#2}(L) = \sqrt{\frac{2}{3L}} \cdot sin(2 \cdot \pi) + \sqrt{\frac{4}{3L}} \cdot cos(6 \cdot \pi) = \sqrt{\frac{2}{3L}} \cdot sin(0) + \sqrt{\frac{4}{3L}} \cdot cos(0) = \sqrt{\frac{4}{3L}}$

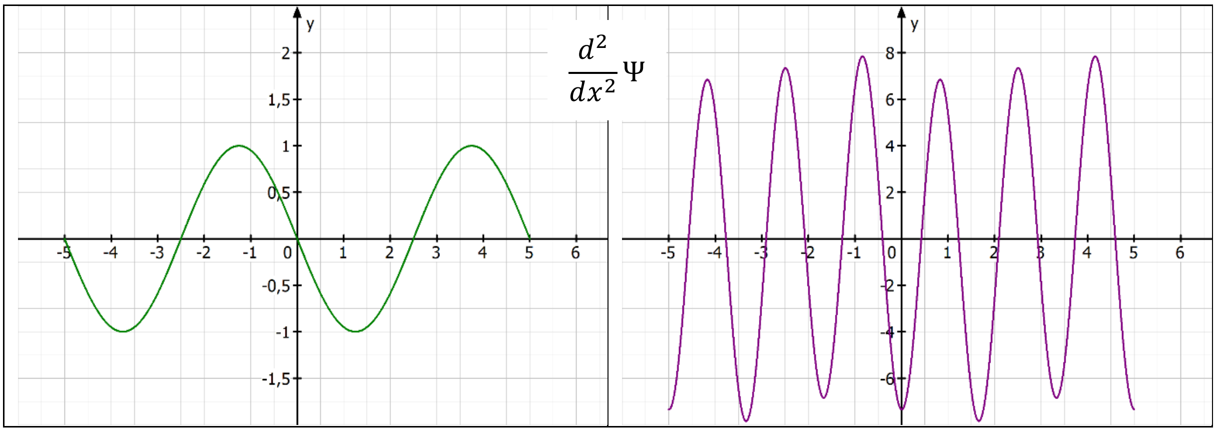
We check the condition $\frac{d}{dx}\,\Psi(0) = \frac{d}{dx}\,\Psi(L)$

$\frac{d}{dx}\Psi_{\#1}(0) = \frac{2\pi}{L} \cdot \sqrt{\frac{2}{L}}$	$\frac{d}{dx}\,\Psi_{\#2}(0) = \sqrt{\frac{2}{3L}} \cdot cos(0)\frac{2\pi}{L} - \sqrt{\frac{4}{3L}} \cdot sin(0)\frac{6\pi}{L} = \sqrt{\frac{2}{3L}} \cdot \frac{2\pi}{L}$
$\frac{d}{dx}\Psi_{\#1}(L) = \frac{2\pi}{L} \cdot \sqrt{\frac{2}{L}}$	$\frac{d}{dx}\,\Psi_{\#2}(L) = \sqrt{\frac{2}{3L}} \cdot cos(2 \cdot \pi)\frac{2\pi}{L} - \sqrt{\frac{4}{3L}} \cdot sin(6 \cdot \pi)\frac{6\pi}{L} = \sqrt{\frac{2}{3L}} \cdot \frac{2\pi}{L}$

Preparing the wave function $\Psi(x)$ this way gives the behavior needed.

We take a closer look to $\Psi(x)$, $\frac{d}{dx}\,\Psi(x)$ and $\frac{d^2}{dx^2}\Psi(x)$. We note that these are real functions – we can plot them. We use the parameter $L = 5$. On the left you find $\Psi_{\#1}$, on the right $\Psi_{\#2}$.





The functions seems to be periodic by period 5.

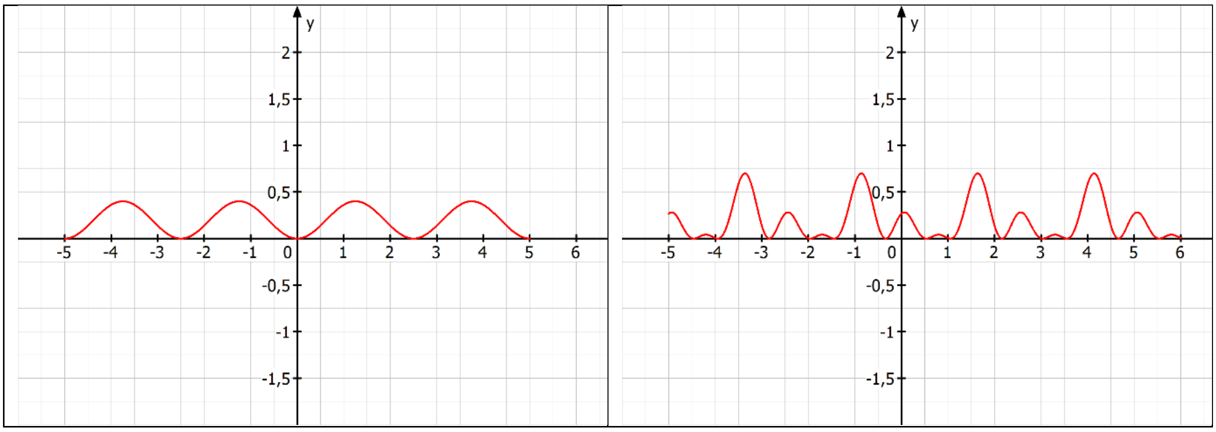
We check whether the normalization is correct, $\int_0^L \left(\Psi(x)\right)^2 dx = 1$ by help of wxmaxima:

$$\int_0^5 \left(\sqrt{\frac{2}{5}} sin\left(\frac{2\pi}{5}\cdot x\right)\right)^2 dx = \left|\frac{x}{5} - \frac{1}{4\pi}\sin\left(\frac{4\pi}{5}x\right)\right|_0^5 = 1$$

$$\int_0^5 \left(\sqrt{\frac{2}{15}}\cdot sin\left(\frac{2\cdot\pi\cdot x}{5}\right) + \sqrt{\frac{4}{15}}\cdot cos\left(\frac{6\cdot\pi\cdot x}{5}\right)\right)^2 dx =$$

$$\left|\frac{sin\left(\frac{12\pi}{5}x\right)}{18\pi} + \frac{2}{15}x + \sqrt{2^5}\left(\frac{1}{24\pi}cos\left(\frac{4\pi}{5}x\right) - \frac{1}{48\pi}cos\left(\frac{8\pi}{5}x\right)\right)\right|_0^5 = 1$$

We plot the square of the function:



Remark: Expanding the length of the interval to $\pm\infty$ leads to Fourier series resp. the Fourier transformation. You find a paper dealing with Fourier series here.

We want to measure momentum and need eigenfunctions and eigenvalues of the momentum operator.

General Ψ- functions

We have no potential, $V(x) = 0$.

We use the Schrödinger equation:

$$\hat{H}\Psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = \mathrm{E}\Psi$$

$$-\frac{\hbar^2}{2m}\Psi^*\frac{d^2}{dx^2}\Psi = \Psi^*\mathrm{E}\Psi$$

and integrate over the whole space, the interval $[0; L)$:

$$-\frac{\hbar^2}{2m}\int_0^L \Psi^*\frac{d^2}{dx^2}\Psi \mathrm{dx} = E\int_0^L \Psi^*\Psi\,\mathrm{dx} = E$$

Remark: For the left integral we apply a standard access in quantum mechanics. You should use it whenever possible, it saves you from calculating often difficult second derivatives and work with the first one only.

Using differential rules, we transform the left integral:

$$\frac{d}{dx}\left(\Psi^*\frac{d}{dx}\Psi\right) = \frac{d}{dx}\Psi^*\frac{d}{dx}\Psi + \Psi^*\frac{d^2}{dx^2}\Psi$$

Thus, we can replace:

$$\Psi^*\frac{d^2}{dx^2}\Psi = \frac{d}{dx}\left(\Psi^*\frac{d}{dx}\Psi\right) - \frac{d}{dx}\Psi^*\frac{d}{dx}\Psi$$

We get:

$$-\frac{\hbar^2}{2m}\int_0^L\left(\Psi^*\frac{d^2}{dx^2}\Psi\right)dx =$$

$$-\frac{\hbar^2}{2m}\int_0^L\left(\frac{d}{dx}\left(\Psi^*\frac{d}{dx}\Psi\right) - \frac{d}{dx}\Psi^*\frac{d}{dx}\Psi\right)dx =$$

$$-\frac{\hbar^2}{2m}\int_0^L\left(\frac{d}{dx}\left(\Psi^*\frac{d}{dx}\Psi\right)\right)dx + \frac{\hbar^2}{2m}\int_0^L\left|\frac{d}{dx}\Psi\right|^2 dx =;$$

First term:

$$-\frac{\hbar^2}{2m}\int_0^L\left(\frac{d}{dx}\left(\Psi^*\frac{d}{dx}\Psi\right)\right)dx = -\frac{\hbar^2}{2m}\left[\Psi\frac{d}{dx}\Psi\right]_0^L = 0$$

Note: Ψ and $\frac{d}{dx}\Psi$ are the same on the boundaries 0 and L.

Second term:

$$\frac{\hbar^2}{2m}\int_0^L\left|\frac{d}{dx}\Psi\right|^2 dx$$

This is an integral of a positive integrand over a positive range, it must give a positive real value.

We combine the result.

From:	$-\frac{\hbar^2}{2m}\int_0^L \Psi^* \frac{d^2}{dx^2}\Psi \mathrm{dx} = \int_0^L \mathrm{E}\Psi^*\Psi \,\mathrm{dx}$
We ended with:	$\frac{\hbar^2}{2m}\int_0^L \left\lvert\frac{d}{dx}\Psi\right\rvert^2 dx = E$

We name:

$$\int_0^L \left\lvert\frac{d}{dx}\Psi\right\rvert^2 dx \coloneqq k^2$$

We get an expression for the energy:

$$\frac{\hbar^2 k^2}{2m} = E$$

Comparing with the classical case we see $\hbar^2 k^2$ is the square of momentum or:

$$k^2 = \frac{2mE}{\hbar^2}$$

Orthonormal wave functions

The differential equation we started with:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = \mathrm{E}\Psi \to \frac{d^2}{dx^2}\Psi = -\frac{2mE}{\hbar^2}\Psi = -k^2\Psi$$

$$\frac{d^2}{dx^2}\Psi = -k^2\Psi$$

Solutions are e^{ikx} or e^{-ikx} or any linear combination.

We choose e^{ikx}:

$$\Psi = e^{ikx}$$

We have the periodic condition:

$$\Psi(x) = \Psi(x+L)$$

$$e^{ikx} = e^{ik(x+L)} = e^{ikx}\cdot e^{ikL} \to e^{ikL} = 1 \to$$

$$k\cdot L = a\cdot 2\pi, a \in \mathbb{N}$$

We use the index a:

$$k_a = \frac{2\pi a}{L}$$

We get the momenta:

$$p_a = \frac{2\pi \mathrm{a}}{L}\hbar$$

We get the energies:

$$\frac{\hbar^2 k^2}{2m} = E \to E_a = \frac{2}{m}\left(\frac{\pi\hbar\mathrm{a}}{L}\right)^2$$

Result: We have a set of orthogonal wave functions:

$$\Psi_a(x) = exp\left(i\frac{2\pi a}{L}\cdot x\right)$$

Note: orthogonality with respect to the interval $[0; L)$.

We check:

$$\int_0^L \Psi^*{}_b(x)\Psi_a(x)dx = \int_0^L exp\left(-i\frac{2\pi b}{L}\cdot x\right) exp\left(i\frac{2\pi a}{L}\cdot x\right) dx =$$

$$\int_0^L exp\left(-i\frac{2\pi b}{L}\cdot x + i\frac{2\pi a}{L}\cdot x\right) dx = \int_0^L exp\left(i\frac{2\pi(a-b)}{L}\cdot x\right) dx =;$$

For $a, b \in \mathbb{N}, a \neq b$ we get $a - b$ is a number and the integral, evaluated at the borders 0 and L gives the same value:	For $a, b \in \mathbb{N}, a = b$ we get:
$\int_0^L exp\left(i\frac{2\pi(a-b)}{L}\cdot x\right) dx = 0$	$\int_0^L exp\left(i\frac{2\pi(a-b)}{L}\cdot x\right) dx = \int_0^L exp(0)dx = L$

We get the normalizing factor:

$$\sqrt{\frac{1}{L}}$$

The orthonormal wave functions are:

$$\Psi_a(x) = \sqrt{\frac{1}{L}} exp\left(i\frac{2\pi a}{L}x\right)$$

The momentum operator

$$-i\hbar\frac{\partial}{\partial x}$$

We act with the momentum operator on the eigenfunctions $\Psi_a(x)$:

$$-i\hbar\frac{\partial}{\partial x}\Psi_a(x) = -i\hbar\frac{\partial}{\partial x}\left(\sqrt{\frac{1}{L}} exp\left(i\frac{2\pi a}{L}x\right)\right) =$$

$$-i\hbar\sqrt{\frac{1}{L}}\left(\frac{\partial}{\partial x} exp\left(i\frac{2\pi a}{L}x\right)\right) = -i\hbar\sqrt{\frac{1}{L}}\left(i\frac{2\pi a}{L}\right) exp\left(i\frac{2\pi a}{L}x\right) =$$

$$\frac{2\hbar\pi a}{L}\Psi_a(x)$$

Result: For every eigenfunction $\Psi_a(x)$ we get the eigenvalue $\frac{2\hbar\pi a}{L}$.

We want to express the wave functions from the beginning in terms of eigenfunctions.

We begin with:

$$\Psi_{\#1}(x) = \sqrt{\frac{2}{L}} sin\left(\frac{2\pi}{L}x\right)$$

We note:

$$sin(x) = \frac{exp(ix) - exp(-ix)}{2i}$$

We express sin by eigenfunctions. We want to get $\sin\left(\frac{2\pi}{L}x\right)$ from them. This is true for the two eigenfunctions $a = 1$ and $a = -1$:

$$exp\left(i\frac{2\pi}{L}(1 \cdot x)\right), exp\left(i\frac{2\pi}{L}(-1 \cdot x)\right)$$

We check sin and use the symmetry $cos(x) = cos(-x)$ and $sin(x) = -sin(-x)$.

$$\frac{exp\left(i\frac{2\pi}{L}(1 \cdot x)\right) - exp\left(i\frac{2\pi}{L}(-1 \cdot x)\right)}{2i} =$$

$$\frac{1}{2i}\left(cos\left(\frac{2\pi}{L}(1 \cdot x)\right) + isin\left(\frac{2\pi}{L}(1 \cdot x)\right) - cos\left(\frac{2\pi}{L}(-1 \cdot x)\right) - isin\left(\frac{2\pi}{L}(-1 \cdot x)\right)\right) =$$

$$sin\left(\frac{2 \cdot \pi \cdot x}{L}\right)$$

We rewrite:

$$\sqrt{\frac{2}{L}} sin\left(\frac{2 \cdot \pi \cdot x}{L}\right) \rightarrow \frac{\sqrt{2}\sqrt{\frac{1}{L}} exp\left(i\frac{2\pi}{L}x\right) - \sqrt{2}\sqrt{\frac{1}{L}} exp\left(-i\frac{2\pi}{L}x\right)}{2i}$$

We work with the next function:

$$\Psi_{\#2}(x) = \sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}} \cdot sin\left(\frac{2\pi}{L}x\right) + \sqrt{\frac{2}{3}} \cdot cos\left(\frac{6\pi}{L}x\right)\right)$$

We note:

$$sin(x) = \frac{exp(ix) - exp(-ix)}{2i}, \qquad cos(x) = \frac{exp(ix) + exp(-ix)}{2}$$

We express sin and cos by the eigenfunctions.

We want to get $\sin\left(\frac{2\pi}{L}x\right)$ out of the eigenfunctions. As above this is true for the two eigenfunctions $a = 1$ and $a = -1$:

$$exp\left(i\frac{2\pi}{L}x\right), \qquad exp\left(-i\frac{2\pi}{L}x\right)$$

We check sin:

$$\frac{exp\left(i\frac{2\pi}{L}x\right)-exp\left(-i\frac{2\pi}{L}x\right)}{2i}=sin\left(\frac{2\pi x}{L}x\right)$$

We want to get $\cos\left(\frac{2\pi}{L}x\right)$ from of the eigenfunctions. This is true for the two eigenfunctions $a=3$ and $a=-3$:

$$exp\left(i\frac{6\pi}{L}x\right),\qquad exp\left(-i\frac{62\pi}{L}x\right)$$

We check $\cos$:

$$\frac{exp\left(i\frac{6\pi}{L}x\right)+exp\left(-i\frac{6\pi}{L}x\right)}{2}=cos\left(\frac{6\pi}{L}x\right)$$

We rewrite $\Psi_{\#2}(x)$:

$$\Psi_{\#2}(x)=\sqrt{\frac{2}{3}}\sqrt{\frac{1}{L}}\left(\frac{exp\left(i\frac{2\pi}{L}x\right)-exp\left(-i\frac{2\pi}{L}x\right)}{2i}+\sqrt{\frac{4}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(i\frac{6\pi}{L}x\right)+exp\left(-i\frac{6\pi}{L}x\right)}{2}\right)$$

The wave function expressed in terms of eigenfunctions

We rewrite our example wave functions $\Psi(x)$ in terms of eigenfunctions. We remember:

$$\Psi_a(x)=\sqrt{\frac{1}{L}}exp\left(i\frac{2\pi a}{L}x\right)$$

$\Psi_{\#1}(x)=\sqrt{\frac{2}{L}}sin\left(\frac{2\pi}{L}\cdot x\right)\rightarrow$	$\Psi_{\#2}(x)=\sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}}sin\left(\frac{2\pi}{L}x\right)+\sqrt{\frac{2}{3}}cos\left(\frac{6\pi}{L}x\right)\right)\rightarrow$
$\sqrt{2}\sqrt{\frac{1}{L}}\frac{exp\left(i\frac{2\pi}{L}x\right)-exp\left(-i\frac{2\pi}{L}x\right)}{2i}=$ $\frac{\Psi_1(x)-\Psi_{-1}(x)}{2i}$	$\sqrt{\frac{2}{3}}\sqrt{\frac{1}{L}}\left(\frac{exp\left(i\frac{2\pi}{L}x\right)-exp\left(-i\frac{2\pi}{L}x\right)}{2i}\right.$ $\left.+\sqrt{\frac{4}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(i\frac{6\pi}{L}x\right)+exp\left(-i\frac{6\pi}{L}x\right)}{2}\right)=$ $\sqrt{\frac{2}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(i\frac{2\pi}{L}x\right)}{2i}-\sqrt{\frac{2}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(-i\frac{2\pi}{L}x\right)}{2i}+$ $\sqrt{\frac{4}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(i\frac{6\pi}{L}x\right)}{2}+\sqrt{\frac{4}{3}}\sqrt{\frac{1}{L}}\frac{exp\left(-i\frac{6\pi}{L}x\right)}{2}=$ $\sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)-\Psi_{-1}(x)}{2i}+\sqrt{\frac{4}{3}}\cdot\frac{\Psi_3(x)+\Psi_{-3}(x)}{2}$

Result:

We express the Wave functions

$$\Psi_{\#1}(x) = \sqrt{\frac{2}{L}} sin\left(\frac{2\pi}{L}\cdot x\right) \quad resp. \;\; \Psi_{\#2}(x) = \sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}}\cdot sin\left(\frac{2\cdot\pi\cdot x}{L}\right) + \sqrt{\frac{2}{3}}\cdot cos\left(\frac{6\cdot\pi\cdot x}{L}\right)\right)$$

by a set of eigenfunctions $\Psi_1(x), \Psi_{-1}(x), \Psi_3(x), \Psi_{-3}(x)$. They build a basis for the wave function and can replace it if we are in search of the results of momentum measurement.

Measuring momentum

Measuring the momentum of $\Psi(x)$ can be done by applying the momentum operator to the eigenfunctions. Applying the momentum operator reproduces the eigenfunctions multiplied by the respective eigenvalue.

Quantum mechanically we get with every measurement one of the possible results.

The probabilities are the squares of the eigenfunctions, $|\Psi_a(x)|^2 = \Psi_a{}^*(x)\Psi_a(x)$.

<table>
<tr><td>$\Psi_{\#1}(x) = \sqrt{2}\frac{\Psi_1(x) - \Psi_{-1}(x)}{2i}$</td><td colspan="3">$\Psi_{\#2}(x) = \sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x) - \Psi_{-1}(x)}{2i} + \sqrt{\frac{4}{3}}\cdot\frac{\Psi_3(x) + \Psi_{-3}(x)}{2}$</td></tr>
<tr><td>For eigenfunction $\frac{\Psi_1(x)}{2i}$ we get:</td><td colspan="3">For eigenfunction $\sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)}{2i}$ we get:</td></tr>
<tr><td>$\sqrt{2}\left(\frac{\Psi_1(x)}{2i}\right)^*\cdot\sqrt{2}\frac{\Psi_1(x)}{2i} =$
$\frac{2}{4}\Psi_1{}^*(x)\cdot\Psi_1(x) =$
$\frac{1}{2}$

The same result we get for the eigenfunction $\frac{\Psi_{-1}(x)}{2i}$.</td><td colspan="3">$\sqrt{\frac{2}{3}}\cdot\left(\frac{\Psi_1(x)}{2i}\right)^*\cdot\sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)}{2i} =$
$\frac{2}{3}\cdot\frac{1}{4}\cdot\Psi_1{}^*(x)\cdot\Psi_1(x) =$
$\frac{2}{12} = \frac{1}{6}$</td></tr>
<tr><td></td><td colspan="3">For the other eigenfunctions $\Psi_{-1}(x), \Psi_3(x), \Psi_{-3}(x)$ we get:</td></tr>
<tr><td rowspan="4"></td><td></td><td>$eigenvalues$</td><td>$probability$</td></tr>
<tr><td>$\Psi_{-1}(x)$</td><td>$-\frac{2\hbar\pi}{L}$</td><td>$\frac{1}{6}$</td></tr>
<tr><td>$\Psi_{-1}(x)$</td><td>$\frac{6\hbar\pi}{L}$</td><td>$\frac{1}{3}$</td></tr>
<tr><td>$\Psi_3(x)$</td><td>$-\frac{6\hbar\pi}{L}$</td><td>$\frac{1}{3}$</td></tr>
</table>

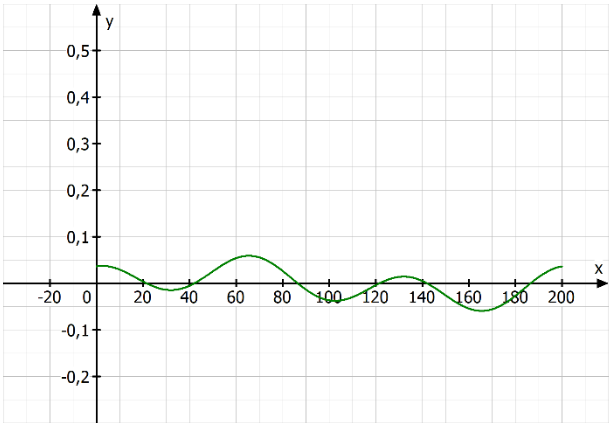
The total probability sums up to one as it should be.

A look at the wave function

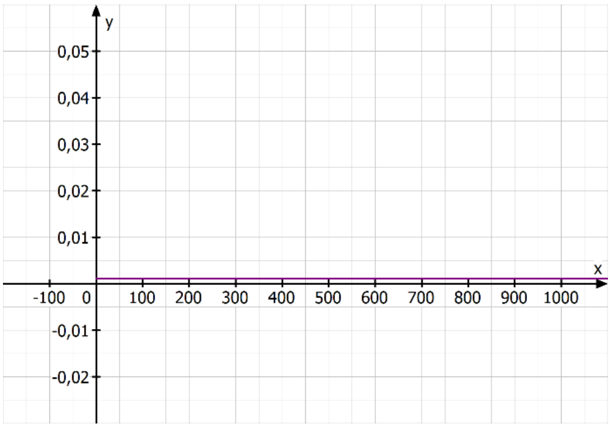
We take a look at the wave function:

$$\Psi_{\#2}(x) = \sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}}\cdot sin\left(\frac{2\cdot\pi\cdot x}{L}\right) + \sqrt{\frac{2}{3}}\cdot cos\left(\frac{6\cdot\pi\cdot x}{L}\right)\right)$$

We use $L = 200$



We use $L = 200{,}000$.



The wave function is expanded onto the new range, the x-value falling.

In the limes $\lim_{L\to\infty} \Psi(x)$ the wave function goes to zero.

Remark: this is a typical behavior for probabilities if you proceed from finite probabilities to continuous probabilities and shows the need for probability densities instead of discrete probabilities because the discrete probabilities will go to zero.

We check $\lim_{L\to\infty} \Psi(x)$:

$$\lim_{L\to\infty} \Psi(x) = \lim_{L\to\infty} \sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}} \cdot sin\left(\frac{2\cdot\pi\cdot x}{L}\right) + \sqrt{\frac{2}{3}} \cdot cos\left(\frac{6\cdot\pi\cdot x}{L}\right)\right) =$$

$$\lim_{L\to\infty} \sqrt{\frac{2}{L}}\left(\sqrt{\frac{1}{3}} \cdot sin(0) + \sqrt{\frac{2}{3}} \cdot cos(0)\right) = \lim_{L\to\infty} \sqrt{\frac{4}{3L}} = 0$$

Measuring energy

We use the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}(\Psi) = E = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\Psi)$$

Note: we have no potential.

The energy operator:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\Psi)$$

We act with the energy operator on the eigenfunctions $\Psi_a(x)$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_a(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\sqrt{\frac{1}{L}} exp\left(i\frac{2\pi a}{L}x\right)\right) =$$

$$-\frac{\hbar^2}{2m}\sqrt{\frac{1}{L}}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} exp\left(i\frac{2\pi a}{L}x\right)\right) = -\frac{\hbar^2}{2m}\sqrt{\frac{1}{L}}\frac{\partial}{\partial x}\left(\left(i\frac{2\pi a}{L}\right) exp\left(i\frac{2\pi a}{L}x\right)\right) =$$

$$-\frac{\hbar^2}{2m}\sqrt{\frac{1}{L}}\left(i\frac{2\pi a}{L}\right)\frac{\partial}{\partial x}\left(exp\left(i\frac{2\pi a}{L}x\right)\right) = -\frac{\hbar^2}{2m}\sqrt{\frac{1}{L}}\left(i\frac{2\pi a}{L}\right)^2\left(exp\left(i\frac{2\pi a}{L}x\right)\right) =$$

$$\frac{\hbar^2}{2m}\sqrt{\frac{1}{L}}\left(\frac{2\pi a}{L}\right)^2\left(exp\left(i\frac{2\pi a}{L}x\right)\right) = \frac{2}{m}\left(\frac{\hbar\pi a}{L}\right)^2\Psi_a(x)$$

Result: For every eigenfunction $\Psi_a(x)$ we get the eigenvalue $\frac{2}{m}\left(\frac{\hbar\pi a}{L}\right)^2$. The eigenfunctions for momentum are eigenfunctions for energy too with different eigenvalues. This corresponds with classical physics, the kinetic energy there is $\frac{p^2}{2m}$.

We express the wave function $\Psi_{\#2}(x)$ in terms of eigenfunctions:

$$\sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)}{2i} - \sqrt{\frac{2}{3}}\cdot\frac{\Psi_{-1}(x)}{2i} + \sqrt{\frac{4}{3}}\cdot\frac{\Psi_3(x)}{2} + \sqrt{\frac{4}{3}}\frac{\Psi_{-3}(x)}{2}$$

Measuring the energy of $\Psi_{\#2}(x)$ can be done by applying the energy operator (the Hamiltonian) to the eigenfunctions. Applying the energy operator reproduces the eigenfunctions multiplied by the respective eigenvalue.

Quantum mechanically we get with every measurement one of the four possible results. The probabilities are the squares of the eigenfunctions, $|\Psi_a(x)|^2 = \Psi_a{}^*(x)\Psi_a(x)$:

For eigenfunction $\Psi_1(x)$ we get:

$$\sqrt{\frac{2}{3}}\cdot\left(\frac{\Psi_1(x)}{2i}\right)^*\cdot\sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)}{2i} = \sqrt{\frac{2}{3}}\cdot\frac{1}{-2i}\cdot\sqrt{\frac{2}{3}}\cdot\frac{1}{2i}\cdot\Psi_1{}^*(x)\cdot\Psi_1(x) =$$

$$\sqrt{\frac{2}{3}}\cdot\frac{1}{-2i}\cdot\sqrt{\frac{2}{3}}\cdot\frac{1}{2i}\cdot 1 = \frac{2}{12} = \frac{1}{6}$$

We get:

	$eigenvalues$	$probability$
$\Psi_1(x) = \Psi_{-1}(x)$	$\frac{2}{m}\left(\frac{\hbar\pi}{L}\right)^2$	$\frac{1}{6}$
$\Psi_3(x) = \Psi_{-3}(x)$	$\frac{18}{m}\left(\frac{\hbar\pi}{L}\right)^2$	$\frac{1}{3}$

The total probability sums up to one as it should be.

Note: We get only two different energy eigenvalues, they are degenerated.

Uncertainty in momentum

We know eigenfunctions and eigenvalues with respect to momentum that construct $\Psi_{\#2}(x)$:

$$\Psi_{\#2}(x) = \sqrt{\frac{2}{3}}\cdot\frac{\Psi_1(x)}{2i} - \sqrt{\frac{2}{3}}\cdot\frac{\Psi_{-1}(x)}{2i} + \sqrt{\frac{4}{3}}\cdot\frac{\Psi_3(x)}{2} + \sqrt{\frac{4}{3}}\frac{\Psi_{-3}(x)}{2}$$

We got the eigenvalues (the momenta) for each eigenfunction (basis function) and their probabilities:

	$eigenvalues\ p_i$	$probability\ P_i$
$\Psi_1(x)$	$\frac{\hbar 2\pi \cdot 1}{L}$	$\frac{1}{6}$
$\Psi_{-1}(x)$	$-\frac{\hbar 2\pi \cdot 1}{L}$	$\frac{1}{6}$
$\Psi_3(x)$	$\frac{\hbar 2\pi \cdot 3}{L}$	$\frac{1}{3}$
$\Psi_{-3}(x)$	$-\frac{\hbar 2\pi \cdot 3}{L}$	$\frac{1}{3}$

We calculate the expectation value for momentum:

$$\langle p \rangle = \sum_{i=1}^{4} p_i \cdot P_i = \frac{2 \cdot \hbar \cdot \pi}{L} \cdot \left(1 \cdot \frac{1}{6} + \left(-1 \cdot \frac{1}{6}\right) + 3 \cdot \frac{1}{3} + \left(-3 \cdot \frac{1}{3}\right)\right) =$$

$$\frac{2 \cdot \hbar \cdot \pi}{L} \cdot 0 = 0$$

The expectation value for momentum is zero.

In order to calculate the uncertainty, we use:

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$$

As we have $\langle p \rangle = 0$ this gives:

$$(\Delta p)^2 = \langle p^2 \rangle$$

We calculate $\langle p^2 \rangle$:

$$\langle p^2 \rangle = \sum_{i=1}^{4} {p_i}^2 \cdot P_i = \left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2 \cdot \left(1^2 \cdot \frac{1}{6} + \left((-1)^2 \cdot \frac{1}{6}\right) + 3^2 \cdot \frac{1}{3} + \left((-3)^2 \cdot \frac{1}{3}\right)\right) =$$

$$\left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{1}{6} + \frac{1}{6} + \frac{9}{3} + \frac{9}{3}\right) = \left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{1 + 1 + 18 + 18}{6}\right) =$$

$$\left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2 \cdot \frac{38}{6} = \left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2 \cdot \frac{19}{3}$$

We get:

$$(\Delta p)^2 = \langle p^2 \rangle = \frac{19}{3}\left(\frac{2 \cdot \hbar \cdot \pi}{L}\right)^2$$

The uncertainty in momentum:

$$\Delta p = \frac{2 \cdot \hbar \cdot \pi}{L} \cdot \sqrt{\frac{19}{3}}$$

Uncertainty in energy

We know eigenfunctions and eigenvalues with respect to energy of $\Psi_{\#2}(x)$:

$$\Psi_{\#2}(x) = \sqrt{\frac{2}{3}} \cdot \frac{\Psi_1(x)}{2i} - \sqrt{\frac{2}{3}} \cdot \frac{\Psi_{-1}(x)}{2i} + \sqrt{\frac{4}{3}} \cdot \frac{\Psi_3(x)}{2} + \sqrt{\frac{4}{3}} \frac{\Psi_{-3}(x)}{2}$$

We got the eigenvalues (the energy) for each eigenfunction (basis function) and their probabilities:

	$eigenvalues$	$probability$
$\Psi_1(x) = \Psi_{-1}(x)$	$\frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2$	$\frac{1}{6}$
$\Psi_3(x) = \Psi_{-3}(x)$	$\frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2$	$\frac{1}{3}$

Note: If two eigenfunctions have the same eigenvalue we call them degenerate.

We calculate the expectation value for energy $\langle e \rangle$:

$$\langle e \rangle = \sum_{i=1}^{4} p_i \cdot P_i = \frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{1}{6} + \frac{1}{6}\right) + \frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{1}{3} + \frac{1}{3}\right) =$$

$$\frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{1}{3}\right) + \frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 \cdot \left(\frac{2}{3}\right) = \frac{2}{3m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 + \frac{36}{3m}\left(\frac{\hbar \cdot \pi}{L}\right)^2 =$$

$$\frac{38}{3m}\left(\frac{\hbar \cdot \pi}{L}\right)^2$$

We get the squared expectation value for the energy:

$$\langle e \rangle^2 = \left(\frac{38}{3 \cdot m}\left(\frac{\hbar \cdot \pi}{L}\right)^2\right)^2 = \frac{1444}{9 \cdot m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4$$

The expectation value for the energy squared:

$$\langle e^2 \rangle = \sum_{i=1}^{4} {p_i}^2 \cdot P_i = \left(\frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2\right)^2 \cdot \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2\right)^2 \cdot \left(\frac{1}{3} + \frac{1}{3}\right) =$$

$$\left(\frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2\right)^2 \cdot \left(\frac{1}{3}\right) + \left(\frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2\right)^2 \cdot \left(\frac{2}{3}\right) = \frac{4}{m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 \cdot \left(\frac{1}{3}\right) + \frac{324}{m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 \cdot \left(\frac{2}{3}\right) =$$

$$\frac{4}{3m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 + \frac{648}{3m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 = \frac{652}{3m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4$$

In order to calculate the uncertainty, we use:

$$(\Delta e)^2 = \langle e^2 \rangle - \langle e \rangle^2$$

$$(\Delta e)^2 = \frac{652}{3m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 - \frac{1444}{9 \cdot m^2}\left(\frac{\hbar \cdot \pi}{L}\right)^4 =$$

$$\left(\frac{\hbar \cdot \pi}{L}\right)^4 \left(\frac{652}{3m^2} - \frac{1444}{9m^2}\right) = \left(\frac{\hbar \cdot \pi}{L}\right)^4 \left(\frac{1956}{9m^2} - \frac{1444}{9m^2}\right) =$$

$$\left(\frac{\hbar \cdot \pi}{L}\right)^4 \left(\frac{512}{9m^2}\right)$$

We get the uncertainty:

$$\Delta e = \left(\frac{\hbar \cdot \pi}{L}\right)^2 \left(\frac{16\sqrt{2}}{3m}\right)$$

Degenerate eigenstates

We found the energy eigenvalues degenerate:

	$eigenvalues\ e_i$	$probabilities\ P_i$
$\Psi_1(x) = \Psi_{-1}(x)$	$\frac{2}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2$	$\frac{1}{6}$
$\Psi_3(x) = \Psi_{-3}(x)$	$\frac{18}{m}\left(\frac{\hbar \cdot \pi}{L}\right)^2$	$\frac{1}{3}$

This is valid for all eigenstates:

$$\Psi_i(x) = \Psi_{-i}(x)\ \forall i \in \mathbb{Z}$$

Degenerate states do not mean the states are identic. They are identic with respect to the energy (the eigenvalues). They differ in the momenta:

	$eigenvalues\ p_i$	$probabilities\ P_i$
$\Psi_1(x)$	$\frac{\hbar 2\pi \cdot 1}{L}$	$\frac{1}{6}$
$\Psi_{-1}(x)$	$-\frac{\hbar 2\pi \cdot 1}{L}$	$\frac{1}{6}$

Physically the momentum of a particle on the circle rotating clockwise or counterclockwise is different, its kinetic energy is not. The kinetic energy is independent from the direction of movement.

Position operator $\hat{x}$

The position operator in this setting is not an easy task. Please see: https://arxiv.org/pdf/2111.12538

You may find this paper directly here.

Summary

Whenever you want to predict the results of quantum measurements you need to work with eigenfunctions. Their eigenvalues are the basis for the probabilities of measuring outcome.