This paper deals with the path integral. It may give a first impression of what path integrals mean.

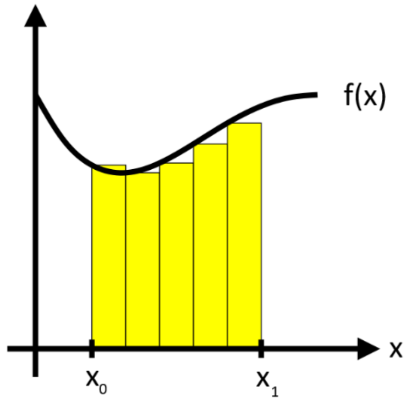
It follows:

https://www.youtube.com/watch?v=UWuhMJZaiZM

Physics with Elliot elliot-schneider@t.kajabimail.com

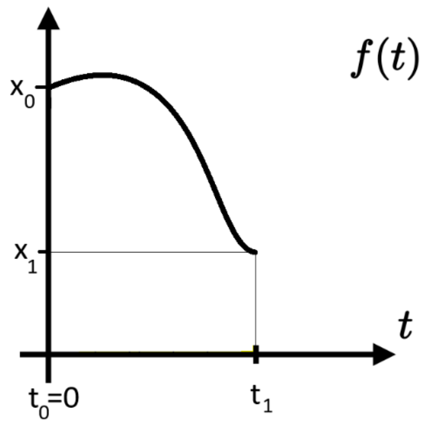
Hope I can help you with learning quantum mechanics.

The ordinary integral



The rectangles below the function $f(x)$ give the approximation of the area between the function and the x-axis. By refining the rectangles, the area tends more and more to the exact value of the area below $f(x)$. In the limit of infinitesimally tall rectangles we get the correct sum.

Feynman reinterpreted this picture.



He interpreted this as a path an object travels in space, starting at position x_0 and ending at x_1 for which the time t_1 was required.

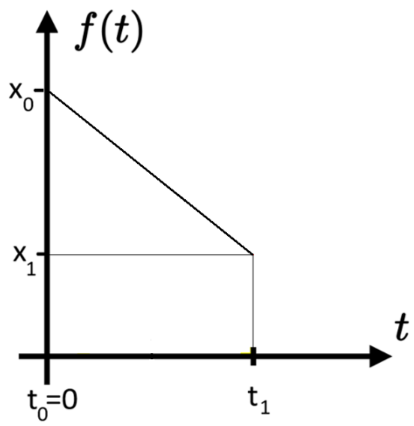
The integral gives the total action of the particle along this path.

Moving particles have kinetic energy, expressed as units of Joule. Multiplying the kinetic energy with time gives action.

In our example we have a free particle with no potential.

The kinetic energy is:

$$\frac{1}{2} \cdot m \cdot \dot{x}^2$$



We integrate the kinetic energy along the path from time t_0 to $t_1 \coloneqq \Delta t$:

$$\int_{t_0}^{t_1} \frac{1}{2} \cdot m \cdot \dot{x}^2 dt = \frac{1}{2} \cdot m \cdot \dot{x}^2 \cdot \Delta t$$

We have no potential, no force, the velocity of the particle is constant:

$$\dot{x} = c \ = \frac{x_1 - x_0}{t_1 - t_0}$$

For the straight line from x_0 to x_1 we get the action:

$$\int_{t_0}^{t_1} \frac{1}{2} \cdot m \cdot \dot{x}^2 dt = \frac{1}{2} \cdot m \cdot \left(\frac{x_1 - x_0}{t_1 - t_0}\right)^2 \int_{t_0}^{t_1} dt = \frac{1}{2} \cdot m \cdot \left(\frac{x_1 - x_0}{t_1 - t_0}\right)^2 \cdot (t_1 - t_0) =$$

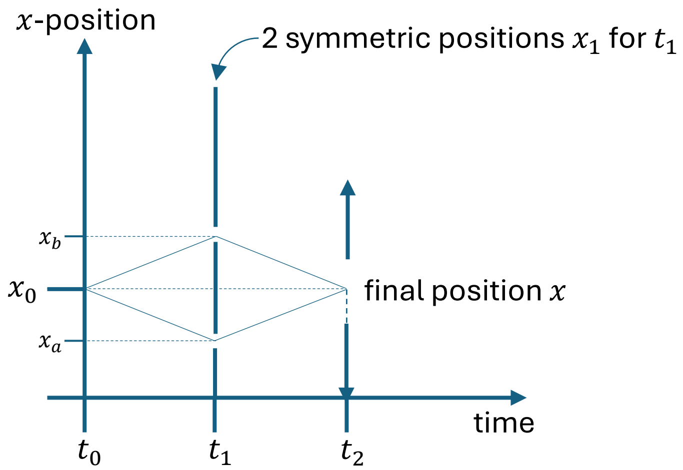
$$\frac{1}{2} \cdot m \cdot \left(\frac{x_1 - x_0}{\Delta t}\right)^2 \cdot \Delta t = \frac{1}{2} \cdot m \cdot \frac{(x_1 - x_0)^2}{\Delta t} = \frac{1}{2} \cdot m \cdot \frac{(\Delta x)^2}{\Delta t}$$

The double slit experiment

We fix x_0, x_a, x_b and vary the x_2-position from $-\infty$ to ∞.

$$x_0 = const., x_a = const., x_b = const., x_2 = x$$

Graphic representation:



Each path contributes to the probability of finding the particle at position x by raising it to the exponential.

The lower path:

$$e^{\frac{i}{\hbar}\cdot S[x_l]} = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_a-x_0)^2+(x-x_a)^2\right)}$$

The upper path:

$$e^{\frac{i}{\hbar}\cdot S[x_u]} = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_b-x_0)^2+(x-x_b)^2\right)}$$

We get the probability amplitude K_{02} for a particle to go from x_0 to x_2:

$$K_{02} = \sum_{paths\ x(t)} e^{\frac{i}{\hbar}\cdot S[x]} \approx \left(e^{\frac{i}{\hbar}\cdot S[x_l]} + e^{\frac{i}{\hbar}\cdot S[x_u]}\right)$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_a-x_0)^2+(x-x_a)^2\right)} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_b-x_0)^2+(x-x_b)^2\right)} =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x_a{}^2-2x_ax_0+x_0{}^2+x^2-2x_ax+x_a{}^2\right)} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x_b{}^2-2x_bx_0+x_0{}^2+x^2-2x_bx+x_b{}^2\right)} =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax} + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_b{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_bx_0}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_bx} =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\left(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax} + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_b{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_bx_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_bx}\right)$$

We use:

$$x_0 = \frac{x_a+x_b}{2} \to x_b = 2x_0 - x_a$$

$$\begin{aligned}&e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\Big(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\\&\qquad + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}(2x_0-x_a)^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}(2x_0-x_a)x_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}(2x_0-x_a)x}\Big) =\end{aligned}$$

$$\begin{aligned}&e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\Big(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\\&\qquad + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}\left(4x_0{}^2-4x_0x_a+x_a{}^2\right)}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}(2x_0-x_a)x_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}(2x_0-x_a)x}\Big)\end{aligned}$$

$$\begin{aligned}&e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\Big(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\\&\qquad + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}4x_0{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}4x_0x_a}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0x}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\Big) =\end{aligned}$$

$$\begin{aligned}&e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\Big(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\\&\qquad + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}3x_0x_a}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0x}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\Big) =\end{aligned}$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x_0{}^2}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_a{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax_0}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}x_ax}\left(1+e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0{}^2}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}4x_0x_a}e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_0x}e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2x_ax}\right) =$$

$$e^{\frac{i}{\hbar}\frac{m}{2\Delta t}{x_0}^2}e^{\frac{i}{\hbar}\frac{m}{\Delta t}{x_a}^2}e^{-\frac{i}{\hbar}\frac{m}{\Delta t}x_a x_0}e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-x_a x)}\left(1+e^{\frac{i}{\hbar}\frac{m}{\Delta t}2{x_0}^2}e^{-\frac{i}{\hbar}\frac{m}{\Delta t}4x_0 x_a}e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)$$

The only factors varying with x are $x^2 - x_a x$ and $(x_a - x_0)x$:

$$e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-x_a x)}\left(1+e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)$$

The probability to find the particle at x then is:

$$Prob(x) \approx |K_{02}|^2$$

$$e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-x_a x)}\left(1+e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)\cdot e^{\frac{-i}{\hbar}\frac{m}{2\Delta t}(x^2-x_a x)}\left(1+e^{\frac{-i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)=$$

$$\left(1+e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)\cdot\left(1+e^{\frac{-i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)=$$

$$2+e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}+e^{\frac{-i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}=$$

$$2+2Re\left(e^{\frac{i}{\hbar}\frac{m}{\Delta t}2(x_a-x_0)x}\right)=$$

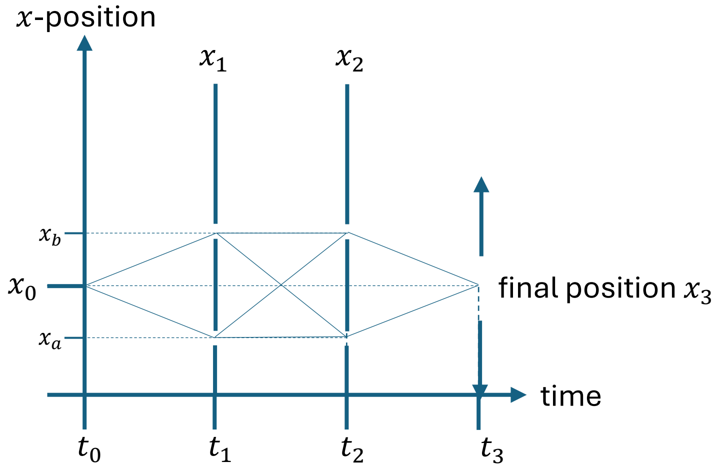
$$2+2cos\left(\frac{m}{\hbar\Delta t}2(x_a-x_0)x\right)$$

We use $x_a - x_0 = \Delta x$

$$2+2cos\left(2\frac{m}{\hbar\Delta t}\Delta x x\right)$$

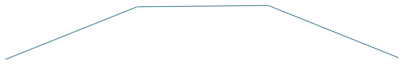
Adding a second double slit barrier

Graphic representation:

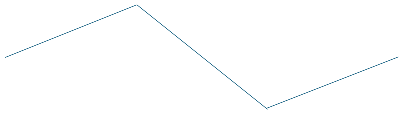


We have four possible paths from x_0 to x_3:

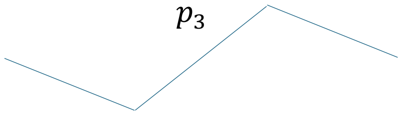
p_1



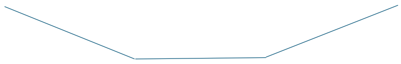
p_2



p_3



p_4



We leave Δt constant as before and examine the case with x_3 varying.

We fix x_0, x_a, x_b and vary the x_3-position from $-\infty$ to ∞.

$$x_0 = const., x_a = const., x_b = const., x_3 = x$$

Each path contributes to the probability of finding the particle at position x by raising it to the exponential.

Path p_1:

$$e^{\frac{i}{\hbar} S[p_1]} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_b-x_0)^2+(x_b-x_b)^2+(x-x_b)^2)} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_b-x_0)^2+(x-x_b)^2)}$$

Path p_2:

$$e^{\frac{i}{\hbar} S[p_2]} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_b-x_0)^2+(x_a-x_b)^2+(x-x_a)^2)}$$

Path p_3:

$$e^{\frac{i}{\hbar} S[p_3]} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_a-x_0)^2+(x_b-x_a)^2+(x-x_b)^2)}$$

Path p_4:

$$e^{\frac{i}{\hbar} S[p_4]} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_a-x_0)^2+(x_a-x_a)^2+(x-x_a)^2)} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_a-x_0)^2+(x-x_a)^2)}$$

We resolve the binomials and omit constant factors in the end:

$$p_1 = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_b-x_0)^2+(x-x_b)^2)} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(2x_b{}^2-2x_bx_0+x_0{}^2+x^2-2x_bx)} \approx e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-2x_bx)}$$

$$p_2 = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_b-x_0)^2+(x_a-x_b)^2+(x-x_a)^2)} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x_b{}^2-2x_bx_0+x_0{}^2+x_a{}^2-2x_ax_b+x_b{}^2+x^2-2xx_a+x_a{}^2)} =$$

$$e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(2x_b{}^2-2x_bx_0+x_0{}^2+2x_a{}^2-2x_ax_b+x^2-2xx_a)} \approx e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-2xx_a)}$$

$$p_3 = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}((x_a-x_0)^2+(x_b-x_a)^2+(x-x_b)^2)} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x_a{}^2-2x_ax_0+x_0{}^2+x_b{}^2-2x_ax_b+x_a{}^2+x^2-2xx_b+x_b{}^2)} =$$

$$e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(2x_a{}^2-2x_ax_0+x_0{}^2+2x_b{}^2-2x_ax_b+x^2-2xx_b)} \approx e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x^2-2xx_b)}$$

$$p_4 = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_a-x_0)^2+(x-x_a)^2\right)} = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(2x_a{}^2-2x_ax_0+x_0{}^2+x^2-2x_ax\right)} \approx e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2x_ax\right)}$$

We get the probability amplitude K_{02} for a particle to go from x_0 to x:

$$K_{03} \approx \sum_{paths\ x(t)} e^{\frac{i}{\hbar}\cdot S[x]} = \left(e^{\frac{i}{\hbar}\cdot S[p_1]} + e^{\frac{i}{\hbar}\cdot S[p_2]} + e^{\frac{i}{\hbar}\cdot S[p_3]} + e^{\frac{i}{\hbar}\cdot S[p_4]}\right) =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2x_bx\right)} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2xx_a\right)} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2xx_b\right)} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2x_ax\right)} =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2x_bx} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_b} + e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2x_ax} =$$

$$e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2x_bx} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_b} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2x_ax}\right) =$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2x_bx} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a}\right)$$

We use:

$$x_0 = \frac{x_a + x_b}{2} \rightarrow x_b = 2x_0 - x_a$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2(2x_0-x_a)x} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a}\right) =$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}4xx_0} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a} + e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a}\right) =$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}x^2} e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}2xx_a}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}4xx_0} + 1\right) =$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2xx_a\right)}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right)$$

The probability to find the particle at x position is:

$$Prob(x) \approx |K_{03}|^2$$

$$2e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2xx_a\right)}\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right) \cdot 2e^{-\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left(x^2-2xx_a\right)}\left(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right) =$$

$$4\left(e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right) \cdot \left(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right) =$$

$$4 \cdot \left(1 + e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + 1\right) =$$

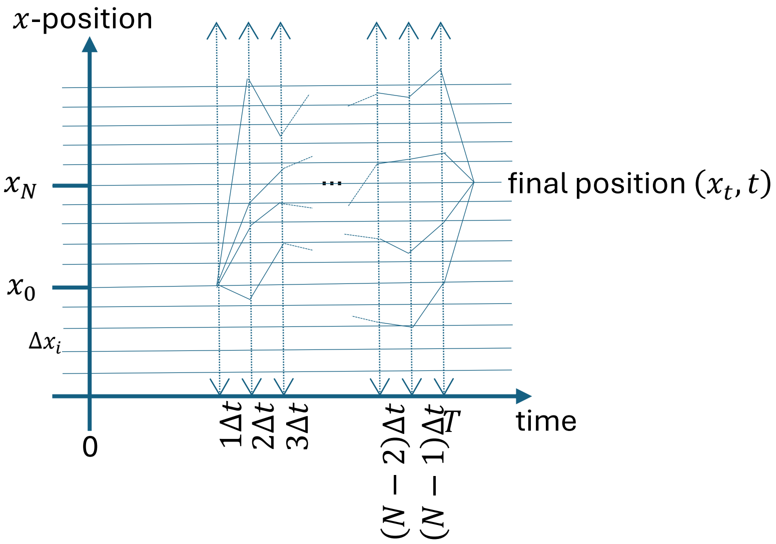
$$4 \cdot \left(2 + e^{-\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0} + e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0}\right) =$$

$$4 \cdot \left(2 + 2Re\left(e^{\frac{i}{\hbar}\cdot\frac{m}{\Delta t}2xx_0}\right)\right) =$$

$$8 \cdot \left(1 + cos\left(\frac{m}{\hbar\Delta t} 2xx_0\right)\right) =$$

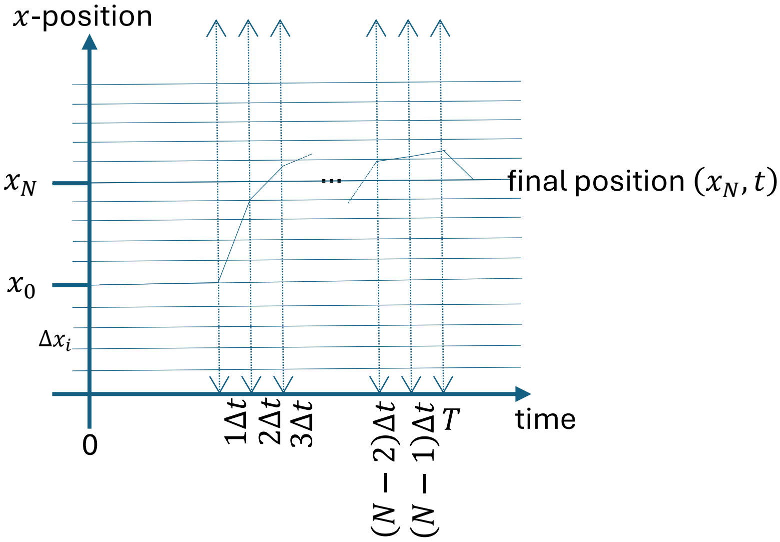
$$8 + 8cos\left(\frac{m}{\hbar\Delta t} 2xx_0\right)$$

N gratings with $N \times N$ possible paths



We divide time into equal parts, $\Delta t = t_{i+1} - t_i = const$. The line segments of the position Δx might be individually different.

We take one of these possibilities.



The action $S[x]$ of this free particle in each segment:

$$S[x_i] = \int_{t_i}^{t_{i+i}} dt \left\{\frac{1}{2} m \left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2\right\} = \frac{m}{2\Delta t} (\Delta x)^2$$

We divided the timeline into N steps. Summing up, we get the action for a single path:

$$S[x] = \sum_{i=1}^{N} S[x_i] = \sum_{i=1}^{N} \frac{m}{2\Delta t}(x_i - x_{i-1})^2 =$$

$$\frac{m}{2\Delta t}((x_1 - x_0)^2 + (x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2) =$$

$$\sum_{i=1}^{N} \frac{m}{2\Delta t}(\Delta x_i)^2$$

This path contributes to the probability of finding the particle at position x_N by raising it to the exponential:

$$e^{\frac{i}{\hbar} \cdot S[x]} = e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t}((x_1 - x_0)^2 + (x_2 - x_1)^2 + \cdots + (x_N - x_{N-1})^2)}$$

We get the probability amplitude K_{0N} for a particle to go from x_0 to x_N:

$$K_{0N} \approx \sum_{paths\ x(t)} e^{\frac{i}{\hbar} \cdot S[x]}$$

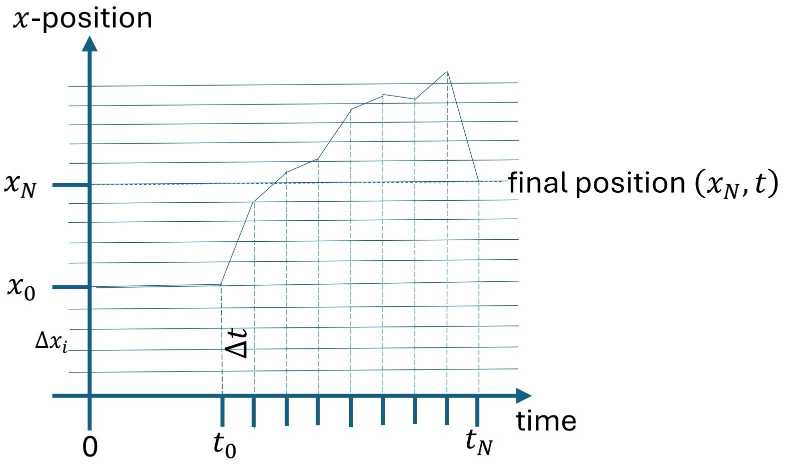
The probability of finding the particle at x_N:

$$Prob(x_N) \approx |K_{0N}|^2$$

Taking the limit

Note: $N \times N$ is countable so we can take the limit.

We took one of these possible paths and split it up in N time slots of equal length:



For every possible path we get discrete approximation.

If we want to sum over all those approximations, we need to sum over all possible values of each x_i:

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\ldots\int_{-\infty}^{\infty}dx_{N-1}=\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_{N-1}=\int_{x_0}^{x_N}Dx$$

We get:

$$\int_{x_0}^{x_N}Dx\,e^{\frac{i}{\hbar}\cdot S[x]}\approx\lim_{N\to\infty}\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_{N-1}e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_1-x_0)^2+(x_2-x_1)^2+\cdots+(x_N-x_{N-1})^2\right)}$$

With this we express the path integral ...

... by a sequence of ordinary integrals

Note: from the viewpoint of dx_1 all expressions $\left(x_j-x_{j-1}\right)^2$ are constant if $j\neq 1{,}2$ etc.

A look at Wikipedia gives:

$$\int_{-\infty}^{\infty}e^{iax^2}dx=\sqrt{\frac{\pi i}{a}}$$

By performing this for all $\left(x_j-x_{j-1}\right)^2$ we get, after lengthy calculations:

$$\int_{x_0}^{x_N}Dx\,e^{\frac{i}{\hbar}\cdot S[x]}\approx\lim_{N\to\infty}\left\{e^{\frac{i}{\hbar}\cdot\frac{m}{2}\frac{(x_N-x_0)^2}{t_N-t_0}}\sqrt{\frac{m}{2\pi i\hbar(t_N-t_0)}}\left(\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\right)^N\right\}$$

We note that the only dependency of N is:

$$\left(\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\right)^N$$

To eliminate this dependency, we choose a factor A as the inverse:

$$A=\frac{1}{\left(\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\right)^N}$$

We write:

$$\int_{x_0}^{x_N}Dx\,e^{\frac{i}{\hbar}\cdot S[x]}\approx\lim_{N\to\infty}\left\{A\cdot e^{\frac{i}{\hbar}\cdot\frac{m}{2}\frac{(x_N-x_0)^2}{t_N-t_0}}\sqrt{\frac{m}{2\pi i\hbar(t_N-t_0)}}\left(\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\right)^N\right\}$$

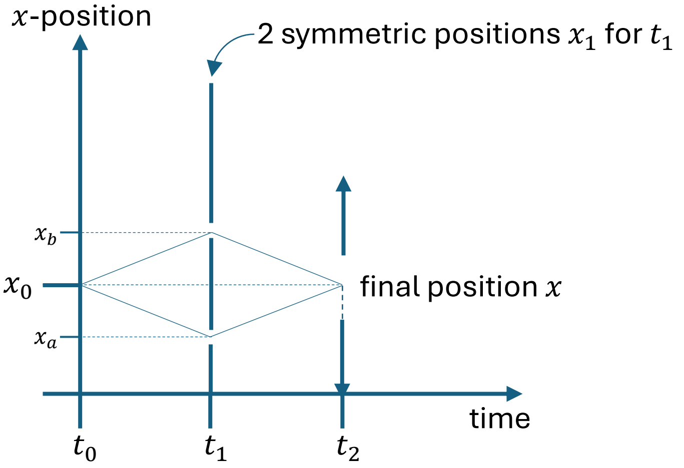
Then the dependency of N cancels out and we get:

$$\boxed{\int_{x_0}^{x_N} Dx\, e^{\frac{i}{\hbar}\cdot S[x]} = e^{\frac{i}{\hbar}\cdot\frac{m}{2}\cdot\frac{(x_N-x_0)^2}{t_N-t_0}}\sqrt{\frac{m}{2\pi i\hbar(t_N-t_0)}} := K_{0N}}$$

This is the probability amplitude for a free particle to propagate from x_0 to x_N.

Addentum: The interference pattern of the double slit and the cascaded double slit.

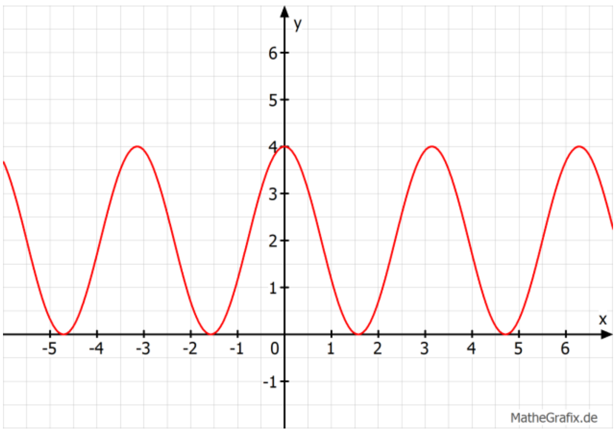
Double slit



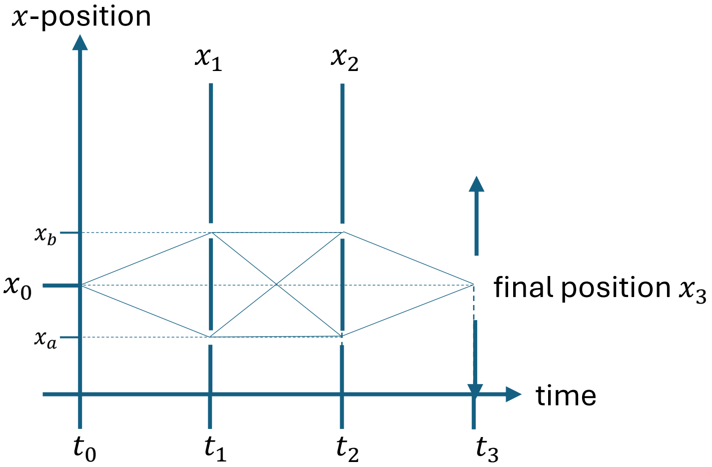
The calculation for the double slit experiment gave:

$$2 + 2cos\left(2\frac{m}{\hbar\Delta t}\Delta x x\right)$$

We set $m = \hbar = \Delta t = \Delta x = 1$ and plot:



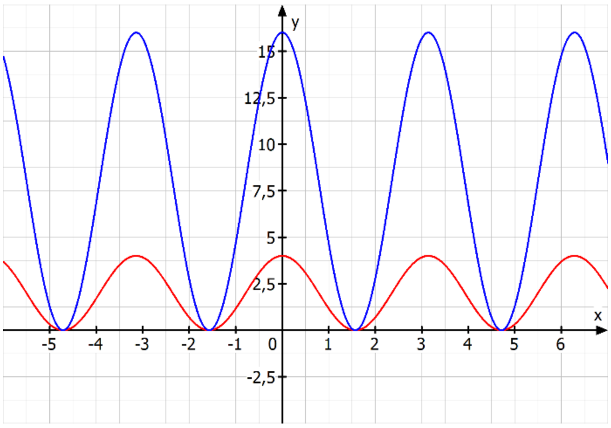
Cascaded double slit



The calculation for the cascaded double slit experiment gave:

$$8 + 8cos\left(\frac{m}{\hbar\Delta t}2xx_0\right)$$

We set $m = \hbar = \Delta t = x_0 = 1$ and plot:

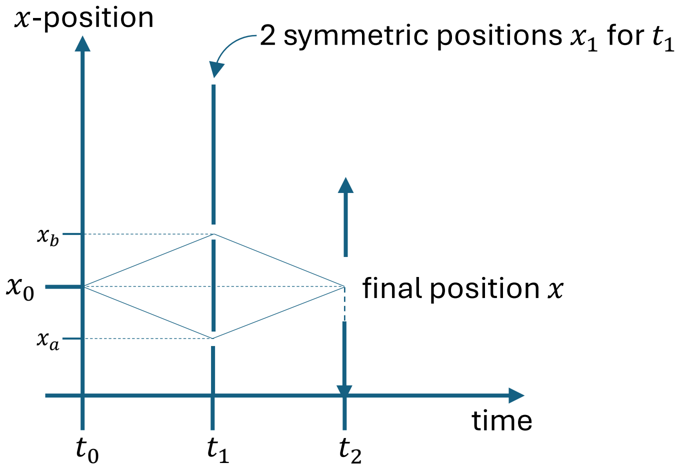


Note: the red plot was the amplitude of the double slit case.

What would happen if we varied x_b only?

We fix x_0, x_a, x_2 and vary the x_b-position from $-\infty$ to ∞ and omit all constant factors

$$x_0 = const., x_a = const., x_2 = const., x_b = x$$



Each path contributes to the probability of finding the particle at position x by raising it to the exponential.

The lower path:

$$e^{\frac{i}{\hbar}\cdot S[x_l]} = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_a - x_0)^2 + (x - x_a)^2\right)}$$

The upper path:

$$e^{\frac{i}{\hbar}\cdot S[x_u]} = e^{\frac{i}{\hbar}\cdot\frac{m}{2\Delta t}\left((x_b - x_0)^2 + (x - x_b)^2\right)}$$

We get the probability amplitude K_{02} for a particle to go from x_0 to x_2:

$$K_{02} = \sum_{paths\ x(t)} e^{\frac{i}{\hbar} \cdot S[x]} \approx \left(e^{\frac{i}{\hbar} \cdot S[x_l]} + e^{\frac{i}{\hbar} \cdot S[x_u]} \right)$$

$$e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t}((x_a - x_0)^2 + (x - x_a)^2)} + e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t}((x_b - x_0)^2 + (x - x_b)^2)} =$$

$$e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t}(x_a{}^2 - 2x_a x_0 + x_0{}^2 + x^2 - 2x_a x + x_a{}^2)} + e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t}(x_b{}^2 - 2x_b x_0 + x_0{}^2 + x^2 - 2x_b x + x_b{}^2)} =$$

$$e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_a{}^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_a x_0} e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t} x_0{}^2} e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t} x^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_a x} + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b{}^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b x_0} e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t} x_0{}^2} e^{\frac{i}{\hbar} \cdot \frac{m}{2\Delta t} x^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b x}$$

We omit all constant factors:

$$1 + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b{}^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b x_0} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b x} =$$

$$1 + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b{}^2} e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t} x_b (x_0 - x)} =$$

$$1 + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t}(x_b{}^2 - x_b x_0 + x_b x)}$$

We get the probability:

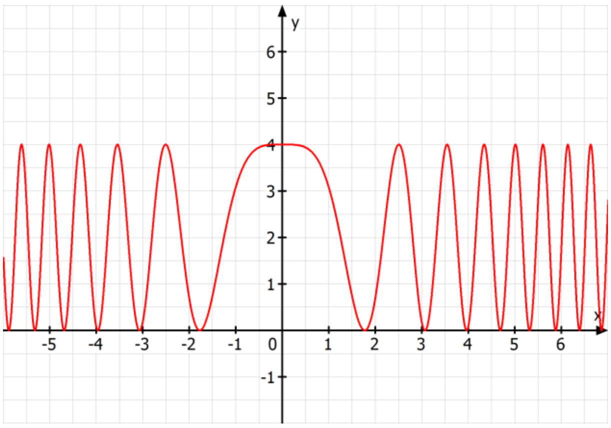
$$\left(1 + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t}(x_b{}^2 - x_b x_0 + x_b x)}\right)\left(1 + e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t}(x_b{}^2 - x_b x_0 + x_b x)}\right) =$$

$$2 + e^{-\frac{i}{\hbar} \cdot \frac{m}{\Delta t}(x_b{}^2 - x_b x_0 + x_b x)} + e^{\frac{i}{\hbar} \cdot \frac{m}{\Delta t}(x_b{}^2 - x_b x_0 + x_b x)} =$$

$$2 + 2cos\left(\frac{m}{\hbar \Delta t} x_b{}^2 - x_b x_0 + x_b x\right)$$

We use that $x_0 = x$:

$$2 + 2cos(x^2)$$



If we move x_b further outwards, we obtain oscillation with increasing frequency as the distance x_b from x_0 increases.