We analyze the model of a general two-level system in a magnetic field using the density matrix formalism instead of pure state vectors.

You may find more information at:

https://ocw.mit.edu/courses/8-05-quantum-physics-ii-fall-2013/7c376a3b5b27e5ea586a7e37b85a05d3_MIT8_05F13_Chap_07.pdf

https://phys.libretexts.org/Bookshelves/Quantum_Mechanics/Essential_Graduate_Physics_-_Quantum_Mechanics_(Likharev)/05%3A_Some_Exactly_Solvable_Problems/5.01%3A_Two-level_Systems

Hope I can help you with learning quantum mechanics.

Recall the Hamiltonian:

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{1}{2}\hbar\begin{pmatrix} \omega_0 & \omega_R e^{-i\omega t} \\ \omega_R e^{i\omega t} & -\omega_0 \end{pmatrix}$$

We can express by help of the Pauli matrix basis:

$$\hat{H} = \frac{1}{2}\hbar\begin{pmatrix} \omega_0 & V_1 - iV_2 \\ V_1 + iV_2 & -\omega_0 \end{pmatrix} = \frac{1}{2}\hbar\left(V_1\hat{\sigma}_x + V_2\hat{\sigma}_y + \omega_0\hat{\sigma}_z\right)$$

Note: $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We calculate the coefficient matrices.

$$\begin{pmatrix} 0 & V_1 \\ V_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iV_2 \\ iV_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_R e^{-i\omega t} \\ \omega_R e^{i\omega t} & 0 \end{pmatrix}$$

We have two equations:

$V_1 - iV_2 = \omega_R e^{-i\omega t}$	$V_1 + iV_2 = \omega_R e^{i\omega t}$
$V_1 = iV_2 + \omega_R e^{-i\omega t}$	
	$iV_2 + \omega_R e^{-i\omega t} + iV_2 = \omega_R e^{i\omega t}$ $2iV_2 = \omega_R\left(e^{i\omega t} - e^{-i\omega t}\right)$ $V_2 = \frac{1}{2i}\omega_R\left(e^{i\omega t} - e^{-i\omega t}\right)$ Can be noted as: $V_2 = \omega_R sin(\omega t)$
$V_1 = i\frac{1}{2i}\omega_R\left(e^{i\omega t} - e^{-i\omega t}\right) + \omega_R e^{-i\omega t}$ $V_1 = \frac{1}{2}\omega_R\left(e^{i\omega t} - e^{-i\omega t} + 2e^{-i\omega t}\right)$ $V_1 = \frac{1}{2}\omega_R\left(e^{i\omega t} + e^{-i\omega t}\right)$ Can be noted as: $V_1 = \omega_R cos(\omega t)$	

Equation of motion for Bloch vector

We take the Pauli matrix representation of the general two-level system Hamiltonian:

$$\hat{H} = \frac{1}{2}\hbar\left(V_1\hat{\sigma}_x + V_2\hat{\sigma}_y + \omega_0\hat{\sigma}_z\right)$$

We get an equation of motion for the Bloch vector, evolving under this Hamiltonian, by computing the evolution of an arbitrary general two-level system density matrix state $\vec{\rho}$, represented using the Pauli matrices:

$$\vec{\rho} = \frac{Id + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\left(Id + r_1\hat{\sigma}_x + r_2\hat{\sigma}_y + r_3\hat{\sigma}_z\right) =$$

$$\frac{1}{2}\begin{pmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix}$$

We use:

$$i\hbar\frac{d\vec{\rho}}{dt}=\left[\hat{H},\vec{\rho}\right]\rightarrow\frac{d\vec{\rho}}{dt}=-\frac{i}{\hbar}\left[\hat{H},\vec{\rho}\right]$$

We calculate the equations of motion for the coordinates r_1, r_2, r_3.

We can do this in two ways.

First by the cross product.

We use:

$$\frac{d\vec{\rho}}{dt}=\vec{\omega}\times\vec{r},\qquad \vec{\omega}=\begin{pmatrix}V_1\\V_2\\\omega_0\end{pmatrix},\qquad \vec{r}=\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}$$

We calculate:

$$\vec{\omega}\times\vec{r}=\begin{pmatrix}V_2r_3-\omega_0r_2\\\omega_0r_1-V_1r_3\\V_1r_2-V_2r_1\end{pmatrix}$$

We get the equations of motion for r_1, r_2, r_3:

$$\dot{r}_1=V_2r_3-\omega_0r_2$$

$$\dot{r}_2=\omega_0r_1-V_1r_3$$

$$\dot{r}_3=V_1r_2-V_2r_1$$

Second by calculating the commutator:

$$\frac{d\vec{\rho}}{dt}=-\frac{i}{\hbar}\left[\hat{H},\vec{\rho}\right]=-\frac{i}{\hbar}\left(\hat{H}\vec{\rho}-\vec{\rho}\hat{H}\right)$$

We use:

$$\hat{H}=\frac{1}{2}\hbar\begin{pmatrix}\omega_0 & \omega_Re^{-i\omega t}\\\omega_Re^{i\omega t} & -\omega_0\end{pmatrix}=\frac{1}{2}\hbar\begin{pmatrix}\omega_0 & V_1-iV_2\\V_1+iV_2 & -\omega_0\end{pmatrix}$$

$$\rho=\begin{pmatrix}\rho_{11} & \rho_{12}\\\rho_{21} & \rho_{22}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}r_0+r_3 & r_1-ir_2\\r_1+ir_2 & r_0-r_3\end{pmatrix}$$

We get:

$$\hat{H}\rho=\frac{1}{4}\hbar\begin{pmatrix}\omega_0 & V_1-iV_2\\V_1+iV_2 & -\omega_0\end{pmatrix}\begin{pmatrix}r_0+r_3 & r_1-ir_2\\r_1+ir_2 & r_0-r_3\end{pmatrix}=$$

$$\frac{1}{4}\hbar\begin{pmatrix}\omega_0(r_0+r_3)+(V_1-iV_2)(r_1+ir_2) & \omega_0(r_1-ir_2)+(V_1-iV_2)(r_0-r_3)\\(V_1+iV_2)(r_0+r_3)-\omega_0(r_1+ir_2) & (V_1+iV_2)(r_1-ir_2)-\omega_0(r_0-r_3)\end{pmatrix}$$

$$\rho\hat{H}=\frac{1}{4}\hbar\begin{pmatrix}r_0+r_3 & r_1-ir_2\\r_1+ir_2 & r_0-r_3\end{pmatrix}\begin{pmatrix}\omega_0 & V_1-iV_2\\V_1+iV_2 & -\omega_0\end{pmatrix}=$$

$$\frac{1}{4}\hbar\begin{pmatrix}(r_0+r_3)\omega_0+(r_1-ir_2)(V_1+iV_2) & (r_0+r_3)(V_1-iV_2)-(r_1-ir_2)\omega_0\\(r_1+ir_2)\omega_0+(r_0-r_3)(V_1+iV_2) & (r_1+ir_2)(V_1-iV_2)-(r_0-r_3)\omega_0\end{pmatrix}$$

We build the differences $\hat{H}\rho - \rho\hat{H}$:

$\hat{H}\rho_{11} - \rho\hat{H}_{11}$:

$$\omega_0(r_0 + r_3) + (V_1 - iV_2)(r_1 + ir_2) - (r_0 + r_3)\omega_0 - (r_1 - ir_2)(V_1 + iV_2) =$$
$$(V_1 - iV_2)(r_1 + ir_2) - (r_1 - ir_2)(V_1 + iV_2) =$$
$$V_1r_1 - iV_2r_1 + iV_1r_2 + V_2r_2 - (V_1r_1 + iV_2r_1 - iV_1r_2 + V_2r_2) =$$
$$V_1r_1 - iV_2r_1 + iV_1r_2 + V_2r_2 - V_1r_1 - iV_2r_1 + iV_1r_2 - V_2r_2 =$$
$$-2iV_2r_1 + 2iV_1r_2$$

$\hat{H}\rho_{12} - \rho\hat{H}_{12}$:

$$\omega_0(r_1 - ir_2) + (V_1 - iV_2)(r_0 - r_3) - \big((r_0 + r_3)(V_1 - iV_2) - (r_1 - ir_2)\omega_0\big) =$$
$$\omega_0r_1 - \omega_0ir_2 + V_1r_0 - V_1r_3 - iV_2r_0 + iV_2r_3 - (r_0V_1 - r_0iV_2 + r_3V_1 - r_3iV_2 - r_1\omega_0 + ir_2\omega_0) =$$
$$\omega_0r_1 - \omega_0ir_2 + V_1r_0 - V_1r_3 - iV_2r_0 + iV_2r_3 - r_0V_1 + r_0iV_2 - r_3V_1 + r_3iV_2 + r_1\omega_0 - ir_2\omega_0 =$$
$$\omega_0r_1 - \omega_0ir_2 + V_1r_0 - V_1r_3 - iV_2r_0 + iV_2r_3 - r_0V_1 + r_0iV_2 - r_3V_1 + r_3iV_2 + r_1\omega_0 - ir_2\omega_0 =$$
$$2\omega_0r_1 - 2\omega_0ir_2 - 2V_1r_3 + 2iV_2r_3$$

$\hat{H}\rho_{21} - \rho\hat{H}_{21}$:

$$(V_1 + iV_2)(r_0 + r_3) - \omega_0(r_1 + ir_2) - \big((r_1 + ir_2)\omega_0 + (r_0 - r_3)(V_1 + iV_2)\big) =$$
$$V_1r_0 + V_1r_3 + iV_2r_0 + iV_2r_3 - \omega_0r_1 - \omega_0ir_2 - (\omega_0r_1 + \omega_0ir_2 + V_1r_0 + iV_2r_0 - V_1r_3 - iV_2r_3) =$$
$$V_1r_0 + V_1r_3 + iV_2r_0 + iV_2r_3 - \omega_0r_1 - \omega_0ir_2 - \omega_0r_1 - \omega_0ir_2 - V_1r_0 - iV_2r_0 + V_1r_3 + iV_2r_3 =$$
$$2V_1r_3 + 2iV_2r_3 - 2\omega_0r_1 - 2\omega_0ir_2$$

$\hat{H}\rho_{22} - \rho\hat{H}_{22}$:

$$(V_1 + iV_2)(r_1 - ir_2) - \omega_0(r_0 - r_3) - \big((r_1 + ir_2)(V_1 - iV_2) - (r_0 - r_3)\omega_0\big) =$$
$$V_1r_1 - V_1ir_2 + iV_2r_1 + V_2r_2 - \omega_0r_0 + \omega_0r_3 - (r_1V_1 - r_1iV_2 + ir_2V_1 + r_2V_2 - \omega_0r_0 + \omega_0r_3) =$$
$$V_1r_1 - V_1ir_2 + iV_2r_1 + V_2r_2 - \omega_0r_0 + \omega_0r_3 - r_1V_1 + r_1iV_2 - ir_2V_1 - r_2V_2 + \omega_0r_0 - \omega_0r_3 =$$
$$V_1r_1 - V_1ir_2 + iV_2r_1 + V_2r_2 - \omega_0r_0 + \omega_0r_3 - r_1V_1 + r_1iV_2 - ir_2V_1 - r_2V_2 + \omega_0r_0 - \omega_0r_3 =$$
$$-2V_1ir_2 + 2iV_2r_1$$

We get the complete commutator:

$$\frac{1}{4}\hbar\left(\cdot - \frac{i}{\hbar}\begin{pmatrix} -2iV_2r_1 + 2iV_1r_2 & 2\omega_0r_1 - 2\omega_0ir_2 - 2V_1r_3 + 2iV_2r_3 \\ 2V_1r_3 + 2iV_2r_3 - 2\omega_0r_1 - 2\omega_0ir_2 & -2V_1ir_2 + 2iV_2r_1 \end{pmatrix}\right)$$
$$-\frac{i}{2}\begin{pmatrix} -iV_2r_1 + iV_1r_2 & \omega_0r_1 - \omega_0ir_2 - V_1r_3 + iV_2r_3 \\ V_1r_3 + iV_2r_3 - \omega_0r_1 - \omega_0ir_2 & -V_1ir_2 + iV_2r_1 \end{pmatrix}$$
$$\frac{1}{2}\begin{pmatrix} -V_2r_1 + V_1r_2 & -i\omega_0r_1 - \omega_0r_2 + iV_1r_3 + V_2r_3 \\ -iV_1r_3 + V_2r_3 + i\omega_0r_1 - \omega_0r_2 & -V_1r_2 + V_2r_1 \end{pmatrix}$$
$$\frac{1}{2}\begin{pmatrix} V_1r_2 - V_2r_1 & i(V_1r_3 - \omega_0r_1) + V_2r_3 - \omega_0r_2 \\ i(\omega_0r_1 - V_1r_3) + V_2r_3 - \omega_0r_2 & V_2r_1 - V_1r_2 \end{pmatrix}$$

We use the equation of motion:

$$\vec{\rho} = \frac{Id + \vec{r} \cdot \vec{\sigma}}{2} = \frac{1}{2}\left(Id + r_1\hat{\sigma}_x + r_2\hat{\sigma}_y + r_3\hat{\sigma}_z\right) \rightarrow$$

$$\frac{d\vec{\rho}}{dt} = \frac{1}{2}\left(\dot{r}_1\hat{\sigma}_x + \dot{r}_2\hat{\sigma}_y + \dot{r}_3\hat{\sigma}_z\right)$$

We use the Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We expand:

$$\dot{r}_1\hat{\sigma}_x + \dot{r}_2\hat{\sigma}_y + \dot{r}_3\hat{\sigma}_z = \begin{pmatrix} 0 & \dot{r}_1 \\ \dot{r}_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\dot{r}_2 \\ i\dot{r}_2 & 0 \end{pmatrix} + \begin{pmatrix} \dot{r}_3 & 0 \\ 0 & -\dot{r}_3 \end{pmatrix} =$$

$$\begin{pmatrix} \dot{r}_3 & \dot{r}_1 - i\dot{r}_2 \\ \dot{r}_1 + i\dot{r}_2 & -\dot{r}_3 \end{pmatrix}$$

We compare:

$$\frac{1}{2}\begin{pmatrix} V_1r_2 - V_2r_1 & i(V_1r_3 - \omega_0r_1) + V_2r_3 - \omega_0r_2 \\ i(\omega_0r_1 - V_1r_3) + V_2r_3 - \omega_0r_2 & V_2r_1 - V_1r_2 \end{pmatrix}$$

We get:

$\dot{r}_1 = V_2r_3 - \omega_0r_2$	$\dot{r}_2 = \omega_0r_1 - V_1r_3$	$\dot{r}_3 = V_1r_2 - V_2r_1$

This fits with the cross product:

$$\vec{\omega} \times \vec{r} = \begin{pmatrix} V_2r_3 - \omega_0r_2 \\ \omega_0r_1 - V_1r_3 \\ V_1r_2 - V_2r_1 \end{pmatrix}$$