The four Bell states are:

$$|\theta^{+}\rangle = \frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle)$$

$$|\theta^{-}\rangle = \frac{1}{\sqrt{2}} \cdot (|00\rangle - |11\rangle)$$

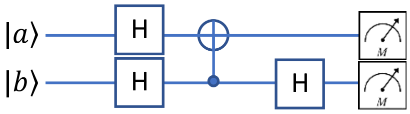
$$|\psi^{+}\rangle = \frac{1}{\sqrt{2}} \cdot (|01\rangle + |10\rangle)$$

$$|\psi^{-}\rangle = \frac{1}{\sqrt{2}} \cdot (|01\rangle - |10\rangle)$$

The Hadamard

- acts on the one qubit only
- changes $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ resp. to $|+\rangle$
- changes $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ resp. to $|-\rangle$.

We use the following circuit:



Note: $|a\rangle$ and $|b\rangle$ represent the input bits of the Bell states. The desired result is:

$$|\theta^{+}\rangle \to |00\rangle, \qquad |\theta^{-}\rangle \to |01\rangle, \qquad |\psi^{+}\rangle \to |10\rangle, \qquad |\psi^{-}\rangle \to |11\rangle$$

Conceptual level

Bell state one:

$$|\theta^{+}\rangle = \frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle)$$

Step 1: We apply the Hadamard:

$$H|\theta^{+}\rangle = \frac{1}{2\sqrt{2}} \cdot \big((|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\big) =$$

$$\frac{1}{2\sqrt{2}} \cdot (|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle) =$$

$$\frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle)$$

Step 2: We apply the topdown-CNOT. The CNOT reverses qubit one if qubit two equals one.

$$\mathrm{CNOT}\frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \cdot (|00\rangle + |01\rangle)$$

Step 3: We apply the Hadamard to qubit two.

$$\mathrm{H}\frac{1}{\sqrt{2}} \cdot (|00\rangle + |01\rangle) = \frac{1}{2} \cdot \big(|0\rangle(|0\rangle + |1\rangle) + |0\rangle(|0\rangle - |1\rangle)\big) = |00\rangle$$

Bell state 2:

Step 1: We apply the Hadamard:

$$H|\theta^{-}\rangle = \frac{1}{2\sqrt{2}} \cdot \big((|0\rangle + |1\rangle)(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\big) =$$

$$\frac{1}{2\sqrt{2}} \cdot (|00\rangle + |01\rangle + |10\rangle + |11\rangle - |00\rangle + |01\rangle + |10\rangle - |11\rangle) =$$

$$\frac{1}{\sqrt{2}} \cdot (|01\rangle + |10\rangle)$$

Step 2: We apply the topdown-CNOT. The CNOT reverses qubit one if qubit two equals one.

$$\mathrm{CNOT}\frac{1}{\sqrt{2}} \cdot (|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \cdot (|11\rangle + |10\rangle)$$

Step 3: We apply the Hadamard to qubit two.

$$\mathrm{H}\frac{1}{\sqrt{2}} \cdot (|11\rangle + |10\rangle) = \frac{1}{2} \cdot \big(|1\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + |1\rangle)\big) = |10\rangle$$

Bell state 3:

$$|\psi^{+}\rangle = \frac{1}{\sqrt{2}} \cdot (|01\rangle + |10\rangle)$$

Step 1: We apply the Hadamard:

$$H|\psi^{+}\rangle = \frac{1}{2\sqrt{2}} \cdot \big((|0\rangle + |1\rangle)(|0\rangle - |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)\big) =$$

$$\frac{1}{2\sqrt{2}} \cdot (|00\rangle - |01\rangle + |10\rangle - |11\rangle + |00\rangle + |01\rangle - |10\rangle - |11\rangle) =$$

$$\frac{1}{\sqrt{2}} \cdot (|00\rangle - |11\rangle)$$

Step 2: We apply the topdown-CNOT. The CNOT reverses qubit one if qubit two equals one.

$$\mathrm{CNOT}\frac{1}{\sqrt{2}} \cdot (|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \cdot (|00\rangle - |01\rangle)$$

Step 3: We apply the Hadamard to qubit two.

$$\mathrm{H}\frac{1}{\sqrt{2}} \cdot (|00\rangle - |01\rangle) = \frac{1}{2} \cdot \big(|0\rangle(|0\rangle + |1\rangle) - |0\rangle(|0\rangle - |1\rangle)\big) = |01\rangle$$

Bell state 4:

$$|\psi^{-}\rangle = \frac{1}{\sqrt{2}} \cdot (|01\rangle - |10\rangle)$$

Step 1: We apply the Hadamard:

$$H|\psi^{-}\rangle = \frac{1}{2\sqrt{2}} \cdot \big((|0\rangle + |1\rangle)(|0\rangle - |1\rangle) - (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)\big) =$$

$$\frac{1}{2\sqrt{2}} \cdot (|00\rangle - |01\rangle + |10\rangle - |11\rangle - |00\rangle - |01\rangle + |10\rangle + |11\rangle) =$$

$$\frac{1}{\sqrt{2}} \cdot (|10\rangle - |01\rangle)$$

Step 2: We apply the topdown-CNOT. The CNOT reverses qubit one if qubit two equals one.

$$\mathrm{CNOT} \frac{1}{\sqrt{2}} \cdot (|10\rangle - |01\rangle) = \frac{1}{\sqrt{2}} \cdot (|10\rangle - |11\rangle)$$

Step 3: We apply the Hadamard to qubit two.

$$\mathrm{H} \frac{1}{\sqrt{2}} \cdot (|10\rangle - |11\rangle) = \frac{1}{2} \cdot \big(|1\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle - |1\rangle)\big) = |11\rangle$$

Basic level

We can do the same calculation on the basic level.

Bell state one:

$$|\theta^{+}\rangle = \frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete Hadamard:

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Step 1: We apply the Hadamard:

$$H \otimes H |\theta^{+}\rangle =$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The topdown CNOT:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Step 2: We apply the topdown-CNOT.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The Hadamard:

$$I \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Step 3: We apply the Hadamard to qubit two.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Bell state 2:

$$|\theta^{-}\rangle=\frac{1}{\sqrt{2}}\cdot(|00\rangle-|11\rangle)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

Step 1: We apply the Hadamard:

$$H\otimes H|\theta^{-}\rangle=$$

$$\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}0\\2\\2\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

Step 2: We apply the topdown-CNOT.

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

Step 3: We apply the Hadamard to qubit two.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\0\\2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Bell state 3:

$$|\psi^{+}\rangle=\frac{1}{\sqrt{2}}\cdot(|01\rangle+|10\rangle)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$H\otimes H|\psi^{+}\rangle=$$

$$\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}2\\0\\0\\-2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

Step 2: We apply the topdown-CNOT.

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

Step 3: We apply the Hadamard to qubit two.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\ 2\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}$$

Bell state 4:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\cdot(|01\rangle - |10\rangle)$$

Step 1: We apply the Hadamard:

$$H \otimes H|\psi^-\rangle =$$

$$\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ -1\\ 0\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}0\\ -2\\ 2\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ -1\\ 1\\ 0\end{pmatrix}$$

Step 2: We apply the topdown-CNOT.

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ -1\\ 1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 1\\ -1\end{pmatrix}$$

Step 3: We apply the Hadamard to qubit two.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 1\\ -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\ 0\\ 0\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}$$

We compare high-level solution with low-level solution.

	high-level	low-level
$\lvert\theta^+\rangle = \frac{1}{\sqrt{2}}\cdot(\lvert 00\rangle + \lvert 11\rangle)$	$\lvert 00\rangle = \lvert 0\rangle \otimes \lvert 0\rangle = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}$
$\lvert\theta^-\rangle = \frac{1}{\sqrt{2}}\cdot(\lvert 00\rangle - \lvert 11\rangle)$	$\lvert 10\rangle = \lvert 1\rangle \otimes \lvert 0\rangle = \begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix}$
$\lvert\psi^+\rangle = \frac{1}{\sqrt{2}}\cdot(\lvert 01\rangle + \lvert 10\rangle)$	$\lvert 01\rangle = \lvert 0\rangle \otimes \lvert 1\rangle = \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}$
$\lvert\psi^-\rangle = \frac{1}{\sqrt{2}}\cdot(\lvert 01\rangle - \lvert 10\rangle)$	$\lvert 11\rangle = \lvert 1\rangle \otimes \lvert 1\rangle = \begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}$