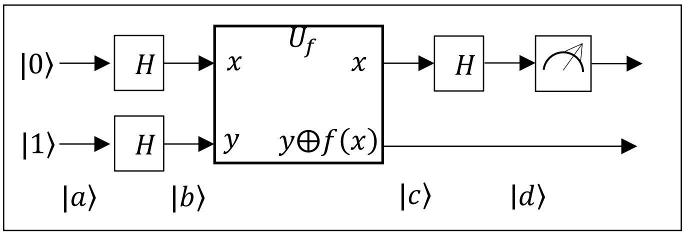
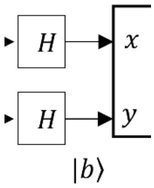
This is an extended version of https://en.wikipedia.org/wiki/Deutsch-Jozsa_algorithm

We interpret the quantum circuit.

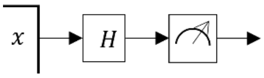


Note: $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ are 4D state vectors.

After the two Hadamard 

we have x and y in superposition.

In superposition the action of $y \oplus f(x)$ does not affect the y-branch only but affects the x-branch too.

With this it makes sense to evaluate 

in the upper x-branch.

In detail

We have a function $f: \{0,1\} \to \{0,1\}$.

The function given, is it constant $f(0) = f(1)$ or variable: $f(0) \neq f(1)$? We know the result of f, not the explicit definition. A classical computer needs two runs to decide. It must calculate $f(0)$ as well as $f(1)$.

What we have is:

$U_f: |x\rangle \otimes |y\rangle \to |x\rangle \otimes |f(x) \oplus y\rangle$

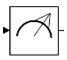
$\oplus$: addition modulo 2

$0 \oplus 0 = 0, \qquad 0 \oplus 1 = 1, \qquad 1 \oplus 0 = 1, \qquad 1 \oplus 1 = 0$

$\boxed{H}$: Hadamard operator:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

 measuring, $\otimes$ tensor product

Steps

Step 1 initializing:

$$|x\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |y\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to |x\rangle \otimes |y\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} =: |a\rangle$$

Step 2 applying the Hadamard transformation to both qubits:

$$|x\rangle \to H|x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$$

$$|y\rangle \to H|y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$$

$$|x\rangle\,|y\rangle \to H|x\rangle \otimes H|y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} =: |b\rangle$$

Alternatively, we could write:

$$|x\rangle|y\rangle \to (H\otimes H)(|x\rangle \otimes |y\rangle)$$

$$H\otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}$$

$$|x\rangle \otimes |y\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$(H\otimes H)(|x\rangle \otimes |y\rangle) = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} =: |b\rangle$$

Step 3 evaluating the function $U_f := |y\rangle \oplus f(|x\rangle)$:

$$|x\rangle \otimes |y\rangle \to |x\rangle \otimes U_f(|y\rangle) =$$

$$\frac{1}{2}(|0\rangle \otimes |0 \oplus f(0)\rangle - |0\rangle \otimes |1 \oplus f(0)\rangle + |1\rangle \otimes |0 \oplus f(1)\rangle - |1\rangle \otimes |1 \oplus f(1)\rangle) =;$$

Note: $0 \oplus f(0) = f(0)$ *and* $0 \oplus f(1) = f(1)$ *due to* $\oplus$*: addition modulo 2.*

$$\frac{1}{2}(|0\rangle \otimes |f(0)\rangle - |0\rangle \otimes |1 \oplus f(0)\rangle + |1\rangle \otimes |f(1)\rangle - |1\rangle \otimes |1 \oplus f(1)\rangle) =$$

$$\frac{1}{2}(|0\rangle \otimes [0 \oplus |f(0)\rangle - 1 \oplus |f(0)\rangle] + |1\rangle \otimes [|f(1)\rangle - 1 \oplus f(1)\rangle]) =$$

$$\frac{1}{2}\left((-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle \otimes (|0\rangle - |1\rangle)\right) =$$

Note: $|f(0)\rangle$ *is a vector,* $f(0)$ *a number,* 0 *or* 1*.*

We have four possibilities for $f(x)$:

$f(0) = 0$ $f(1) = 0$	$f(0) = 1$ $f(1) = 1$	$f(0) = 0$ $f(1) = 1$	$f(0) = 1$ $f(1) = 0$

We check all possibilities:

$f(0)=0$ and $f(1)=0$: $\frac{1}{2}\Big(\big((-1)^{f(0)}\|0\rangle\otimes(\|0\rangle-\|1\rangle\big) + (-1)^{f(1)}\|1\rangle\otimes(\|0\rangle-\|1\rangle)\Big) = \frac{1}{2}\big(\|0\rangle\otimes(\|0\rangle-\|1\rangle)+\|1\rangle\otimes(\|0\rangle-\|1\rangle)\big)$	$f(0)=1$ and $f(1)=1$: $\frac{1}{2}\Big(\big((-1)^{f(0)}\|0\rangle\otimes(\|0\rangle-\|1\rangle\big) + (-1)^{f(1)}\|1\rangle\otimes(\|0\rangle-\|1\rangle)\Big) = \frac{1}{2}\big(-\|0\rangle\otimes(\|0\rangle-\|1\rangle)-\|1\rangle\otimes(\|0\rangle-\|1\rangle)\big)$
$f(0)=0$ and $f(1)=1$: $\frac{1}{2}\Big(\big((-1)^{f(0)}\|0\rangle\otimes(\|0\rangle-\|1\rangle\big) + (-1)^{f(1)}\|1\rangle\otimes(\|0\rangle-\|1\rangle)\Big) = \frac{1}{2}\big(\|0\rangle\otimes(\|0\rangle-\|1\rangle)-\|1\rangle\otimes(\|0\rangle-\|1\rangle)\big)$	$f(0)=1$ and $f(1)=0$: $\frac{1}{2}\Big(\big((-1)^{f(0)}\|0\rangle\otimes(\|0\rangle-\|1\rangle\big) + (-1)^{f(1)}\|1\rangle\otimes(\|0\rangle-\|1\rangle)\Big) = -\frac{1}{2}\big(\|0\rangle\otimes(\|0\rangle-\|1\rangle)-\|1\rangle\otimes(\|0\rangle-\|1\rangle)\big)$

We compare with:

$$\frac{1}{2}\big(|0\rangle\otimes[0\oplus|f(0)\rangle-1\oplus|f(0)\rangle]+|1\rangle\otimes[|f(1)\rangle-1\oplus f(1)\rangle]\big)$$

The last case $f(0)=1$ and $f(1)=0$ gives an overall phase of -1 that is not measurable in quantum mechanics.

The step from

$$\frac{1}{2}\big(|0\rangle\otimes[0\oplus|f(0)\rangle-1\oplus|f(0)\rangle]+|1\rangle\otimes[|f(1)\rangle-1\oplus f(1)\rangle]\big)$$

to

$$\frac{1}{2}\Big(\big((-1)^{f(0)}|0\rangle\otimes(|0\rangle-|1\rangle\big)+(-1)^{f(1)}|1\rangle\otimes(|0\rangle-|1\rangle)\Big)$$

is correct.

We rewrite:

$$\frac{1}{2}\Big(\big((-1)^{f(0)}|0\rangle\otimes(|0\rangle-|1\rangle\big)+(-1)^{f(1)}|1\rangle\otimes(|0\rangle-|1\rangle)\Big) =$$

$$\frac{1}{2}\big((-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle\big)\otimes(|0\rangle-1\rangle) := |c\rangle$$

If f is constant, $f(0)=f(1)$ we get:	If f is variable, $f(0)\neq f(1)$ we get:
$\frac{1}{2}\big((-1)^{f(0)}\|0\rangle+(-1)^{f(1)}\|1\rangle\big)\otimes(\|0\rangle-1\rangle) = \frac{(-1)^{f(0)}}{2}\big((\|0\rangle+\|1\rangle)\big)\otimes(\|0\rangle-1\rangle) = \pm\frac{1}{2}\big((\|0\rangle+\|1\rangle)\big)\otimes(\|0\rangle-1\rangle) = \pm\left(\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)\right)\otimes\frac{1}{\sqrt{2}}(\|0\rangle-1\rangle)$	$\frac{1}{2}\big((-1)^{f(0)}\|0\rangle+(-1)^{f(1)}\|1\rangle\big)\otimes(\|0\rangle-1\rangle) = \pm\frac{1}{2}\big((\|0\rangle-\|1\rangle)\big)\otimes(\|0\rangle-1\rangle)$ $\pm\left(\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)\right)\otimes\frac{1}{\sqrt{2}}(\|0\rangle-1\rangle)$

Note: for f variable we get alternative signs from $(-1)^{f(0)}$ resp. $(-1)^{f(1)}$.

In terms of superposition, we get:

$\pm\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right)\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}=\pm\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$	$\pm\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right)\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}=\pm\frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$

Step 4 applying the Hadamard transformation to the first qubit $\vert x\rangle \rightarrow H\vert x\rangle$:

Case f constant:	Case f variable:
$\vert x\rangle \rightarrow H\vert x\rangle$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}((\vert 0\rangle+\vert 1\rangle))=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}\left(\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)\right)=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}=$ $\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=$ $\frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}=\vert 0\rangle \coloneqq \vert d\rangle$	$\vert x\rangle \rightarrow H\vert x\rangle$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}((\vert 0\rangle-\vert 1\rangle))=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}\left(\left(\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\right)\right)=$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}=$ $\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=$ $\frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}=\vert 1\rangle \coloneqq \vert d\rangle$

In terms of superposition applying the Hadamard transformation to the first qubit $\vert x\rangle \rightarrow H\vert x\rangle$:

$$H\otimes Id=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}$$

$\pm\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\pm\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\pm\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$
We disassemble the 4D vector $\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}=\vert 00\rangle-\vert 01\rangle=$ $\vert 0\rangle(\vert 0\rangle-\vert 1\rangle)$	We disassemble the 4D vector $\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}=\vert 10\rangle-\vert 11\rangle=$ $\vert 1\rangle(\vert 0\rangle-\vert 1\rangle)$
Measuring the first qubit will always give zero.	Measuring the first qubit will always give one.

Result

If f is constant we get the qubit $\vert 0\rangle$ as result when measuring the first qubit.

If f is variable we get the qubit $\vert 1\rangle$ as result when measuring the first qubit.

With one run we can distinguish whether f is constant or variable.