This paper follows a lecture Peter Shor has given at

https://openlearninglibrary.mit.edu/courses/course-
v1:MITx+8.370.2x+1T2018/courseware/Week2/lectures_U2_2_dj_qcirc/6?activate_block_id=block-
v1%3AMITx%2B8.370.2x%2B1T2018%2Btype%40vertical%2Bblock%40Classical_algorithm_solving_Deutschs_problem

We use a function $f(x): \{0,1\} \rightarrow \{0,1\}$.

$f(x)$ is either "constant" or "balanced".

$f(x)$ is constant: $f(x) = 0 \ \forall x \in \{0,1\}$ or $f(x) = 1 \ \forall x \in \{0,1\}$

$f(x)$ is balanced: $f(x_1) \neq f(x_2)$.

We use a three-bit quantum computer starting in the state $|+++\rangle$.

$\oplus$ denotes addition modulo 2.

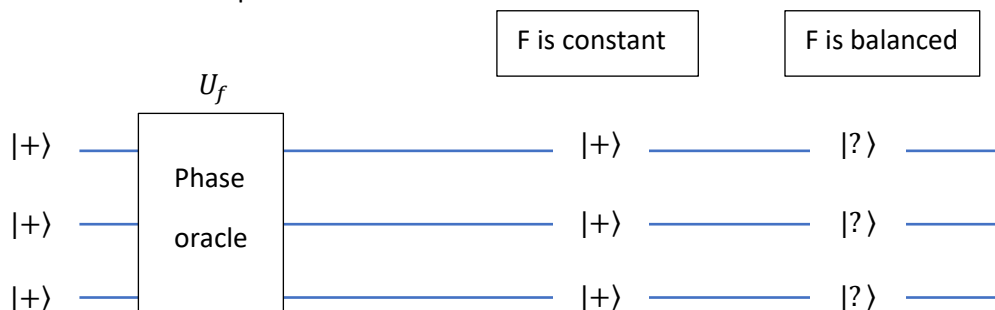$\otimes$ denotes the Kronecker product or tensor product.

We call the unitary transformation matrix U_f the phase-oracle.

The oracle acts on qubits:

$$(-1)^{f(x)}|0\rangle \text{ resp. } (-1)^{f(x)}|1\rangle$$

From this definition of the oracle we see that:

- If the function is constant, the input remains unchanged because all qubits are multiplied by either $+1$ or -1 which gives a constant additional phase that doesn't change measurements.
- If the function is balanced, there is at least one qubit that changes sign, the input function is different to the output function.

Note: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Input:

$$|+++\rangle = |+\rangle \otimes |+\rangle \otimes |+\rangle =$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$$

$$\frac{1}{\sqrt{8}}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) =$$

$$\frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

Output:

$$(-1)^{f(x)}|+\rangle(-1)^{f(x)}|+\rangle(-1)^{f(x)}|+\rangle =$$

$$\frac{1}{\sqrt{2}}(-1)^{f(x)}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(-1)^{f(x)}(|0\rangle + |1\rangle)(-1)^{f(x)}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$$

$$\frac{1}{\sqrt{8}}\left((-1)^{f(x)}|0\rangle + (-1)^{f(x)}|1\rangle\right)\left((-1)^{f(x)}|0\rangle + (-1)^{f(x)}|1\rangle\right)\left((-1)^{f(x)}|0\rangle + (-1)^{f(x)}|1\rangle\right) =$$

$$\frac{1}{\sqrt{8}}\left((-1)^{3\cdot f(0)}|000\rangle + (-1)^{2\cdot f(0)+f(1)}|001\rangle + (-1)^{2\cdot f(0)+f(1)}|010\rangle + (-1)^{f(0)+2\cdot f(1)}|011\rangle\right.$$
$$+ (-1)^{2\cdot f(0)+f(1)}|100\rangle + (-1)^{f(0)+2\cdot f(1)}|101\rangle + (-1)^{f(0)+2\cdot f(1)}|110\rangle$$
$$\left.+ (-1)^{3\cdot f(1)}|111\rangle\right)$$

The possibility to get output $|+ + +\rangle$ is the inner product of input and output:

$$p(|+ + +\rangle) = \left|\langle + + +|U_f|+ + +\rangle\right|^2$$

This is too cumbersome to write out. We use that the vectors $(|000\rangle, |001\rangle, \dots, |111\rangle)$ are orthonormal, meaning that $\langle 000|000\rangle = 1, \langle 000|001\rangle = 0$ etc. for all other combinations.

The scalar product $\langle + + +|U_f|+ + +\rangle$ then gives:

$$\frac{1}{8}\left((-1)^{3\cdot f(0)} + (-1)^{2\cdot f(0)+f(1)} + (-1)^{2\cdot f(0)+f(1)} + (-1)^{f(0)+2\cdot f(1)} + (-1)^{2\cdot f(0)+f(1)}\right.$$
$$\left.+ (-1)^{f(0)+2\cdot f(1)} + (-1)^{f(0)+2\cdot f(1)} + (-1)^{f(0)+2\cdot f(1)}\right) =$$

$$\frac{1}{8}\left((-1)^{3\cdot f(0)} + 3\cdot(-1)^{2\cdot f(0)+f(1)} + 3\cdot(-1)^{f(0)+2\cdot f(1)} + (-1)^{3\cdot f(1)}\right)$$

Constant case (all modulo 2):

$f(0) = 0, f(1) = 0$	$f(0) = 1, f(1) = 1$
$f(0) + f(0) = 0 + 0 = 0$ $f(1) + f(1) = 0 + 0 = 0$	$f(0) + f(0) = 1 + 1 = 0$ $f(1) + f(1) = 0 + 0 = 0$
$f(0) + f(0) + f(0) = 0 + 0 + 0 = 0$ $f(1) + f(1) + f(1) = 0 + 0 + 0 = 0$	$f(0) + f(0) + f(0) = 1 + 1 + 1 = 1$ $f(1) + f(1) + f(1) = 1 + 1 + 1 = 1$

Balanced case (all modulo 2):

| $f(0) = 0, f(1) = 1$	$f(0) = 1, f(1) = 0$
$f(0) + f(0) = 0 + 0 = 0$ $f(1) + f(1) = 1 + 1 = 0$	$f(0) + f(0) = 1 + 1 = 0$ $f(1) + f(1) = 0 + 0 = 0$
$f(0) + f(0) + f(0) = 0 + 0 + 0 = 0$ $f(1) + f(1) + f(1) = 1 + 1 + 1 = 1$	$f(0) + f(0) + f(0) = 1 + 1 + 1 = 1$ $f(1) + f(1) + f(1) = 0 + 0 + 0 = 0$

We check the constant case $f(0) = 0, f(1) = 0$:	We check the constant case $f(0) = 1, f(1) = 1$:
$\frac{1}{8}\big((-1)^{3\cdot f(0)} + 3\cdot(-1)^{2\cdot f(0)+f(1)} + 3\cdot(-1)^{f(0)+2\cdot f(1)}$ $+ (-1)^{3\cdot f(1)}\big) =$ $\frac{1}{8}\big((-1)^0 + 3\cdot(-1)^0 + 3\cdot(-1)^0 + (-1)^0\big) =$ $\frac{1}{8}(1 + 3 + 3 + 1) = 1$	$\frac{1}{8}\big((-1)^{3\cdot f(0)} + 3\cdot(-1)^{2\cdot f(0)+f(1)} + 3\cdot(-1)^{f(0)+2\cdot f(1)}$ $+ (-1)^{3\cdot f(1)}\big) =$ $\frac{1}{8}\big((-1)^1 + 3\cdot(-1)^1 + 3\cdot(-1)^1 + (-1)^1\big) =$ $\frac{1}{8}(-1 - 3 - 3 - 1) = -1$

We check the balanced case $f(0) = 0, f(1) = 1$:	We check the balanced case $f(0) = 1, f(1) = 0$:
$\frac{1}{8}\big((-1)^{3\cdot f(0)} + 3\cdot(-1)^{2\cdot f(0)+f(1)} + 3\cdot(-1)^{f(0)+2\cdot f(1)}$ $+ (-1)^{3\cdot f(1)}\big) =$ $\frac{1}{8}\big((-1)^0 + 3\cdot(-1)^1 + 3\cdot(-1)^0 + (-1)^1\big) =$ $\frac{1}{8}(1 - 3 + 3 - 1) = 0$	$\frac{1}{8}\big((-1)^{3\cdot f(0)} + 3\cdot(-1)^{2\cdot f(0)+f(1)} + 3\cdot(-1)^{f(0)+2\cdot f(1)}$ $+ (-1)^{3\cdot f(1)}\big) =$ $\frac{1}{8}\big((-1)^1 + 3\cdot(-1)^0 + 3\cdot(-1)^1 + (-1)^0\big) =$ $\frac{1}{8}(-1 + 3 - 3 + 1) = 0$

The probability to get output $|+++\rangle$ is the square of the expectation value, $\left|\langle+++|U_f|+++\rangle\right|^2$ and is one if the function is constant, zero if the function is balanced.

We can decide with one application of the oracle whether the function is constant or balanced by measuring the result in the $|+\rangle / |-\rangle$ basis.

Usually we work in the $|0\rangle / |1\rangle$ basis. We modify the circuit:
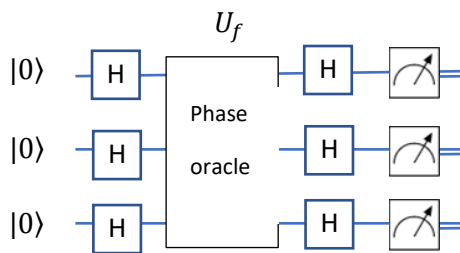
The measurement will give 0,0,0 only if the function is constant.

Basic level

We use the input $|000\rangle = |000\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

We apply the Hadamard on each line:

$$H\otimes H\otimes H =$$

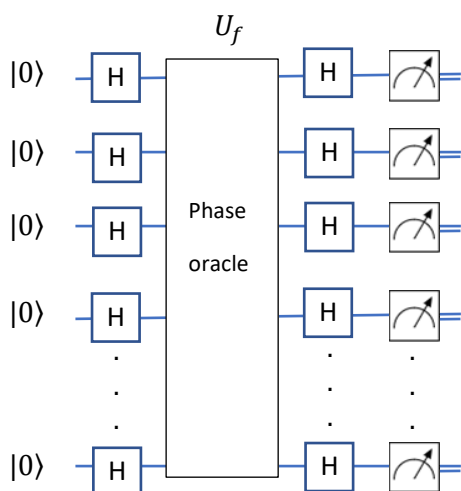$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\otimes\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\otimes\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) =$$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)\otimes\frac{1}{\sqrt{2^2}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Basic level

We use parallel a balanced and a constant function.

Balanced	Constant
In both cases we get as entrance to the oracle: $$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$	
The phase oracle applies phase $+1$ or -1, e. g.:	The phase oracle applies constant phase $+1$:

Note: zeros omitted for better readability.

We apply the balanced oracle: $$\frac{1}{\sqrt{8}}\begin{pmatrix} +1 \\ & +1 \\ & & -1 \\ & & & +1 \\ & & & & -1 \\ & & & & & -1 \\ & & & & & & -1 \\ & & & & & & & +1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$	We apply the constant oracle: $$\frac{1}{\sqrt{8}}\begin{pmatrix} +1 \\ & +1 \\ & & +1 \\ & & & +1 \\ & & & & +1 \\ & & & & & +1 \\ & & & & & & +1 \\ & & & & & & & +1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
We apply the Hadamard after the oracle: $$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}\frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$ $$\frac{1}{8}\begin{pmatrix} 0 \\ -4 \\ 0 \\ 4 \\ 4 \\ 0 \\ 4 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$	We apply the Hadamard after the oracle: $$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}\frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$ $$\frac{1}{8}\begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
This obviously is not $|000\rangle$. Comparing with the table "Kronecker_product" on this website we see that it is even not a Kronecker product of basis states. Measuring the output in the $|0\rangle$ / $|1\rangle$ basis does not give 000.	This obviously is $|000\rangle$. Measuring the output in the $|0\rangle$ / $|1\rangle$ basis gives 000.

We can easily expand this for any number of qubits:

Given any number of input qubits the circuit answers the question "balanced/constant" with one pass.

Remarks

The story behind this algorithm starts with Alice that uses a function $f(x)$:

$$f(x): \{0,1\} \times \{0,1\} \times \dots \times \{0,1\} \to \{0,1\}$$

The domain of $f(x)$ are n-digit binary vectors, the range is 0 and 1.

There are 2^n elements in the domain, 2 elements in the range.

We distinguish two possibilities.

Possibility one, part of the elements of the domain maps to the element 0 of the range, part of the elements maps to the element 1 of the range. The function is surjective.	Possibility two, $2^n - 1$ elements of the domain maps to the element 0 of the range, one element maps to the element 1 of the range. The function is neither injective nor surjective.	Possibility two*, $2^n - 1$ elements of the domain maps to the element 1 of the range, one element maps to the element 0 of the range. The function is neither injective nor surjective.

Now Bob enters the scene. He is supposed to find out (via a program) what kind of function Alice uses. He has the following possibilities:

a) His program queries the function values of one vector after the other. Possibility two needs 2^n queries, possibility one less depending of the kind of function.
b) He uses 2^n parallel CPU's and queries all function values in one run.
c) He uses a quantum computer with n qubits and queries all function values in one run.

The function must be available in the program as a black box.

Method a) and b) are classic. We pay in time or hardware currencies.

Method c) is quantum. We pay in the currency qubits. Superposition reduces the cost from 2^n to n.

Example

Let us check our understanding with an example.

We use the circuit:
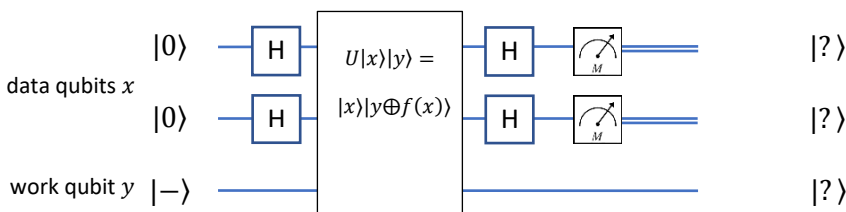
Note: $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Note: $\equiv$ represent classical bits.

We use $f(x)$:

$$f(x) : \{0\ 1\} \times \{0\ 1\} \to \{0\ 1\}$$

First case: $f(x)$ is constant with $f(x) = 0 \ \forall x$.

Conceptual level

The input:

$$|0\rangle|0\rangle|-\rangle = |0\rangle|0\rangle \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right)$$

We apply the Hadamards:

$$H|0\rangle H|0\rangle \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right) =$$

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right) \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right) \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right) =$$

$$\frac{1}{\sqrt{2^3}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle)$$

We apply the oracle U. As the function is constant, $f(x) = 0$, there is no change in the last qubit:

$$\frac{1}{\sqrt{2^3}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle)$$

We get the state of the data qubits after the application of the oracle in basis $\{00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\frac{1}{2}|00\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) + \frac{1}{2}|01\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) + \frac{1}{2}|10\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) + \frac{1}{2}|11\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

We apply the Hadamards. The oracle did no change to the data qubits, so applying twice the Hadamards lead to identity. We get back the original state:

$$|0\rangle|0\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

The probability for measuring 00 is 1.

Basic level

We refer to "Kronecker_product" on this website and get the input state:

$$|0\rangle|0\rangle|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We calculate the Hadamards:

$$H \otimes H \otimes I =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} =$$

$$\frac{1}{2}\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

We apply:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

The oracle changes nothing, the corresponding matrix is the identity.

We apply the Hadamards again:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{4\sqrt{2}}\begin{pmatrix} 4 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Obviously, this is the input state we can decompose into $|0\rangle|0\rangle|-\rangle$. The probability to measure $|0\rangle|0\rangle$ is one.

Second case: $f(00) = 0, f(01) = 1, f(10) = 0, f(11) = 1$, the function is balanced.

Conceptual level
The input:

$$|0\rangle|0\rangle|-\rangle = |0\rangle|0\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

We apply the Hadamards:

$$H|0\rangle H|0\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\frac{1}{\sqrt{2^3}}(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle)$$

We apply the oracle U. The function is balanced, $f(00) = 0, f(01) = 1, f(10) = 0, f(11) = 1$, the working qubit changes for $|01\rangle$ and $|11\rangle$:

$$\frac{1}{\sqrt{2^3}}(|000\rangle - |001\rangle + |011\rangle - |010\rangle + |100\rangle - |101\rangle + |111\rangle - |111\rangle)$$

We get the state of the data qubits after the application of the oracle in basis $\{00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\frac{1}{2}|00\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) - \frac{1}{2}|01\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) + \frac{1}{2}|10\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) - \frac{1}{2}|11\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\left(\frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle) + \frac{1}{\sqrt{2}}|1\rangle(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =;$$

We note: $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

We get:

$$|+\rangle|-\rangle|-\rangle$$

We note: $H|-\rangle = |1\rangle, H|+\rangle = |0\rangle$

We apply the Hadamards:

$$H|+\rangle H|-\rangle|-\rangle = |0\rangle|1\rangle|-\rangle$$

The probability for measuring 00 then is zero.

Basic level

We refer to "Kronecker_product" on this website and get the input state:

$$|0\rangle|0\rangle|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We calculate the Hadamards:

$$H \otimes H \otimes I =$$

$$\frac{1}{2}\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

We apply:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

We construct the matrix for the oracle using the conceptual level:

$$f(00) = 0, f(01) = 1, f(10) = 0, f(11) = 1$$

$\quad\quad\quad\quad\quad\quad$ $|000\rangle\ |001\rangle\ |010\rangle\ |011\rangle\ |100\rangle\ |101\rangle\ |110\rangle\ |111\rangle$ $\quad\longleftarrow$ output

input $\longrightarrow$

$$
\begin{array}{c}
|000\rangle \\
|001\rangle \\
|010\rangle \\
|011\rangle \\
|100\rangle \\
|101\rangle \\
|110\rangle \\
|111\rangle
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

We apply the oracle:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\frac{1}{2\sqrt{2}}
\begin{pmatrix}
1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1
\end{pmatrix}
=
\frac{1}{2\sqrt{2}}
\begin{pmatrix}
1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1
\end{pmatrix}
$$

We apply the Hadamards again:

$$
\frac{1}{2}
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 1
\end{pmatrix}
\frac{1}{2\sqrt{2}}
\begin{pmatrix}
1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1
\end{pmatrix}
=
\frac{1}{4\sqrt{2}}
\begin{pmatrix}
0 \\ 0 \\ 4 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
=
\frac{1}{\sqrt{2}}
\begin{pmatrix}
0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

We refer to "Kronecker_product" on this website and get the state:

$$|0\rangle|1\rangle|-\rangle$$

Remark:

The functions used are different. In the beginning we used function $f(x): \{0,1\} \to \{0,1\}$. Therefore, we worked with a phase modification, $(-1)^{f(x)}|0\rangle$ resp. $(-1)^{f(x)}|1\rangle$.

The example uses function $f(x): \{0\ 1\} \times \{0\ 1\} \to \{0\ 1\}$ that directly can be applied to the qubits.