This paper shows a simple case of entanglement. It follows
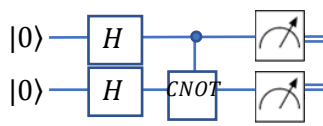
https://lab.quantumflytrap.com/lab?mode=waves

Contents

Entanglement

We use the following circuit:

Note: Single lines denote qubits.

Note: Double lines denote classical bits.

Qubit in state zero	$	0\rangle$
Hadamard gate	H
CNOT gate	$CNOT$
Measuring	

Conceptual level

We start with the state $|00\rangle = |0\rangle|0\rangle = |0\rangle \otimes |0\rangle$.

The Hadamard acting on the first qubit:

$$|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Note: After the first Hadamard, the first qubit is in superposition.

The Hadamard acting on the second qubit:

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Note: After the Hadamards, both qubits are in superposition.

If we measure the first qubit, we get with a probability of $\frac{1}{4}$ each:

$$|00\rangle \; or \; |01\rangle \; or \; |10\rangle \; or \; |11\rangle$$

Summing up for the first qubit $|0\rangle$ we get the probability of $\frac{1}{2}$ and the same probability for $|1\rangle$.

Note: This is the conditional probability:

$$P(first\ qubit = |0\rangle)\ under\ the\ condition\ that\ (second\ qubit = \ |1\rangle)$$

We calculate:

$$\frac{P\big((first\ qubit = |0\rangle)\ and\ (second\ qubit = \ |1\rangle)\ \big)}{P(second\ qubit = \ |1\rangle)} = \frac{^1/_4}{^1/_2} = \frac{1}{2}$$

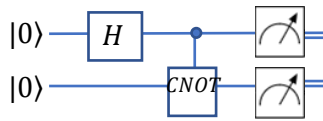The same holds if we measure the second qubit.

Result: The qubits are independent.

We apply the CNOT-gate on qubit 2:

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \rightarrow \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle)$$

With the exception of a permutation nothing happens.

We change the circuit:

The Hadamard acting on the first qubit:

$$|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

The CNOT-gate acting on the second qubit:

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The two qubits are entangled. Entanglement is a special case of superposition.

- If we measure the first qubit and get the result 0, measuring the second qubit also gives the result 0.
- If we measure the first qubit and get the result 1, measuring the second qubit also gives the result 1.

This holds whether we measure the second qubit at a later time or at another place.
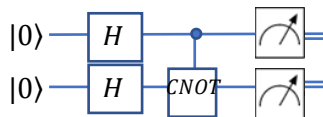
Note: Entanglement does not hold for eternity. Typical times you may find at

https://phys.org/news/2022-09-longest-quantum-standard-silicon-chip.html

that are up to now in the range of milliseconds.

Basic level
We use the following circuit:

We start with the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We apply the first pair of Hadamards.

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

We get:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Note: We can apply $H \otimes Id$ and $Id \otimes H$ giving the same result:

$$H \otimes Id = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$Id \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We apply the $CNOT$ -gate.

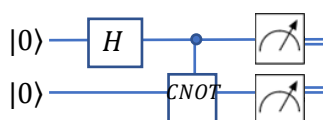|	00	01	10	11
00	1	0	0	0
01	0	1	0	0
10	0	0	0	1
11	0	0	1	0

We get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$\frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Note: These are the four basis vectors we already got from the conceptual level.

We change the circuit:

We start with the state

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

We apply the first Hadamard.

Note: In fact, we apply the Kronecker product $H \otimes I$ onto $|00\rangle$:

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{pmatrix}$$

We get:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

We apply the CNOT and get:

$$\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$$

These are the two basis vectors $|00\rangle$ and $|11\rangle$. The mixed vectors $|01\rangle$ and $|10\rangle$ vanished. The qubits are entangled.
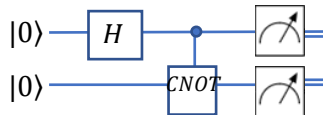
Bell states

The state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is one of the four maximally entangled Bell states. Usually they are written as:
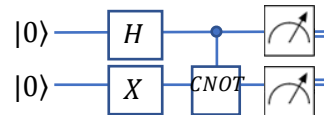
$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$|\Phi^-\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$|\Psi^+\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$|\Psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

Circuits creating the four Bell states

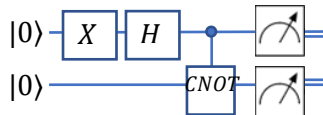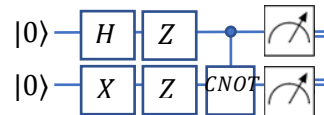Bell state $|\Phi^+\rangle$:

Bell state $|\Psi^+\rangle$:

Note: $\boxed{X}$ is the Pauli X-gate.

Bell state $|\Phi^-\rangle$:

Note: $\boxed{X}$ is the Pauli X-gate.

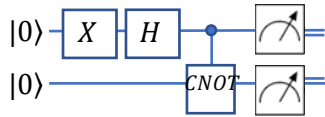Bell state $|\Psi^-\rangle$:

Note: $\boxed{Z}$ is the Pauli Z-gate.

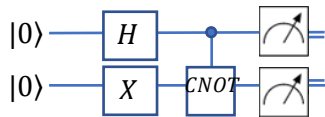## Creating Bell states on conceptual level

Bell state $|\Phi^+\rangle$

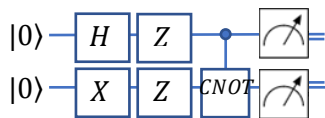… is already done.

Bell state $|\Phi^-\rangle$

$$|00\rangle \to |10\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Bell state $|\Psi^+\rangle$

$$|00\rangle \to |01\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|1\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Bell state $|\Psi^-\rangle$

$$|00\rangle \to |01\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(-|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(-|01\rangle + |11\rangle) \to \frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle) = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
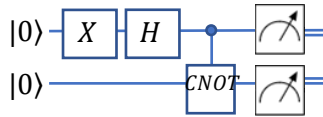
Note: The "minus" in front of $\frac{1}{\sqrt{2}}$ is a global phase that has no effect on the measurement of single qubits.

Creating Bell states on basic level

Bell state $|\Phi^+\rangle$

… is already done.

Bell state $|\Phi^-\rangle$

We start with the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We apply the Pauli X-gate. Note: In fact, we apply the Kronecker product $X \otimes I$ onto $|00\rangle$:

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We get:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

We apply the Hadamard. Note: In fact, we apply the Kronecker product $H \otimes I$:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

We get:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

We apply the CNOT and get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Bell state $|\Psi^+\rangle$

$|0\rangle$ —[H]——●——[📷]=

$|0\rangle$ —[X]—[$CNOT$]—[📷]=

We start with the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We apply the Pauli X-gate. Note: In fact, we apply the Kronecker product $I \otimes X$ onto $|00\rangle$:

$$I \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We get:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We apply the Hadamard. Note: In fact, we apply the Kronecker product $H \otimes I$:

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$
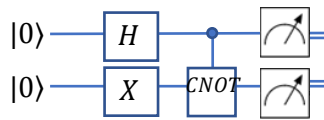
We get:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Note: We can apply $H \otimes X$ giving the same result:

$$H \otimes X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$
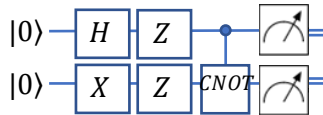
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Note: This holds for the other pairs too.

We apply the CNOT and get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

Bell state $|\Psi^-\rangle$

We start with the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We apply the Pauli X-gate. Note: In fact, we apply the Kronecker product $I \otimes X$ onto $|00\rangle$:

$$I \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We get:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We apply the Hadamard. Note: In fact, we apply the Kronecker product $H \otimes I$:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

We get:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

We apply the upper Pauli Z-gate.

Note: In fact, we apply the Kronecker product $Z \otimes I$:

$$Z \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

We apply the lower Pauli Z-gate.

Note: In fact, we apply the Kronecker product $I \otimes Z$:

$$I \otimes Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Note: We can apply $Z \otimes Z$ giving the same result:

$$Z \otimes Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Note: The "minus" in front of $\frac{1}{\sqrt{2}}$ is a global phase that has no effect on the measurement of single qubits.

Note: Working with the Kronecker product $Z \otimes Z$ leads to the correct result without the need of changing the global phase.

We apply the CNOT and get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) = -\frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right)$$