

This paper follows Alessandro Berti and his didactical access to Grover's algorithm:

<https://towardsdatascience.com/behind-oracles-grovers-algorithm-amplitude-amplification-46b928b46f1e>

SAT problem

We have a set of Boolean variables that should fulfill a given combination, e. g.:

$$(x \vee y) \wedge \bar{y} = 1 \tag{1}$$

The solution:

$$x = 1, y = 0$$

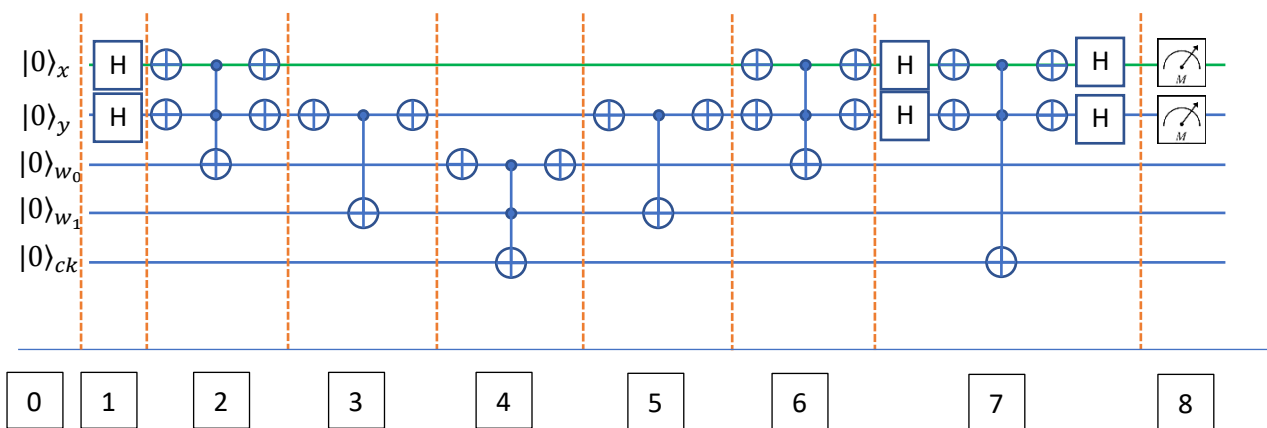
Quantum circuits can handle " \wedge " better than " \vee ", so we use the de Morgan formula:

$$\overline{(x \vee y)} = \bar{x} \wedge \bar{y}$$

We get the equivalent to our problem (1)

$$\overline{(\bar{x} \wedge \bar{y})} \wedge \bar{y} = 1$$

The Quantum circuit



Details

0 We need five qubits to calculate the correct values for x and y .

These are the qubit $|0\rangle_x$ and $|0\rangle_y$. We set them to $|0\rangle$ as start value.

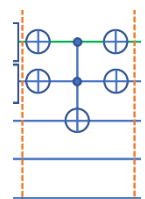
We need two working qubits, $|0\rangle_{w_0}$ and $|0\rangle_{w_1}$. We set them to $|0\rangle$ as start value. The working qubits store intermediate results.

We need a "checker" qubit $|0\rangle_{ck}$. We set it to $|0\rangle$ as start value. The checker qubit transports the solution to the superposition (sorry for having no other words for this ...).

1 We bring the input qubits x and y into superposition by help of the Hadamard gate.

2 We calculate the first part of the logic problem, $(\bar{x} \wedge \bar{y})$ by help of a CCNOT gate:

Note that the two NOT gates on the right are needed to undo the effect of the two NOT gates on the left. The two NOT gates on the left are necessary to perform \bar{x} and \bar{y} . The CCNOT performs a not on line three if the values on line one and line two are "1". The working qubit w_0 then holds the result of the first operation.

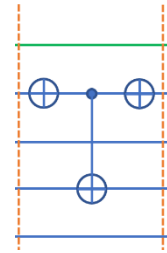


3 In the same way we calculate the second part of the logic problem, \bar{y} and store the result in the working qubit w_1 .

The NOT gate on the left side transforms y to \bar{y} .

The following CNOT then inverts the working bit w_1 .

The NOT gate on the right undo the action of the NOT before.



4 Now we apply the solution function to the check qubit. Note that we must negate the first working qubit w_0 because it contains $(\bar{x} \wedge \bar{y})$.

The first NOT on the left side performs this, the second NOT on the right side undo this action.

We use the second working qubit w_1 as it is.

The check qubit, then, is modified. This modification is not simply the solution because the two input bits are in superposition. The modification is a kind of superposition of all solutions.

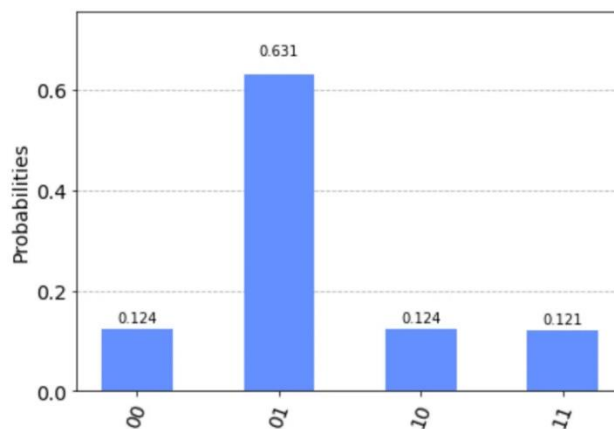
5 and 6 are needed to undo all calculations so far. This is a property of quantum computing.

1 through 6 form the oracle.

7 We use the diffusor to apply the solution in search to the qubits x and y . We apply the Hadamard to both qubits again, invert them and perform a CCNOT to the checker qubit. This brings the solution searched into contact with the superposition of the input qubits x and y .

We then undo the invert and the Hadamard.

8 We measure both qubits. With a single measurement we will get either 0 or 1. We must repeat the whole process for e. g. 100 times and get a sample of results:



Note: Results from the website of the author. They seem to be the result of running a simulation. The values are close to $1/8$ and $5/8$ we get out of the calculation.

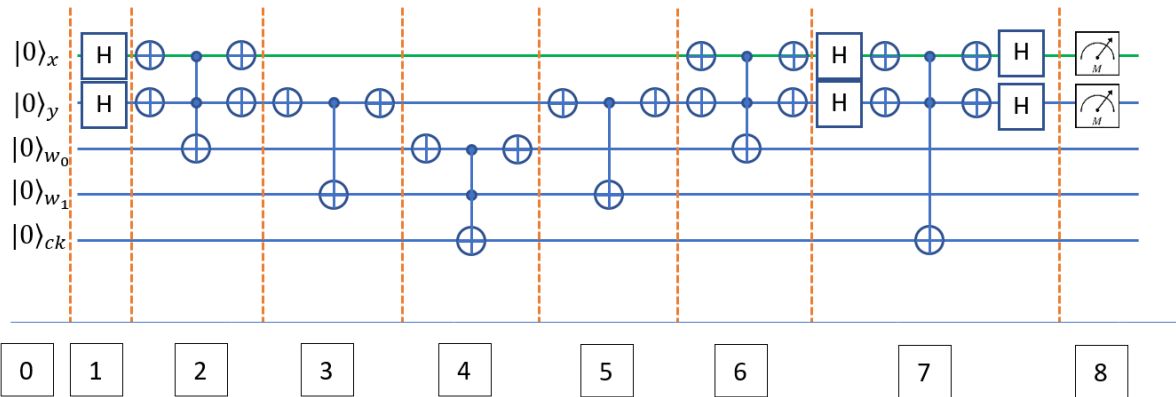
The measurement result with the highest probability is the combination $x = 1, y = 0$, the solution for our problem.

Remark: We did not insert the mere solution into our circuit but we inserted the Boolean formula for it and we never modified the input qubits x and y directly.

We take a look at the whole process on conceptual level.

Conceptual level

We used the circuit:



0 We have five input qubits we write as $|00000\rangle$.

1 We apply the Hadamard to qubits one and two. The Hadamard applied to $|0\rangle$ gives $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the Hadamard applied to $|1\rangle$ gives $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$\begin{aligned} |00000\rangle &= H|0\rangle H|0\rangle |000\rangle = \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot |000\rangle = \\ &= \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|000\rangle = \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)|000\rangle = \\ &= \frac{1}{2}(|00000\rangle + |01000\rangle + |10000\rangle + |11000\rangle) \end{aligned}$$

2 We apply the X-gate to qubits one and two. The X-gate changes $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

$$\frac{1}{2}(|11000\rangle + |10000\rangle + |01000\rangle + |00000\rangle)$$

Then we apply the CCNOT from lines one and two to line three. The CCNOT changes line three from $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$ if lines one and two both are $|1\rangle$.

$$\frac{1}{2}(|11100\rangle + |10000\rangle + |01000\rangle + |00000\rangle)$$

We undo the X-gate:

$$\frac{1}{2}(|00100\rangle + |01000\rangle + |10000\rangle + |11000\rangle)$$

3 We apply the X-gate to qubit 2, then apply the CNOT from line two to line 4 and undo the X-gate:

$$\frac{1}{2}(|01100\rangle + |00000\rangle + |11000\rangle + |10000\rangle)$$

$$\frac{1}{2}(|01110\rangle + |00000\rangle + |11010\rangle + |10000\rangle)$$

$$\frac{1}{2}(|00110\rangle + |01000\rangle + |10010\rangle + |11000\rangle)$$

4 We apply the X -gate to qubit 3, then apply the CCNOT from line three and four to line five and undo the X -gate:

$$\frac{1}{2}(|00010\rangle + |01100\rangle + |10110\rangle + |11100\rangle)$$

$$\frac{1}{2}(|00010\rangle + |01100\rangle + |10111\rangle + |11100\rangle)$$

$$\frac{1}{2}(|00110\rangle + |01000\rangle + |10011\rangle + |11000\rangle)$$

5 We apply the X -gate to qubit 2, then apply the CNOT from line two to line four and undo the X -gate. This is an undo to 3 :

$$\frac{1}{2}(|01110\rangle + |00000\rangle + |11011\rangle + |10000\rangle)$$

$$\frac{1}{2}(|01100\rangle + |00000\rangle + |11001\rangle + |10000\rangle)$$

$$\frac{1}{2}(|00100\rangle + |01000\rangle + |10001\rangle + |11000\rangle)$$

6 We apply the X -gate to qubits one and two. Then we apply the CCNOT from lines one and two to line three. Then we undo the X -gate. This is an undo of 2 :

$$\frac{1}{2}(|11100\rangle + |10000\rangle + |01001\rangle + |00000\rangle)$$

$$\frac{1}{2}(|11000\rangle + |10000\rangle + |01001\rangle + |00000\rangle)$$

$$\frac{1}{2}(|00000\rangle + |01000\rangle + |10001\rangle + |11000\rangle)$$

7 The diffusor with the application of the Hadamard, X -gate, CCNOT and the undo.

The Hadamards:

$$\begin{aligned} & \frac{1}{2}(H|0\rangle H|0\rangle|000\rangle + H|0\rangle H|1\rangle|000\rangle + H|1\rangle H|0\rangle|001\rangle + H|1\rangle H|1\rangle|000\rangle) = \\ & \frac{1}{4}((|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|000\rangle + (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)|000\rangle + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)|001\rangle \\ & \quad + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)|000\rangle) = \end{aligned}$$

$$\frac{1}{4}(|00000\rangle + |01000\rangle + |10000\rangle + |11000\rangle + |00000\rangle - |01000\rangle + |10000\rangle - |11000\rangle + |00001\rangle + |01001\rangle - |10001\rangle - |11001\rangle + |00000\rangle - |01000\rangle - |10000\rangle + |11000\rangle) =$$

$$\frac{1}{4}(3|00000\rangle - |01000\rangle + |10000\rangle + |11000\rangle + |00001\rangle + |01001\rangle - |10001\rangle - |11001\rangle)$$

The X-gates:

$$\frac{1}{4}(3|11000\rangle - |10000\rangle + |01000\rangle + |00000\rangle + |11001\rangle + |10001\rangle - |01001\rangle - |00001\rangle)$$

The CCNOT:

$$\frac{1}{4}(3|11001\rangle - |10000\rangle + |01000\rangle + |00000\rangle + |11000\rangle + |10001\rangle - |01001\rangle - |00001\rangle)$$

$$\frac{1}{4}(|11000\rangle - |10000\rangle + |01000\rangle + |00000\rangle + 3|11001\rangle + |10001\rangle - |01001\rangle - |00001\rangle)$$

The X-gates:

$$\frac{1}{4}(|00000\rangle - |01000\rangle + |10000\rangle + |11000\rangle + 3|00001\rangle + |01001\rangle - |10001\rangle - |11001\rangle)$$

$$\frac{1}{4}((|00\rangle - |01\rangle + |10\rangle + |11\rangle)|000\rangle + (3|00\rangle + |01\rangle - |10\rangle - |11\rangle)|001\rangle)$$

The Hadamards:

$$\frac{1}{8}(((|0\rangle + |1\rangle)(|0\rangle + |1\rangle) - (|0\rangle + |1\rangle)(|0\rangle - |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle - |1\rangle))|000\rangle + (3(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle + |1\rangle)(|0\rangle - |1\rangle) - (|0\rangle - |1\rangle)(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)(|0\rangle - |1\rangle))|001\rangle) =$$

$$\frac{1}{8}((|00\rangle + |01\rangle + |10\rangle + |11\rangle - |00\rangle + |01\rangle - |10\rangle + |11\rangle + |00\rangle + |01\rangle - |10\rangle - |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle)|000\rangle + (3|00\rangle + 3|01\rangle + 3|10\rangle + 3|11\rangle + |00\rangle - |01\rangle + |10\rangle - |11\rangle - |00\rangle - |01\rangle + |10\rangle + |11\rangle - |00\rangle + |01\rangle + |10\rangle - |11\rangle)|001\rangle) =$$

$$\frac{1}{8}((2|00\rangle + 2|01\rangle - 2|10\rangle + 2|11\rangle)|000\rangle + (2|00\rangle + 2|01\rangle + 6|10\rangle + 2|11\rangle)|001\rangle) =$$

$$\frac{1}{4}((|00\rangle + |01\rangle - |10\rangle + |11\rangle)|000\rangle) + \frac{1}{4}((|00\rangle + |01\rangle + 3|10\rangle + |11\rangle)|001\rangle)$$

8 We measure qubits one and two. The probabilities are:

00⟩	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$
01⟩	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$
10⟩	$\frac{1}{16} + \frac{9}{16} = \frac{5}{8}$
11⟩	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

Note: The sum of probabilities is 1.

This shows that the distribution of the probability for $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ that was $\frac{1}{4}$ after the first Hadamard changed. The probabilities for $|00\rangle, |01\rangle, |11\rangle$ lowered to $\frac{1}{8}$, the probability for $|10\rangle$ raised to $\frac{5}{8}$.

Basic level

The calculation on the basic level is "unreadable stuff". For your convenience I added in the end the matrices belonging to the scheme we calculate.

The result of calculation on the basic level:

$$\frac{1}{4} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We resolve the kets of the conceptual access and compare with the basic access.

$$\frac{1}{4}((|00\rangle + |01\rangle - |10\rangle + |11\rangle)|000\rangle) + \frac{1}{4}((|00\rangle + |01\rangle + 3|10\rangle + |11\rangle)|001\rangle) =$$

$$\frac{1}{4}(|00000\rangle + |01000\rangle - |10000\rangle + |11000\rangle + |00001\rangle + |01001\rangle + 3|10001\rangle + |11001\rangle)$$

