This paper follows Alessandro Berti and his didactical access to Grover's algorithm:

https://towardsdatascience.com/behind-oracles-grovers-algorithm-amplitude-amplification-46b928b46f1e

SAT problem

We have a set of Boolean variables that should fulfill a given combination, e. g.:

$$(x \lor y) \land \overline{y} = 1$$

The solution:

$$x = 1, y = 0$$

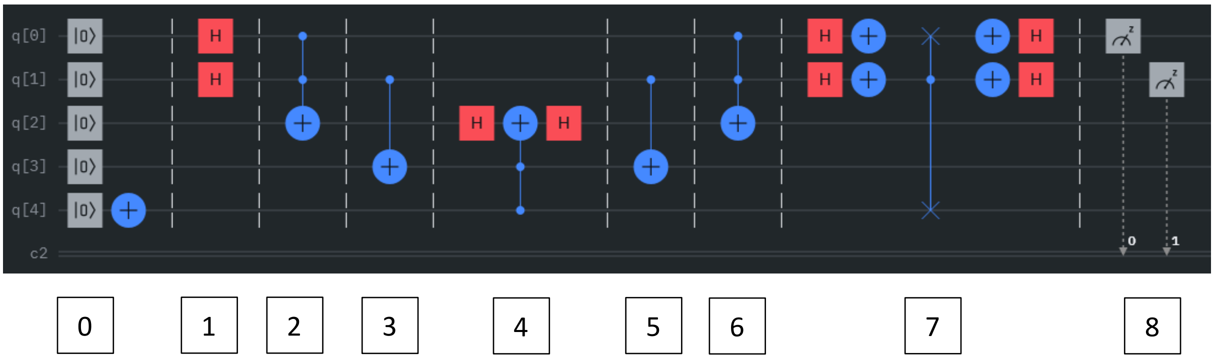
Quantum circuits can handle " $\land$ " better than " $\lor$ ", so we use the de Morgan formula:

$$\overline{(x \lor y)} = \overline{x} \land \overline{y}$$

We get the equivalent to our problem $(x \lor y) \land \overline{y} = 1$

$$\overline{(\overline{x} \land \overline{y})} \land \overline{y} = 1$$

The Quantum circuit



Details

0 We use five qubits to calculate the correct values for x and y.

These are the qubit $q[0]$ and $q[1]$. We set them to $|0\rangle$ as start value.

We need two working qubits, $q[2]$and $q[3]$. We set them to $|0\rangle$ as start value. The working qubits store intermediate results.

We need a "checker" qubit $q[4]$. We set it to $|1\rangle$ via the swap as start value. The checker qubit transports the solution to the superposition in 8 .

1 We bring the input qubits $q[0]$ and $q[1]$ into superposition by help of the $Hadamard$ gate.

2 We calculate the first part of the logic problem, $(\overline{x} \land \overline{y})$ by help of a $CCNOT$ gate:

The $CCNOT$ performs a not on line three if the values on line one and line two are "1". The working qubit $q[2]$ then holds the result of the first operation.

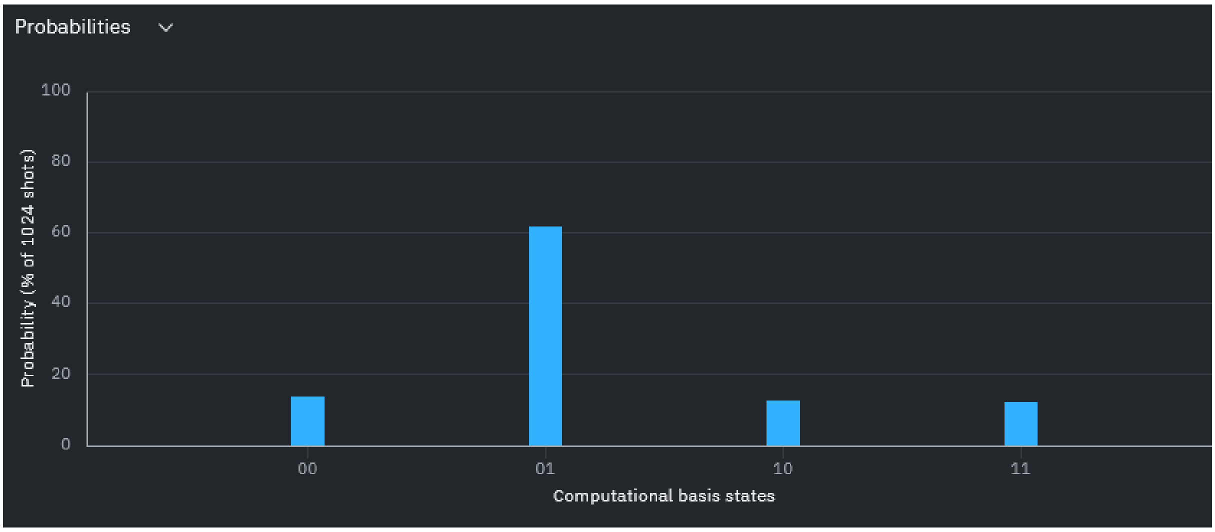
3 In the same way we calculate the second part of the logic problem, $\overline{y}$ and store the result in the working qubit $q[3]$.

4 Now we apply a CCZ to the working qubit $q[2]$.

5 and 6 are needed to undo 2 and 3 . This is a property of quantum computing.
1 through 6 form the oracle.

7 We use the diffusor to apply the solution in search to the qubits x and y. We apply the Hadamards to both qubits again, invert them and perform a $CSWAP$ to the checker qubit. This brings the solution searched into contact with the superposition of the input qubits x and y. We then undo the invert and the Hadamards.

8 We measure both qubits. With a single measurement we will get either 0 or 1. We must repeat the whole process for e. g. 100 times and get a sample of results:



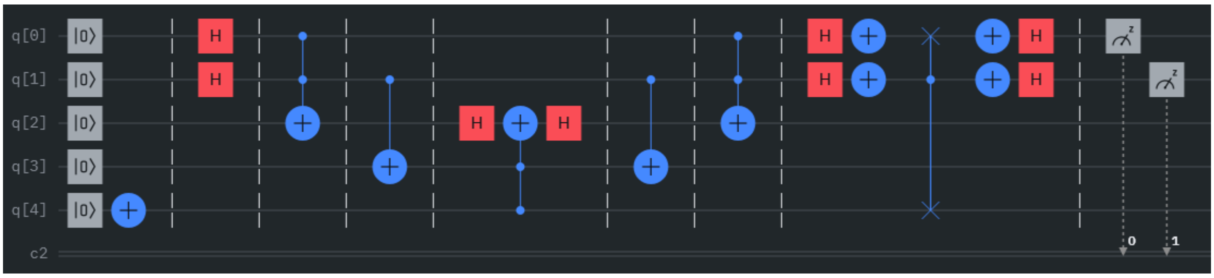
The measurement result with the highest probability is the combination $x = 1, y = 0$, the solution for our problem. Please note that the composer uses little endian.

Remark: We did not insert the solution into our circuit but we inserted the Boolean formula for it.

We take a look at the whole process on conceptual level.

Conceptual level

We used the circuit:



0 1 2 3 4 5 6 7 8

0 We have five input qubits we write as $|00001\rangle$ (the 1 made by the NOT).

1 We apply the $Hadamard$ to qubits one and two. The $Hadamard$ applied to $|0\rangle$ gives $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the $Hadamard$ applied to $|1\rangle$ gives $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$|00001\rangle \rightarrow H|0\rangle H|0\rangle|001\rangle =$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot |001\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|001\rangle =$$

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)|001\rangle = \frac{1}{2}(|00001\rangle + |01001\rangle + |10001\rangle + |11001\rangle)$$

2 Then we apply the $CCNOT$ from lines one and two to line three. The $CCNOT$ swaps position three if positions one and two both are $|1\rangle$.

$$\frac{1}{2}(|00001\rangle + |01001\rangle + |10001\rangle + |11101\rangle)$$

3 We apply the $CNOT$-gate from position two to position four:

$$\frac{1}{2}(|00001\rangle + |01011\rangle + |10001\rangle + |11111\rangle)$$

4 We apply the CCZ-gate from position four and three to qubit 2:

$$\frac{1}{2}(|00001\rangle + |01011\rangle + |10001\rangle - |11111\rangle)$$

Note: The circuit shown represents the CCZ as the CCZ is not directly available in IBM composer.

Note: This is the sign-swap wanted.

5 We undo operations 2 and 3 by applying them again:

Operation 2 was a $CCNOT$ from lines one and two to line three. The $CCNOT$ swaps position three if positions one and two both are $|1\rangle$.

$$\frac{1}{2}(|00001\rangle + |01011\rangle + |10001\rangle - |11011\rangle)$$

6 Operation 3 was a $CNOT$-gate from position two to position four:

$$\frac{1}{2}(|00001\rangle + |01001\rangle + |10001\rangle - |11001\rangle)$$

Note: The first two qubits are in the same state as in the beginning after the first Hadamard.

7 The diffusor with the application of the $Hadamard$, X-gate, $CSWAP$ and the undo.

The $Hadamards$:

$$\frac{1}{2}(H|0\rangle H|0\rangle|001\rangle + H|0\rangle H|1\rangle|001\rangle + H|1\rangle H|0\rangle|001\rangle - H|1\rangle H|1\rangle|001\rangle) =$$

$$\frac{1}{4}\big((|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|001\rangle + (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)|001\rangle + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)|001\rangle - (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)|001\rangle\big) =$$

$$\frac{1}{4}(|00001\rangle + |01001\rangle + |10001\rangle + |11001\rangle + |00001\rangle - |01001\rangle + |10001\rangle - |11001\rangle + |00001\rangle + |01001\rangle - |10001\rangle - |11001\rangle - |00001\rangle + |01001\rangle + |10001\rangle - |11001\rangle) =$$

$$\frac{1}{4}(2|00001\rangle + 2|01001\rangle + 2|10001\rangle - 2|11001\rangle)$$

The X-gates:

$$\frac{1}{4}(2|11001\rangle + 2|10001\rangle + 2|01001\rangle - 2|00001\rangle)$$

The $CSWAP$ from position two swaps position zero and four:

$$\frac{1}{4}(2|11001\rangle + 2|10001\rangle + 2|11000\rangle - 2|00001\rangle)$$

The X-gates:

$$\frac{1}{4}(2|00001\rangle + 2|01001\rangle + 2|00000\rangle - 2|11001\rangle) =$$

$$\frac{1}{2}(|00001\rangle + |01001\rangle + |00000\rangle - |11001\rangle) =$$

The $Hadamards$:

$$\frac{1}{4}\big((|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|001\rangle + (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)|001\rangle + (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|000\rangle - (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)|001\rangle\big) =$$

$$\frac{1}{4}(|00001\rangle + |01001\rangle + |10001\rangle + |11001\rangle + |00001\rangle - |01001\rangle + |10001\rangle - |11001\rangle + |00000\rangle + |01000\rangle + |10000\rangle + |11000\rangle - |00001\rangle + |01001\rangle + |10001\rangle - |11001\rangle) =$$

$$\frac{1}{4}(|00001\rangle + |01001\rangle + 3|10001\rangle - |11001\rangle + |00000\rangle + |01000\rangle + |10000\rangle + |11000\rangle) =$$

8 We measure qubits one and two. The probabilities are:

$\|00xxx\rangle$	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$
$\|01xxx\rangle$	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$
$\|10xxx\rangle$	$\frac{1}{16} + \frac{9}{16} = \frac{5}{8}$
$\|11xxx\rangle$	$\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

Note: When measuring target qubits in a sequence of qubits, we concentrate on the target and ignore the rest.

Note: The sum of probabilities is 1.

This shows that the distribution of the probability for $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ that was $\frac{1}{4}$ after the first $Hadamard$. The probabilities for $|00\rangle, |01\rangle, |11\rangle$ lowered to $\frac{1}{8}$, the probability for $|10\rangle$ raised to $\frac{5}{8}$.