This paper follows a book of Kurt Martin, "Vom Bit zum Qubit", ISBN 9781718983250.

It shows two numeric examples for Grover's algorithm.

Example one

We start with three qubits:

$$|0\rangle|0\rangle|0\rangle = |000\rangle$$

We apply the Hadamard to each qubit:

$$H|0\rangle \cdot H|0\rangle \cdot H|0\rangle = H|000\rangle$$

This gives:

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \cdot \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \cdot \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) =$$

$$\frac{1}{\sqrt{8}}\big((|0\rangle + |1\rangle)\big) \cdot \big((|0\rangle + |1\rangle)\big) \cdot \big((|0\rangle + |1\rangle)\big) =$$

$$\frac{1}{\sqrt{8}}\big((|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)\big)$$

We calculate the average:

$$8 \cdot \sqrt{\frac{1}{8}} : 8 = \sqrt{\frac{1}{8}}$$

We apply the 011-CNOT that inverts the qubit if and only if it has the value $|011\rangle$.

Result:

$$\frac{1}{\sqrt{8}}\big((|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)\big)$$

Every qubit has the probability amplitude $\sqrt{\frac{1}{8}}$ with the exception of $|011\rangle$ and its probability amplitude $-\sqrt{\frac{1}{8}}$.	Every qubit has the probability density $\frac{1}{8}$.

We calculate the average of the probability amplitude **after** the application of the CNOT:

$$\left((7-1) \cdot \sqrt{\frac{1}{8}}\right) : 8 = \frac{3}{4} \cdot \sqrt{\frac{1}{8}}$$

The average lowers caused by the negative value of qubit $|011\rangle$.

We swap all probability amplitudes relative to this new average.

The first qubit has a probability amplitude that is a little bit too high with respect to the new average.

Now Grover's algorithm is working.

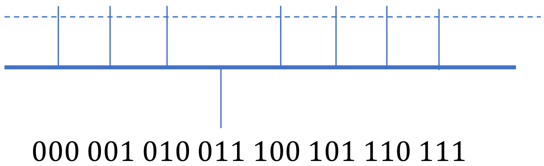
We calculate the difference of the probability *amplitude* of one of the seven qubits with the same probability amplitude and the new average:	In case of the negative one we calculate the difference of the probability *amplitude* and the new average:
$\sqrt{\frac{1}{8}} - \frac{3}{4} \cdot \sqrt{\frac{1}{8}} = \frac{1}{4} \cdot \sqrt{\frac{1}{8}}$	$-\sqrt{\frac{1}{8}} - \frac{3}{4} \cdot \sqrt{\frac{1}{8}} = -\frac{7}{4} \cdot \sqrt{\frac{1}{8}}$
We subtract this difference from the average in order to swap the value around it:	In the same way we subtract this value from the average:
$\frac{3}{4} \cdot \sqrt{\frac{1}{8}} - \frac{1}{4} \cdot \sqrt{\frac{1}{8}} = \frac{1}{2} \cdot \sqrt{\frac{1}{8}}$	$\frac{3}{4} \cdot \sqrt{\frac{1}{8}} - \left(-\frac{7}{4} \cdot \sqrt{\frac{1}{8}}\right) = \frac{10}{4} \cdot \sqrt{\frac{1}{8}}$
This is the case for the seven qubits with positive probability *amplitude*, they get the new amplitude $\frac{1}{2} \cdot \sqrt{\frac{1}{8}}$	This is the case for the one qubit with negative probability *amplitude*, it gets the new amplitude $\frac{10}{4} \cdot \sqrt{\frac{1}{8}}$

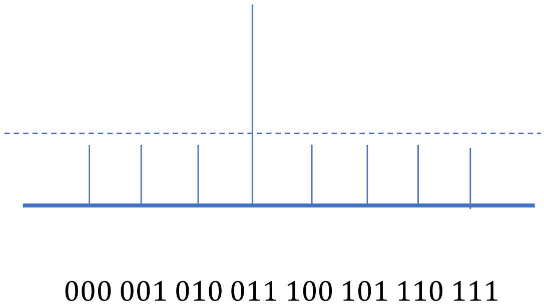
We check whether the probability *densities* add to 1:

$$7 \cdot \frac{1}{32} + \frac{100}{128} = \frac{7 + 25}{32} = 1$$

We have valid probability amplitudes and a correct probability density.

We graphically compare “before” and “after” the swap around the new average of the probability amplitude (not to scale …):





Example two

We repeat this procedure.

We use the qubits we got from example one:

$$\left(\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|000\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|001\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|010\rangle+\frac{10}{4}\cdot\sqrt{\frac{1}{8}}|011\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|100\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|101\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|110\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|111\rangle\right)$$

We apply the 011-CNOT that inverts the qubit if and only if it has the value $|011\rangle$.

Result:

$$\left(\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|000\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|001\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|010\rangle-\frac{10}{4}\cdot\sqrt{\frac{1}{8}}|011\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|100\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|101\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|110\rangle+\frac{1}{2}\cdot\sqrt{\frac{1}{8}}|111\rangle\right)$$

Every qubit has the probability amplitude $\frac{1}{2}\cdot\sqrt{\frac{1}{8}}$ with the exception of $|011\rangle$ and its probability amplitude $-\frac{10}{4}\cdot\sqrt{\frac{1}{8}}$.

We calculate the average of the probability amplitude **before** the application of the CNOT: $$\frac{\left(\frac{14}{4}\cdot\sqrt{\frac{1}{8}}+\frac{10}{4}\cdot\sqrt{\frac{1}{8}}\right)}{8}=\frac{6}{8}\cdot\sqrt{\frac{1}{8}}$$	We calculate the average of the probability amplitude **after** the application of the CNOT: $$\frac{\left(\frac{14}{4}\cdot\sqrt{\frac{1}{8}}-\frac{10}{4}\cdot\sqrt{\frac{1}{8}}\right)}{8}=\frac{1}{8}\cdot\sqrt{\frac{1}{8}}$$

The average becomes lower caused by the negative value of qubit $|011\rangle$.

Now we swap all probability amplitudes relative to this new average.

The first qubit has a probability amplitude that is a little bit too high with respect to the new average.

We calculate the difference of the probability amplitude of one of the seven similar qubits and the new average:	In case of the negative one we calculate the difference of the probability amplitude and the new average:
$\frac{1}{2}\cdot\sqrt{\frac{1}{8}}-\frac{1}{8}\cdot\sqrt{\frac{1}{8}}=\frac{3}{8}\cdot\sqrt{\frac{1}{8}}$	$-\frac{20}{8}\cdot\sqrt{\frac{1}{8}}-\frac{1}{8}\cdot\sqrt{\frac{1}{8}}=-\frac{21}{8}\cdot\sqrt{\frac{1}{8}}$
We subtract this difference from the new average in order to swap the value below it:	We subtract this difference from the new average in order to swap the value around it:
$\frac{1}{8}\cdot\sqrt{\frac{1}{8}}-\frac{3}{8}\cdot\sqrt{\frac{1}{8}}=-\frac{2}{8}\cdot\sqrt{\frac{1}{8}}$	$\frac{1}{8}\cdot\sqrt{\frac{1}{8}}-\left(-\frac{21}{8}\cdot\sqrt{\frac{1}{8}}\right)=\frac{22}{8}\cdot\sqrt{\frac{1}{8}}$
This is the case for the seven qubits with positive probability amplitude, they get the new amplitude:	This is the case for the one qubit with negative probability amplitude, it gets the new amplitude:
$-\frac{2}{8}\cdot\sqrt{\frac{1}{8}}$	$\frac{22}{8}\cdot\sqrt{\frac{1}{8}}$

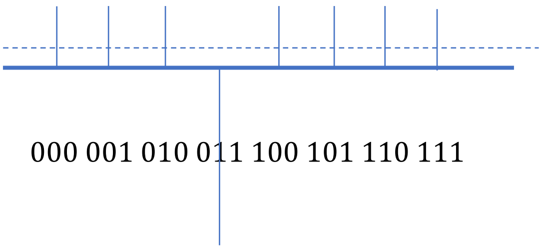
We check whether the probability densities add to one:

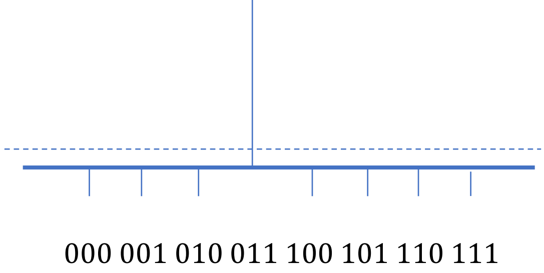
$$7 \cdot \left(-\frac{2}{8} \cdot \sqrt{\frac{1}{8}}\right)^2 + \left(\frac{22}{8} \cdot \sqrt{\frac{1}{8}}\right)^2 =$$

$$7 \cdot \frac{4}{64 \cdot 8} + \frac{484}{64 \cdot 8} = \frac{28 + 484}{64 \cdot 8} = \frac{512}{512} = 1$$

We have valid probability amplitudes.

We graphically compare “before” and “after” the swap around the new average of the probability amplitude (not to scale …):





Remark

In quantum mechanics the wave function Ψ gives the probability amplitude. The probability amplitude has no explicit physical meaning. It can be positive or negative.

The square of the probability amplitude gives the probability density. The probability density is always positive. The integral over the probability density from $-\infty$ to ∞ must give 1. You may find more information at https://en.wikipedia.org/wiki/Probability_distribution

The integral over an interval of the probability density gives the physical probability for: finding a particle within …, getting a measurement in the range of … etc.