No-cloning theorem

The *no-cloning theorem* states that it is impossible to create a perfect copy of an unknown quantum state.

Let Σ be a classical state set having at least two elements and let X and Y be systems sharing the same classical state set Σ.

Then there does not exist a quantum state $|\phi\rangle$ of Y and a unitary operation U on the pair (X, Y) such that:

$$U(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for every state $|\psi\rangle$ of X.

Note: There might exist pairs $|\psi\rangle, |\phi\rangle$ for which this is possible, but not for all.

There is no way to initialize the system Y (to any state $|\phi\rangle$ whatsoever) and perform a unitary operation U on the joint system (X, Y) so that the effect for the state $|\psi\rangle$ of X is to be *cloned* — resulting in (X, Y) being in the state $|\psi\rangle \otimes |\psi\rangle$.

Proof:

$$|\psi\rangle \otimes |\phi\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

is not linear in $|\psi\rangle$.

In particular, because Σ has at least two elements, we may choose $a, b \in \Sigma$ with $a \neq b$.

If there exist a quantum state $|\phi\rangle$ of Y and a unitary operation U on the pair (X, Y) with:

$$U(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for every quantum state $|\psi\rangle$ of X, then it would be the case:

$$U(|a\rangle \otimes |\phi\rangle) = |a\rangle \otimes |a\rangle$$

and

$$U(|b\rangle \otimes |\phi\rangle) = |b\rangle \otimes |b\rangle$$

We use that Tensor product is linear in the first argument, matrix-vector multiplication is linear in the second (vector) argument.

We get:

$$U\left(\left(\frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle\right) \otimes |\phi\rangle\right) =$$

$$U\left(\frac{1}{\sqrt{2}}|a\rangle \otimes |\phi\rangle + \frac{1}{\sqrt{2}}|b\rangle \otimes |\phi\rangle\right) =$$

$$U\left(\frac{1}{\sqrt{2}}|a\rangle \otimes |\phi\rangle\right) + U\left(\frac{1}{\sqrt{2}}|b\rangle \otimes |\phi\rangle\right) =$$

$$\frac{1}{\sqrt{2}}|a\rangle \otimes |a\rangle + \frac{1}{\sqrt{2}}|b\rangle \otimes |b\rangle$$

However, the requirement that $U(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\psi\rangle$ for every quantum state $|\psi\rangle$ demands:

$$U\left(\left(\frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle\right) \otimes |\phi\rangle\right) = \left(\frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle\right) =$$

$$\frac{1}{\sqrt{2}}|a\rangle \otimes \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|a\rangle \otimes \frac{1}{\sqrt{2}}|b\rangle + \frac{1}{\sqrt{2}}|b\rangle \otimes \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle \otimes \frac{1}{\sqrt{2}}|b\rangle =$$

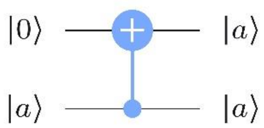
$$\frac{1}{2}|a\rangle \otimes |a\rangle + \frac{1}{2}|a\rangle \otimes |b\rangle + \frac{1}{2}|b\rangle \otimes |a\rangle + \frac{1}{2}|b\rangle \otimes |b\rangle$$

Obviously, this leads to a contradiction.

Note: The no-cloning theorem is a statement about the impossibility of cloning *any arbitrary* state $|\psi\rangle$.

In contrast, we can easily create a clone of any standard basis state.

For example, we can clone a qubit standard basis state using a controlled-NOT operation:



While there is no difficulty in creating a clone of a standard basis state, this does not contradict the no-cloning theorem — the use of the same controlled-NOT gate would not succeed in creating a clone of the state $|+\rangle$, for instance.

Note: The statement of the no-cloning theorem deals with *perfect* cloning. Cloning with limited accuracy might be possible.

Note: You find more information at: https://learning.quantum.ibm.com/course/basics-of-quantum-information/quantum-circuits