This paper deals with the quantum Fourier transform. It is based on:

Nielsen/Chuang, Quantum Computation and Quantum Information

Peter Shor at https://openlearninglibrary.mit.edu/courses/course-v1:MITx+8.370.2x+1T2018/courseware/Week3/lectures_U2_4_qft/6?activate_block_id=block-v1%3AMITx%2B8.370.2x%2B1T2018%2Btype%40vertical%2Bblock%40N-qubit_quantum_Fourier_transform

In this paper we will explicit perform the quantum Fourier transform for 1, 2, 3 and 4 qubits. This should be enough to see how to expand this to n qubits.

Some remarks

Nielsen/Chuang define the Fourier transformation with the positive coefficient:

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{2\pi i \frac{jk}{n}}$$

In literature the discrete Fourier transformation sometimes is defined with the negative coefficient:

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{-2\pi i \frac{jk}{n}}$$

Both versions are the complex conjugated of each other. The effect is a flip of the complex plane around the real axis, so all relative results remain the same. Maybe some absolute results (absolute phase) may be affected but this is beyond the scope of this paper.

Quantum Fourier transformation as well as digital Fourier transformation deal with binary fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Nielsen/Chuang write them in the same way as decimal numbers, 1.123… This is a very compact way of writing expressions. In this paper we use the traditional notation as sums, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$

You find a paper dealing with the basics of Fourier transformation on this website in the quantum mechanics part: Fourier series.

In this paper the part with one, two and three qubits follows the style you find in the lecture of Shor above. This representation is good for learning and understanding the principle.

In the part dealing with four qubits we strictly follow the wiring diagram. This allows to easy apply the method to 5 or more qubits.

Contents

Quantum Fourier transform

The matrix for the quantum Fourier transform:

$$\begin{array}{cc|c}
 & & \overset{\displaystyle k}{\downarrow} \\
 & & \begin{array}{cccccc} 0 & 1 & 2 & \dots & n-2 & n-1 \end{array} \\ \hline
j \rightarrow & \begin{array}{c} 0 \\ 1 \\ 2 \\ \dots \\ n-2 \\ n-1 \end{array} & \frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
1 & e^{-2\pi i\frac{1}{n}} & e^{-2\pi i\frac{2}{n}} & \dots & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-1)}{n}} \\
1 & e^{-2\pi i\frac{2}{n}} & e^{-2\pi i\frac{4}{n}} & \dots & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
1 & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-2)(n-2)}{n}} & e^{-2\pi i\frac{(n-2)(n-1)}{n}} \\
1 & e^{-2\pi i\frac{(n-1)}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-1)(n-2)}{n}} & e^{-2\pi i\frac{(n-1)(n-1)}{n}}
\end{pmatrix}
\end{array}$$

Note: The development of the matrix and the proof of unitarity you find in the appendix.

Single qubit

We begin with a single qubit, $n = 2^1$.

To be consistent with 2, … n qubits we write this a little bit more complicated:

$$|j\rangle \rightarrow \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i\frac{jk}{n}} |k\rangle \rightarrow \frac{1}{\sqrt{2^1}} \sum_{k=0}^{2^1-1} e^{-2\pi i\frac{jk}{2^1}} |k\rangle \rightarrow$$

$$\frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{-\pi ijk} |k\rangle$$

We calculate:

$$|j\rangle \rightarrow \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{-\pi ijk} |k\rangle$$

Note: $|j\rangle$ can be either $|0\rangle$ or $|1\rangle$, the index j in the exponential is 0 or 1.

$|k\rangle$ is the target qubit $|0\rangle$ or $|1\rangle$ and parallel the index k in the exponent.

We get:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{-\pi ijk} |k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{-\pi ijk} |k\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The unitary matrix reduces to a 2 × 2-matrix:

$$\frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & e^{-2\pi i\frac{1}{n}} & e^{-2\pi i\frac{2}{n}} & \dots & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-1)}{n}} \\ 1 & e^{-2\pi i\frac{2}{n}} & e^{-2\pi i\frac{4}{n}} & \dots & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-2)(n-2)}{n}} & e^{-2\pi i\frac{(n-2)(n-1)}{n}} \\ 1 & e^{-2\pi i\frac{(n-1)}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-1)(n-2)}{n}} & e^{-2\pi i\frac{(n-1)(n-1)}{n}} \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Quantum Fourier transform for a single qubit becomes the Hadamard transform.

Conceptual level

$$|j\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-\pi j}|1\rangle\right)$$

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + e^{-\pi 0}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

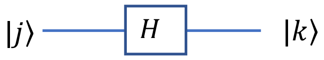
$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + e^{-\pi}|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The corresponding matrix is the Hadamard:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Basic level

The Hadamard acting on one qubit:



$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Two qubits

We use the 2 qubit Quantum Fourier transform.

$$|j_0j_1\rangle \to \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi i\frac{jk}{n}}|k_0k_1\rangle \to \frac{1}{\sqrt{2^2}}\sum_{k=0}^{2^2-1} e^{-2\pi i\frac{jk}{2^2}}|k_0k_1\rangle \to$$

$$\frac{1}{2}\sum_{k=0}^{3} e^{-\pi i\frac{jk}{2}}|k_0k_1\rangle$$

We get:

$$|j_0j_1\rangle \to \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i\frac{jk}{2}}|k_0k_1\rangle$$

The expression is somewhat confusing. The index ranges from 0 to 3, the qubits are represented by $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

We examine the structure of the entries in the matrix. We can build the indices $j \in \{0,1,2,3\}$ out of the particles of the qubits:

$$j = 2j_0 + j_1, k = 2k_0 + k_1$$

With $j_0, j_1, k_0, k_1 \in \{0,1\}$ we get:

$$0 = 2\cdot 0 + 0; \quad 1 = 2\cdot 0 + 1; \quad 2 = 2\cdot 1 + 0; \quad 3 = 2\cdot 1 + 1$$

This is equivalent to:

$$0 \mathrel{\widehat{=}} |00\rangle, 1 \mathrel{\widehat{=}} |01\rangle, 2 \mathrel{\widehat{=}} |10\rangle, 3 \mathrel{\widehat{=}} |11\rangle$$

0 corresponds to $|00\rangle$, 1 corresponds to $|01\rangle$, 2 corresponds to $|10\rangle$, 3 corresponds to $|11\rangle$.

We rewrite the entry (jk):

$$e^{-\pi i\frac{jk}{2}} = e^{-\pi i\frac{(2j_0+j_1)(2k_0+k_1)}{2}} =$$

$$e^{-\pi i\frac{4j_0k_0+2j_0k_1+2j_1k_0+j_1k_1}{2}} =$$

$$e^{-2\pi i j_0k_0}\cdot e^{-\pi i j_0k_1}\cdot e^{-\pi i j_1k_0}\cdot e^{-\frac{\pi i}{2}j_1k_1}$$

We get:

$$|j_0j_1\rangle \to \frac{1}{2}\sum_{k=0}^{3} e^{-2\pi i j_0k_0}\cdot e^{-\pi i j_0k_1}\cdot e^{-\pi i j_1k_0}\cdot e^{-\frac{\pi i}{2}j_1k_1}|k_0k_1\rangle$$

We calculate four possibilities (all modulo 2π):

$\lvert 00\rangle \to \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i\frac{4j_0k_0+2j_0k_1+2j_1k_0+j_1k_1}{2}}\lvert k_0k_1\rangle =$ $\frac{1}{2}\sum_{k=0}^{3}\lvert k_0k_1\rangle = \frac{1}{2}(\lvert 00\rangle + \lvert 01\rangle + \lvert 10\rangle + \lvert 11\rangle)$	$\lvert 01\rangle \to \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i\frac{4j_0k_0+2j_0k_1+2j_1k_0+j_1k_1}{2}}\lvert k_0k_1\rangle =$ $\frac{1}{2}\sum_{k=0}^{3} e^{-\pi i\frac{2k_0+k_1}{2}}\lvert k_0k_1\rangle =$ $\frac{1}{2}\left(\lvert 00\rangle + e^{-\frac{\pi i}{2}}\lvert 01\rangle + e^{-\pi i}\lvert 10\rangle + e^{-\frac{3\pi i}{2}}\lvert 11\rangle\right)$

$\|10\rangle \rightarrow \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i \frac{4j_0k_0+2j_0k_1+2j_1k_0+j_1k_1}{2}}\|k_0k_1\rangle =$ $\frac{1}{2}\sum_{k=0}^{3} e^{-\pi i \frac{4k_0+2k_1}{2}}\|k_0k_1\rangle =$ $\frac{1}{2}(\|00\rangle + e^{-\pi i}\|01\rangle + e^{-2\pi i}\|10\rangle + e^{-\pi i}\|11\rangle)$	$\|11\rangle \rightarrow \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i \frac{4j_0k_0+2j_0k_1+2j_1k_0+j_1k_1}{2}}\|k_0k_1\rangle =$ $\|11\rangle \rightarrow \frac{1}{2}\sum_{k=0}^{3} e^{-\pi i \frac{6k_0+3k_1}{2}}\|k_0k_1\rangle =$ $\frac{1}{2}\left(\|00\rangle + e^{-\frac{3\pi i}{2}}\|01\rangle + e^{-\pi i}\|10\rangle + e^{-\frac{\pi i}{2}}\|11\rangle\right)$

We get the matrix:

$$\frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{\pi i}{2}} & e^{-\pi i} & e^{-\frac{3\pi i}{2}} \\ 1 & e^{-\pi i} & e^{-2\pi i} & e^{-\pi i} \\ 1 & e^{-\frac{3\pi i}{2}} & e^{-\pi i} & e^{-\frac{\pi i}{2}} \end{pmatrix}$$

The matrix corresponds to the general matrix reduced to rank 4.

We rewrite:

$$\begin{array}{c|c} & k \\ & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \hline j \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \end{array}$$

In order to build a quantum circuit, we need operators that can be represented by physical processes. We need to build this matrix out of a series (product) of physically existent gates (matrices) in order to let it run on a quantum computer.

Basic level

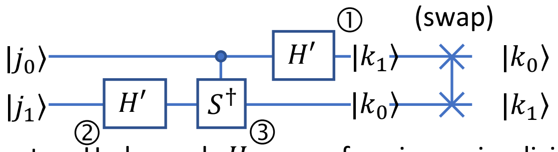
An inspiration, given by peter Shor, lets us try the following.

We translate:

$$|j_0j_1\rangle \rightarrow \frac{1}{2}\sum_{k=0}^{3} \overset{⓪}{e^{-2\pi i j_0k_0}} \cdot \overset{①}{e^{-\pi i j_0k_1}} \cdot \overset{②}{e^{-\pi i j_1k_0}} \cdot \overset{③}{e^{-\frac{\pi i}{2}j_1k_1}}\, |k_0k_1\rangle$$

⓪ is the identity that has no effect so far.	② is the Hadamard, converting j_1 to k_0.
① is the Hadamard, converting j_0 to k_1.	③ is the phase gate $S^\dagger$, from j_1 to k_1.

The circuit, three gates acting on two qubits:



Note: The two Hadamards H are performing an implicit swap as they produce $|k_0\rangle$ out of $|j_1\rangle$ and $k_1\rangle$ out of $|j_0\rangle$.

We therefore write H'.

The matrices:

$$\overset{\text{swap}}{\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}}\overset{\text{Hadamard ①}}{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}}\overset{S^\dagger\ ③}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-i\end{pmatrix}}\overset{\text{Hadamard ②}}{\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}}=$$

$$\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&-i&0&i\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&-i&-1&i\\1&i&-1&-i\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{pmatrix}$$

Note: $S^\dagger$ changes $|11\rangle$ to $|1(-i)\rangle = -i|11\rangle$.

Note: Controlled gates cannot be constructed by simple Kronecker product.

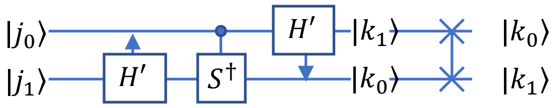
Note: The Hadamard gates are constructed traditionally but used in reverse order:

$$H \otimes Id = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}$$

$$Id \otimes H = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}$$

Conceptual level

Before the swap we have three gates acting on two qubits:



Note: The two Hadamards H perform an implicit swap as they produce $|k_0\rangle$ out of $|j_1\rangle$ and $k_1\rangle$. We mark them with an appropriate arrow to the target line.

We have the transitions:

$$|j_0\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\pi(j_0)}|1\rangle\right)$$

$$|j_1\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\pi(j_1)}|1\rangle\right) \to \left(|0\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|1\rangle\right)$$

We calculate the transitions:

$j_0 = 0, j_1 = 0$

$$|00\rangle \rightarrow \frac{1}{2}\left(|0\rangle + e^{-i\pi(j_0)}|1\rangle\right)\left(|0\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|1\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|01\rangle + e^{-i\pi(j_0)}|10\rangle + e^{-i\pi(j_0)}e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$j_0 = 0, j_1 = 1$

$$|01\rangle \rightarrow \frac{1}{2}\left(|0\rangle + e^{-i\pi(j_0)}|1\rangle\right)\left(|0\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|1\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|01\rangle + e^{-i\pi(j_0)}|10\rangle + e^{-i\pi(j_0)}e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi}|01\rangle + |10\rangle + e^{-i\pi}|11\rangle\right) =$$

$$\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$j_0 = 1, j_1 = 0$

$$|10\rangle \rightarrow \frac{1}{2}\left(|0\rangle + e^{-i\pi(j_0)}|1\rangle\right)\left(|0\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|1\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|01\rangle + e^{-i\pi(j_0)}|10\rangle + e^{-i\pi(j_0)}e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi\left(\frac{1}{2}\right)}|01\rangle + e^{-i\pi}|10\rangle + e^{-i\pi}e^{-i\pi\left(\frac{1}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)$$

$j_0 = 1, j_1 = 1$

$$|11\rangle \rightarrow \frac{1}{2}\left(|0\rangle + e^{-i\pi(j_0)}|1\rangle\right)\left(|0\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|1\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|01\rangle + e^{-i\pi(j_0)}|10\rangle + e^{-i\pi(j_0)}e^{-i\pi(j_1)}e^{-i\pi\left(\frac{j_0}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}\left(|00\rangle + e^{-i\pi}e^{-i\pi\left(\frac{1}{2}\right)}|01\rangle + e^{-i\pi}|10\rangle + e^{-i\pi}e^{-i\pi}e^{-i\pi\left(\frac{1}{2}\right)}|11\rangle\right) =$$

$$\frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle)$$

We get the matrix:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix}$$

This is the state before the swap. We add the swap, we interchange $|01\rangle$ with $|10\rangle$ and get finally:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{array}{l} \\ \leftarrow \\ \leftarrow \\ \\ \end{array}$$

This is the general quantum Fourier transform for a rank 4 matrix.

Three qubits

We apply to the 3 qubit Quantum Fourier transform, $n = 2^3$.

$$|j_0 j_1 j_2\rangle \to \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i \frac{jk}{n}} |k_0 k_1 k_2\rangle \to \frac{1}{\sqrt{2^3}} \sum_{k=0}^{2^3-1} e^{-2\pi i \frac{jk}{2^3}} |k_0 k_1 k_2\rangle \to$$

$$\frac{1}{\sqrt{8}} \sum_{k=0}^{7} e^{-\pi i \frac{jk}{4}} |k_0 k_1 k_2\rangle$$

We get:

$$|j_0 j_1 j_2\rangle \to \frac{1}{\sqrt{8}} \sum_{k=0}^{7} e^{-\pi i \frac{jk}{4}} |k_0 k_1 k_2\rangle$$

The expression is somewhat confusing. The index is running from 0 to 7, the qubits are represented by $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$.

We examine the structure of the entries in the matrix. We can build the indices $j \in \{0,1,2,3,4,5,6,7\}$ of the particles of the qubits:

$$j = 4j_0 + 2j_1 + j_2, k = 4k_0 + 2k_1 + k_2$$

With $j_0, j_1, j_2, k_0, k_1, k_2 \in \{0,1\}$ we get:

$$0 = 4 \cdot 0 + 2 \cdot 0 + 0;\ \ 1 = 4 \cdot 0 + 2 \cdot 0 + 1;\ \ 2 = 4 \cdot 0 + 2 \cdot 1 + 0;\ \ 3 = 4 \cdot 0 + 2 \cdot 1 + 1$$

$$4 = 4 \cdot 1 + 2 \cdot 0 + 0;\ \ 5 = 4 \cdot 1 + 2 \cdot 0 + 1;\ \ 6 = 4 \cdot 1 + 2 \cdot 1 + 0;\ \ 7 = 4 \cdot 1 + 2 \cdot 1 + 1$$

This is equivalent to:

$$0 \mathrel{\widehat{=}} |000\rangle, 1 \mathrel{\widehat{=}} |001\rangle, 2 \mathrel{\widehat{=}} |010\rangle, 3 \mathrel{\widehat{=}} |011\rangle$$

$$4 \mathrel{\widehat{=}} |100\rangle, 5 \mathrel{\widehat{=}} |101\rangle, 6 \mathrel{\widehat{=}} |110\rangle, 7 \mathrel{\widehat{=}} |111\rangle$$

We rewrite the entry (jk)

$$e^{-\pi i \frac{jk}{4}} = e^{-\pi i \frac{(4j_0+2j_1+j_2)(4k_0+2k_1+k_2)}{4}} =$$

$$e^{-\pi i \frac{16j_0k_0+8j_0k_1+4j_0k_2+8j_1k_0+4j_1k_1+2j_1k_2+4j_2k_0+2j_2k_1+j_2k_2}{4}} =$$

$$e^{-2\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}}$$

Note: We calculate modulo 2π.

We get:

$$|j_0 j_1 j_2\rangle \to \frac{1}{\sqrt{8}} \sum_{k=0}^{7} e^{-\pi i \frac{jk}{4}} |k_0 k_1 k_2\rangle \to$$

$$\frac{1}{\sqrt{8}} \sum_{k=0}^{7} e^{-2\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle$$

We calculate eight possibilities:

$$|000\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} |k_0 k_1 k_2\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

$$|001\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\pi i k_0} \cdot e^{-\pi i \frac{k_1}{2}} \cdot e^{-\pi i \frac{k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{1}{4}}|001\rangle + e^{-\pi i \frac{2}{4}}|010\rangle + e^{-\pi i \frac{3}{4}}|011\rangle + e^{-\pi i \frac{4}{4}}|100\rangle + e^{-\pi i \frac{5}{4}}|101\rangle + e^{-\pi i \frac{6}{4}}|110\rangle + e^{-\pi i \frac{7}{4}}|111\rangle\right)$$

$$|010\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{1}{2}}|001\rangle + e^{-\pi i \frac{2}{2}}|010\rangle + e^{-\pi i \frac{3}{2}}|011\rangle + e^{-\pi i \frac{4}{2}}|100\rangle + e^{-\pi i \frac{5}{2}}|101\rangle + e^{-\pi i \frac{6}{2}}|110\rangle + e^{-\pi i \frac{7}{2}}|111\rangle\right)$$

$$|011\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} \cdot e^{-\pi i k_0} \cdot e^{-\pi i \frac{k_1}{2}} \cdot e^{-\pi i \frac{k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-3\pi i k_0} \cdot e^{-\pi i \frac{3k_1}{2}} \cdot e^{-\pi i \frac{3k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{3}{4}}|001\rangle + e^{-\pi i \frac{6}{4}}|010\rangle + e^{-\pi i \frac{9}{4}}|011\rangle + e^{-\pi i \frac{12}{4}}|100\rangle + e^{-\pi i \frac{15}{4}}|101\rangle + e^{-\pi i \frac{18}{4}}|110\rangle + e^{-\pi i \frac{21}{4}}|111\rangle\right)$$

$$|100\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}(|000\rangle + e^{-\pi i}|001\rangle + e^{-2\pi i}|010\rangle + e^{-3\pi i}|011\rangle + e^{-4\pi i}|100\rangle + e^{-5\pi i}|101\rangle + e^{-6\pi i}|110\rangle + e^{-7\pi i}|111\rangle)$$

$$|101\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\frac{20}{4}\pi i k_0} \cdot e^{-\pi i \frac{10k_1}{4}} \cdot e^{-\pi i \frac{5k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{5}{4}}|001\rangle + e^{-\pi i \frac{10}{4}}|010\rangle + e^{-\pi i \frac{15}{4}}|011\rangle + e^{-\pi i \frac{20}{4}}|100\rangle + e^{-\pi i \frac{25}{4}}|101\rangle + e^{-\pi i \frac{30}{4}}|110\rangle + e^{-\pi i \frac{35}{4}}|111\rangle\right)$$

$$|110\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-4\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-\pi i j_0 k_2} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-\pi i j_1 k_1} \cdot e^{-\pi i \frac{j_1 k_2}{2}} \cdot e^{-\pi i j_2 k_0} \cdot e^{-\pi i \frac{j_2 k_1}{2}} \cdot e^{-\pi i \frac{j_2 k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\frac{12}{2}\pi i k_0} \cdot e^{-\frac{6}{2}\pi i k_1} \cdot e^{-\pi i \frac{3k_2}{2}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{3}{2}}|001\rangle + e^{-\pi i \frac{6}{2}}|010\rangle + e^{-\pi i \frac{9}{2}}|011\rangle + e^{-\pi i \frac{12}{2}}|100\rangle + e^{-\pi i \frac{15}{2}}|101\rangle + e^{-\pi i \frac{18}{2}}|110\rangle + e^{-\pi i \frac{21}{2}}|111\rangle\right)$$

$$|111\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\frac{16}{4}\pi i k_0} \cdot e^{-\frac{8}{4}\pi i k_1} \cdot e^{-\pi i \frac{4}{4} k_2} \cdot e^{-\frac{8}{4}\pi i k_0} \cdot e^{-\pi i \frac{4}{4} k_1} \cdot e^{-\pi i \frac{2k_2}{4}} \cdot e^{-\pi i \frac{4k_0}{4}} \cdot e^{-\pi i \frac{2k_1}{4}} \cdot e^{-\pi i \frac{k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\frac{28}{2}\pi i k_0} \cdot e^{-\frac{14}{4}\pi i k_1} \cdot e^{-\pi i \frac{7k_2}{4}} |k_0 k_1 k_2\rangle =$$

$$\frac{1}{\sqrt{8}}\left(|000\rangle + e^{-\pi i \frac{7}{4}}|001\rangle + e^{-\pi i \frac{14}{4}}|010\rangle + e^{-\pi i \frac{21}{4}}|011\rangle + e^{-\pi i \frac{28}{4}}|100\rangle + e^{-\pi i \frac{35}{4}}|101\rangle + e^{-\pi i \frac{42}{4}}|110\rangle + e^{-\pi i \frac{49}{4}}|111\rangle\right)$$

We get the matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\pi i\frac{1}{4}} & e^{-\pi i\frac{2}{4}} & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{7}{4}} \\
1 & e^{-\pi i\frac{2}{4}} & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{8}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{14}{4}} \\
1 & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{9}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{18}{4}} & e^{-\pi i\frac{21}{4}} \\
1 & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{8}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{16}{4}} & e^{-\pi i\frac{20}{4}} & e^{-\pi i\frac{24}{4}} & e^{-\pi i\frac{28}{4}} \\
1 & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{20}{4}} & e^{-\pi i\frac{25}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{35}{4}} \\
1 & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{18}{4}} & e^{-\pi i\frac{24}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{36}{4}} & e^{-\pi i\frac{42}{4}} \\
1 & e^{-\pi i\frac{7}{4}} & e^{-\pi i\frac{14}{4}} & e^{-\pi i\frac{21}{4}} & e^{-\pi i\frac{28}{4}} & e^{-\pi i\frac{35}{4}} & e^{-\pi i\frac{42}{4}} & e^{-\pi i\frac{49}{4}}
\end{pmatrix}$$

The matrix corresponds to the general matrix reduced to rank 8.

We resolve $\pm 1, \pm i$ and modulo 2π:

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} & & & k_0k_1k_2 & & & & \\ & & & \downarrow & & & & \\ 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\
\hline
j_0j_1j_2 \rightarrow \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} & \frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{5}{4}} & i & e^{-\pi i\frac{7}{4}} \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{7}{4}} & -i & e^{-\pi i\frac{5}{4}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{5}{4}} & -1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{1}{4}}
\end{pmatrix}
\end{array}$$

Again, we have the task to disassemble this matrix into gates.

We go back to the entry (jk):

$$|j_0j_1j_2\rangle \rightarrow \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{-\pi i\frac{(4j_0+2j_1+j_2)(4k_0+2k_1+k_2)}{4}}|k_0k_1k_2\rangle$$

We examine the structure of the entries in the matrix:

$$e^{-\pi i\frac{(4j_0+2j_1+j_2)(4k_0+2k_1+k_2)}{4}} =$$

$$e^{-\pi i\frac{16j_0k_0+8j_0k_1+4j_0k_2+8j_1k_0+4j_1k_1+2j_1k_2+4j_2k_0+2j_2k_1+j_2k_2}{4}} =$$

	⓪	①	②	
	$e^{-2\pi i j_0k_0}\cdot$	$e^{-2\pi i j_0k_1}\cdot$	$e^{-\pi i j_0k_2}\cdot$	
③	$e^{-2\pi i j_1k_0}\cdot$	$e^{-\pi i j_1k_1}\cdot$ ④	$e^{-\pi i\frac{j_1k_2}{2}}\cdot$	⑤
	$e^{-\pi i j_2k_0}\cdot$	$e^{-\pi i\frac{j_2k_1}{2}}\cdot$	$e^{-\pi i\frac{j_2k_2}{4}}$	
	⑥	⑦	⑧	

Note: We will find this structure in the four-qubit case again.

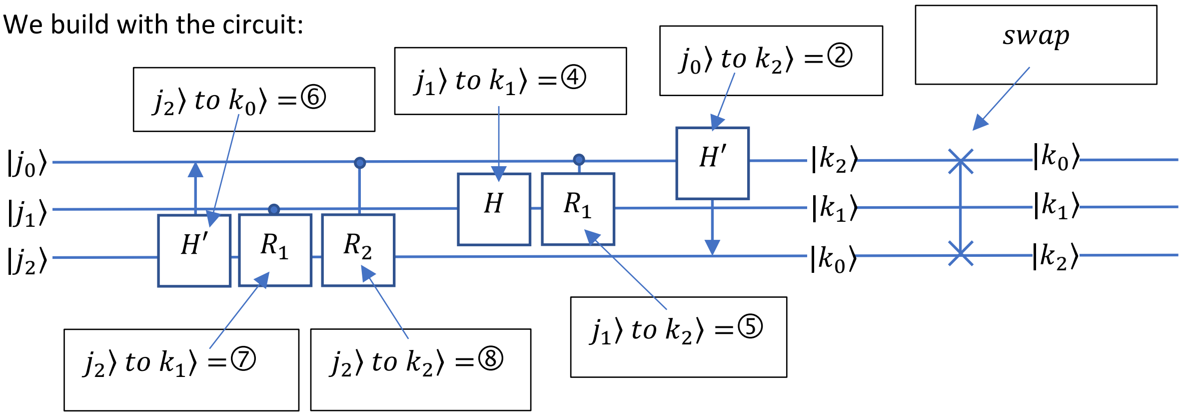
We interpret:

⑥ is the Hadamard converting j_2 to k_0.	④ is the Hadamard converting j_1 to k_1.	② is the Hadamard converting j_0 to k_2.
⑤ is the controlled $S^{\dagger}$ from j_1 to k_2	⑧ is the controlled $T^{\dagger}$ from j_2 to k_2	⑦ is the controlled $S^{\dagger}$ from j_2 to k_1

⓪, ① and ③ are identity transformations that have so far no effect.

Note: Remember the implicit swap with the Hadamards.

We build with the circuit:



Note: We name the Hadamards as H' and point to the line they perform an implicit swap with.

Note: The gates named R_1 and R_2 are controlled $S^{\dagger}$ and $T^{\dagger}$ gates. They are special forms of the single qubit unitary rotation gate R_k:

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi i}{2^k}} \end{pmatrix}$$

Conceptual level

From the circuit above we can derive the state before the swap gate:

The Hadamard, acting on $\lvert j_0\rangle$:	$\lvert j_0\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle}{\sqrt{2}}$

The Hadamard, acting on $\lvert j_1\rangle$:	$\lvert j_1\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_1}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_1-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_1}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_1} e^{-\pi i \frac{j_0}{2}}\lvert 1\rangle}{\sqrt{2}}$

The Hadamard, acting on $\lvert j_2\rangle$:	$\lvert j_2\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_1-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_2}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_2-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}} e^{-\pi i \frac{j_0}{4}}\lvert 1\rangle}{\sqrt{2}}$

We combine:

$$\left(\frac{|0\rangle + e^{-\pi i j_0}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}|1\rangle}{\sqrt{2}}\right) =$$

$$\frac{1}{\sqrt{8}}\left(|0\rangle + e^{-\pi i j_0}|1\rangle\right)\Big(|00\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}|01\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}|10\rangle$$
$$+ e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}|11\rangle\Big) =$$

$$\frac{1}{\sqrt{8}}\left(|0\rangle + e^{-\pi i\frac{4}{4}j_0}|1\rangle\right)\Big(|00\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|01\rangle + e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}|10\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|11\rangle\Big) =$$

$$\frac{1}{\sqrt{8}}\left(|0\rangle + e^{-\pi i\frac{4}{4}j_0}|1\rangle\right)\Big(|00\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|01\rangle + e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}|10\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{6j_1}{4}}e^{-\pi i\frac{3j_0}{4}}|11\rangle\Big) =$$

$$\frac{1}{\sqrt{8}}\Big(|000\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|001\rangle + e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}|010\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{6j_1}{4}}e^{-\pi i\frac{3j_0}{4}}|011\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_0}|100\rangle + e^{-\pi i\frac{4}{4}j_0}e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|101\rangle + e^{-\pi i\frac{4}{4}j_0}e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}|110\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_0}e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{6j_1}{4}}e^{-\pi i\frac{3j_0}{4}}|111\rangle\Big) =$$

$$\frac{1}{\sqrt{8}}\Big(|000\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{j_0}{4}}|001\rangle + e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{2j_0}{4}}|010\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{6j_1}{4}}e^{-\pi i\frac{3j_0}{4}}|011\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_0}|100\rangle + e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{2j_1}{4}}e^{-\pi i\frac{5j_0}{4}}|101\rangle + e^{-\pi i\frac{4}{4}j_1}e^{-\pi i\frac{6j_0}{4}}|110\rangle$$
$$+ e^{-\pi i\frac{4}{4}j_2}e^{-\pi i\frac{6j_1}{4}}e^{-\pi i\frac{7j_0}{4}}|111\rangle\Big)$$

We get the transitions:

$\|000\rangle$ $j_0 = 0,$ $j_1 = 0,$ $j_2 = 0$	$\frac{1}{\sqrt{8}}(\|000\rangle + \|001\rangle + \|010\rangle + \|011\rangle + \|100\rangle + \|101\rangle + \|110\rangle + \|111\rangle)$
$\|001\rangle$ $j_0 = 0,$ $j_1 = 0,$ $j_2 = 1$	$\frac{1}{\sqrt{8}}(\|000\rangle + e^{-\pi i}\|001\rangle + \|010\rangle + e^{-\pi i}\|011\rangle + \|100\rangle + e^{-\pi i}\|101\rangle + \|110\rangle + e^{-\pi i}\|111\rangle)$
$\|010\rangle$ $j_0 = 0,$ $j_1 = 1,$ $j_2 = 0$	$\frac{1}{\sqrt{8}}(\|000\rangle + e^{-\pi i\frac{1}{2}}\|001\rangle + e^{-\pi i}\|010\rangle + e^{-\pi i\frac{3}{2}}\|011\rangle + \|100\rangle + e^{-\pi i\frac{1}{2}}\|101\rangle + e^{-\pi i}\|110\rangle$ $+ e^{-\pi i\frac{3}{2}}\|111\rangle)$
$\|011\rangle$ $j_0 = 0,$ $j_1 = 1,$ $j_2 = 1$	$\frac{1}{\sqrt{8}}(\|000\rangle + e^{-\pi i\frac{3}{2}}\|001\rangle + e^{-\pi i}\|010\rangle + e^{-\pi i\frac{4}{2}}\|011\rangle + \|100\rangle + e^{-\pi i\frac{3}{2}}\|101\rangle + e^{-\pi i}\|110\rangle$ $+ e^{-\pi i\frac{1}{2}}\|111\rangle)$

$\vert 100\rangle$ $j_0 = 1,$ $j_1 = 0,$ $j_2 = 0$	$\frac{1}{\sqrt{8}}\Big(\vert 000\rangle + e^{-\pi i\frac{1}{4}}\vert 001\rangle + e^{-\pi i\frac{1}{2}}\vert 010\rangle + e^{-\pi i\frac{3}{4}}\vert 011\rangle + e^{-\pi i}\vert 100\rangle + e^{-\pi i\frac{5}{4}}\vert 101\rangle + e^{-\pi i\frac{3}{2}}\vert 110\rangle + e^{-\pi i\frac{7}{4}}\vert 111\rangle\Big)$
$\vert 101\rangle$ $j_0 = 1,$ $j_1 = 0,$ $j_2 = 1$	$\frac{1}{\sqrt{8}}\Big(\vert 000\rangle + e^{-\pi i\frac{5}{4}}\vert 001\rangle + e^{-\pi i\frac{1}{2}}\vert 010\rangle + e^{-\pi i\frac{7}{4}}\vert 011\rangle + e^{-\pi i}\vert 100\rangle + e^{-\pi i\frac{1}{4}}\vert 101\rangle + e^{-\pi i\frac{3}{2}}\vert 110\rangle + e^{-\pi i\frac{3}{4}}\vert 111\rangle\Big)$
$\vert 110\rangle$ $j_0 = 1,$ $j_1 = 1,$ $j_2 = 0$	$\frac{1}{\sqrt{8}}\Big(\vert 000\rangle + e^{-\pi i\frac{3}{4}}\vert 001\rangle + e^{-\pi i\frac{3}{2}}\vert 010\rangle + e^{-\pi i\frac{1}{4}}\vert 011\rangle + e^{-\pi i}\vert 100\rangle + e^{-\pi i\frac{7}{4}}\vert 101\rangle + e^{-\pi i\frac{1}{2}}\vert 110\rangle + e^{-\pi i\frac{5}{4}}\vert 111\rangle\Big)$
$\vert 111\rangle$ $j_0 = 1,$ $j_1 = 1,$ $j_2 = 1$	$\frac{1}{\sqrt{8}}\Big(\vert 000\rangle + e^{-\pi i\frac{7}{4}}\vert 001\rangle + e^{-\pi i\frac{3}{2}}\vert 010\rangle + e^{-\pi i\frac{5}{4}}\vert 011\rangle + e^{-\pi i}\vert 100\rangle + e^{-\pi i\frac{3}{4}}\vert 101\rangle + e^{-\pi i\frac{1}{2}}\vert 110\rangle + e^{-\pi i\frac{1}{4}}\vert 111\rangle\Big)$

From the transitions we calculate the matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{3}{4}} \\
1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\
1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{7}{4}} & -i & e^{-\pi i\frac{5}{4}} \\
1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{5}{4}} & -1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{1}{4}}
\end{pmatrix}$$

We need the final swap, interchanging row 2 with row 5: $001 \leftrightarrow 100$ and row 4 with row 7: $011 \leftrightarrow 110$.

We get the matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{3}{4}} \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{7}{4}} & -i & e^{-\pi i\frac{5}{4}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{5}{4}} & -1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{1}{4}}
\end{pmatrix}$$

This is the matrix for the Fourier transformation for three qubits.

Basic level

We use the 3 qubit Quantum Fourier transform. The matrix becomes 8×8:

We use the unitary matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\pi i\frac{1}{4}} & e^{-\pi i\frac{2}{4}} & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{7}{4}} \\ 1 & e^{-\pi i\frac{2}{4}} & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{8}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{14}{4}} \\ 1 & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{9}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{18}{4}} & e^{-\pi i\frac{21}{4}} \\ 1 & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{8}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{16}{4}} & e^{-\pi i\frac{20}{4}} & e^{-\pi i\frac{24}{4}} & e^{-\pi i\frac{28}{4}} \\ 1 & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{20}{4}} & e^{-\pi i\frac{25}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{35}{4}} \\ 1 & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{18}{4}} & e^{-\pi i\frac{24}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{36}{4}} & e^{-\pi i\frac{42}{4}} \\ 1 & e^{-\pi i\frac{7}{4}} & e^{-\pi i\frac{14}{4}} & e^{-\pi i\frac{21}{4}} & e^{-\pi i\frac{28}{4}} & e^{-\pi i\frac{35}{4}} & e^{-\pi i\frac{42}{4}} & e^{-\pi i\frac{49}{4}} \end{pmatrix} =$$

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{5}{4}} & i & e^{-\pi i\frac{7}{4}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{7}{4}} & -i & e^{-\pi i\frac{5}{4}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{5}{4}} & -1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{1}{4}} \end{pmatrix}$$

This is a symmetric matrix.

We need to build this matrix out of a series (product) of gates (matrices) in order to let it run on a quantum computer. We will calculate all gates “from left to the right”.

The first Hadamard is the Kronecker product. We remember that they perform an implicit swap and calculate:

$$H \otimes Id \otimes Id = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The first R_1-gate:

$$
\begin{array}{cc|c}
 & & k \\
 & & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \hline
j & \begin{array}{c} 000\\ 001\\ 010\\ 011\\ 100\\ 101\\ 110\\ 111 \end{array} & \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{pmatrix}
\end{array}
$$

Note: As the first Hadamard does an implicit switch, we need to modify the first R_1-gate.

The first R_2-gate:

$$
\begin{array}{cc|c}
 & & k \\
 & & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \hline
j & \begin{array}{c} 000\\ 001\\ 010\\ 011\\ 100\\ 101\\ 110\\ 111 \end{array} & \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-\pi i \frac{1}{4}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\pi i \frac{1}{4}}
\end{pmatrix}
\end{array}
$$

Note: no modification needed because this gate is positioned symmetric.

Note: Controlled gates cannot be disassembled in simple Kronecker products.

The Hadamard on qubit $|j_1\rangle$ is the Kronecker product:

$$
Id \otimes H \otimes Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =
$$

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix}
$$

The second R_1-gate:

$$
\begin{array}{cc|c}
 & & k \\
 & & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \hline
j & \begin{array}{c} 000\\ 001\\ 010\\ 011\\ 100\\ 101\\ 110\\ 111 \end{array} & \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{pmatrix}
\end{array}
$$

Note: Controlled gates cannot be disassembled in simple Kronecker products.

The last Hadamard is the Kronecker product. We remember the implicit swap and calculate:

$$Id \otimes Id \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The swap matrix interchanges qubit 0 with qubit 2:

$$\begin{array}{c|cccccccc} & \multicolumn{8}{c}{k} \\ & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline 000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 001 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ j \quad 011 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 100 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 110 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

We calculate the product of all matrices (see appendix) and get:

$$\frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\pi i \frac{1}{4}} & -i & e^{-\pi i \frac{3}{4}} & -1 & e^{-\pi i \frac{5}{4}} & i & e^{-\pi i \frac{7}{4}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & e^{-\pi i \frac{3}{4}} & i & e^{-\pi i \frac{1}{4}} & -1 & e^{-\pi i \frac{7}{4}} & -i & e^{-\pi i \frac{5}{4}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{-\pi i \frac{5}{4}} & -i & e^{-\pi i \frac{7}{4}} & -1 & e^{-\pi i \frac{1}{4}} & i & e^{-\pi i \frac{3}{4}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & e^{-\pi i \frac{7}{4}} & i & e^{-\pi i \frac{5}{4}} & -1 & e^{-\pi i \frac{3}{4}} & -i & e^{-\pi i \frac{1}{4}} \end{pmatrix}$$

This is the matrix for the Fourier transformation for three qubits.

Four qubits

We apply to the 4 qubit Quantum Fourier transform, $n = 2^4$.

$$|j_0 j_1 j_2 j_3\rangle \to \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i \frac{jk}{n}} |k_0 k_1 k_2 k_3\rangle \to \frac{1}{\sqrt{2^4}} \sum_{k=0}^{2^4-1} e^{-2\pi i \frac{jk}{2^4}} |k_0 k_1 k_2 k_3\rangle \to$$

$$\frac{1}{4} \sum_{k=0}^{15} e^{-\pi i \frac{jk}{8}} |k_0 k_1 k_2 k_3\rangle$$

We get:

$$|j_0 j_1 j_2 j_3\rangle \to \frac{1}{4} \sum_{k=0}^{15} e^{-\pi i \frac{jk}{8}} |k_0 k_1 k_2 k_3\rangle$$

The index is running from 0 to 15, the qubits are represented by:

$$|0000\rangle, |0001\rangle, |0010\rangle, |0011\rangle, |0100\rangle, |0101\rangle, |0110\rangle, |0111\rangle,$$

$$|1000\rangle, |1001\rangle, |1010\rangle, |1011\rangle, |1100\rangle, |1101\rangle, |1110\rangle, |1111\rangle$$

We examine the structure of the entries in the matrix.

We can build the indices $j \in \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ from the particles of the qubits:

$$j = 8j_0 + 4j_1 + 2j_2 + j_3, k = 8k_0 + 4k_1 + 2k_2 + k_3$$

With $j_0, j_1, j_2, j_3, k_0, k_1, k_2, k_3 \in \{0,1\}$ we get:

$$0 = 8 \cdot 0 + 4 \cdot 0 + 2 \cdot 0 + 0; \quad 1 = 8 \cdot 0 + 4 \cdot 0 + 2 \cdot 0 + 1; \quad 2 = 8 \cdot 0 + 4 \cdot 0 + 2 \cdot 1 + 0; \quad 3 = 8 \cdot 0 + 4 \cdot 0 + 2 \cdot 1 + 1$$

$$4 = 8 \cdot 0 + 4 \cdot 1 + 2 \cdot 0 + 0; \quad 5 = 8 \cdot 0 + 4 \cdot 1 + 2 \cdot 0 + 1; \quad 6 = 8 \cdot 0 + 4 \cdot 1 + 2 \cdot 1 + 0; \quad 7 = 8 \cdot 0 + 4 \cdot 1 + 2 \cdot 1 + 1$$

$$8 = 8 \cdot 1 + 4 \cdot 0 + 2 \cdot 0 + 0; \quad 9 = 8 \cdot 1 + 4 \cdot 0 + 2 \cdot 0 + 1; \quad 10 = 8 \cdot 1 + 4 \cdot 0 + 2 \cdot 1 + 0; \quad 11 = 8 \cdot 1 + 4 \cdot 0 + 2 \cdot 1 + 1$$

$$12 = 8 \cdot 1 + 4 \cdot 1 + 2 \cdot 0 + 0; \quad 13 = 8 \cdot 1 + 4 \cdot 1 + 2 \cdot 0 + 1; \quad 14 = 8 \cdot 1 + 4 \cdot 1 + 2 \cdot 1 + 0; \quad 15 = 8 \cdot 1 + 4 \cdot 1 + 2 \cdot 1 + 1$$

This is equivalent to:

$$0 \triangleq |0000\rangle, 1 \triangleq |0001\rangle, 2 \triangleq |0010\rangle, 3 \triangleq |0011\rangle, 4 \triangleq |0100\rangle, 5 \triangleq |0101\rangle, 6 \triangleq |0110\rangle, 7 \triangleq |0111\rangle$$

$$8 \triangleq |1000\rangle, 9 \triangleq |1001\rangle, 10 \triangleq |1010\rangle, 11 \triangleq |1011\rangle, 12 \triangleq |1100\rangle, 13 \triangleq |1101\rangle, 14 \triangleq |1110\rangle, 15 \triangleq |1111\rangle$$

We rewrite the entry (jk)

$$e^{-\pi i \frac{jk}{8}} = e^{-\pi i \frac{(8j_0+4j_1+2j_2+j_3)(8k_0+4k_1+2k_2+k_3)}{8}} =$$

$$e^{-\pi i \frac{64j_0k_0+32j_0k_1+16j_0k_2+8j_0k_3+32j_1k_0+16j_1k_1+8j_1k_2+4j_1k_3+16j_2k_0+8j_2k_1+4j_2k_2+2j_2k_3+8j_3k_0+4j_3k_1+2j_3k_2+j_3k_3}{8}} =$$

$$e^{-\pi i \left(8j_0k_0+4j_0k_1+2j_0k_2+j_0k_3+4j_1k_0+2j_1k_1+j_1k_2+\frac{j_1k_3}{2}+2j_2k_0+j_2k_1+\frac{j_2k_2}{2}+\frac{j_2k_3}{4}+j_3k_0+\frac{j_3k_1}{2}+\frac{j_3k_2}{4}+\frac{j_3k_3}{8}\right)} =$$

$$e^{-2\pi i j_0 k_0} \cdot e^{-2\pi i j_0 k_1} \cdot e^{-2\pi i j_0 k_2} \cdot e^{-\pi i j_0 k_3} \cdot e^{-2\pi i j_1 k_0} \cdot e^{-2\pi i j_1 k_1} \cdot e^{-\pi i j_1 k_2} \cdot e^{-\pi i \frac{j_1 k_3}{2}} \cdot e^{-2\pi i j_2 k_0}$$
$$\cdot e^{-\pi i j_2 k_1} \cdot e^{-\pi i \frac{j_2 k_2}{2}} \cdot e^{-\pi i \frac{j_2 k_3}{4}} \cdot e^{-\pi i j_3 k_0} \cdot e^{-\pi i \frac{j_3 k_1}{2}} \cdot e^{-\pi i \frac{j_3 k_2}{4}} \cdot e^{-\pi i \frac{j_3 k_3}{8}}$$

Note: We calculate modulo 2π.

We get:

$$|j_0j_1j_2j_3\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{15} e^{-\pi i\frac{jk}{8}}\,|k_0k_1k_2k_3\rangle \rightarrow$$

$$\frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i j_0k_0}\cdot e^{-2\pi i j_0k_1}\cdot e^{-2\pi i j_0k_2}\cdot e^{-\pi i j_0k_3}\cdot e^{-2\pi i j_1k_0}\cdot e^{-2\pi i j_1k_1}\cdot e^{-\pi i j_1k_2}\cdot e^{-\pi i\frac{j_1k_3}{2}}\cdot e^{-2\pi i j_2k_0} \cdot e^{-\pi i j_2k_1}\cdot e^{-\pi i\frac{j_2k_2}{2}}\cdot e^{-\pi i\frac{j_2k_3}{4}}\cdot e^{-\pi i j_3k_0}\cdot e^{-\pi i\frac{j_3k_1}{2}}\cdot e^{-\pi i\frac{j_3k_2}{4}}\cdot e^{-\pi i\frac{j_3k_3}{8}}\,|k_0k_1k_2k_3\rangle$$

We calculate 16 possibilities (note: inconsequently sometimes calculated modulo 2π):

$|0000\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i j_0k_0}\cdot e^{-2\pi i j_0k_1}\cdot e^{-2\pi i j_0k_2}\cdot e^{-\pi i j_0k_3}\cdot e^{-2\pi i j_1k_0}\cdot e^{-2\pi i j_1k_1}\cdot e^{-\pi i j_1k_2}\cdot e^{-\pi i\frac{j_1k_3}{2}}\cdot e^{-2\pi i j_2k_0} \cdot e^{-\pi i j_2k_1}\cdot e^{-\pi i\frac{j_2k_2}{2}}\cdot e^{-\pi i\frac{j_2k_3}{4}}\cdot e^{-\pi i j_3k_0}\cdot e^{-\pi i\frac{j_3k_1}{2}}\cdot e^{-\pi i\frac{j_3k_2}{4}}\cdot e^{-\pi i\frac{j_3k_3}{8}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}(|0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle + |0101\rangle + |0110\rangle + |0111\rangle + |1000\rangle + |1001\rangle + |1010\rangle + |1011\rangle + |1100\rangle + |1101\rangle + |1110\rangle + |1111\rangle)$
$|0001\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-\pi i k_0}\cdot e^{-\pi i\frac{k_1}{2}}\cdot e^{-\pi i\frac{k_2}{4}}\cdot e^{-\pi i\frac{k_3}{8}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{1}{8}}|0001\rangle + e^{-\pi i\frac{2}{8}}|0010\rangle + e^{-\pi i\frac{3}{8}}|0011\rangle + e^{-\pi i\frac{4}{8}}|0100\rangle + e^{-\pi i\frac{5}{8}}|0101\rangle + e^{-\pi i\frac{6}{8}}|0110\rangle + e^{-\pi i\frac{7}{8}}|0111\rangle + e^{-\pi i\frac{8}{8}}|1000\rangle + e^{-\pi i\frac{9}{8}}|1001\rangle + e^{-\pi i\frac{10}{8}}|1010\rangle + e^{-\pi i\frac{11}{8}}|1011\rangle + e^{-\pi i\frac{12}{8}}|1100\rangle + e^{-\pi i\frac{13}{8}}|1101\rangle + e^{-\pi i\frac{14}{8}}|1110\rangle + e^{-\pi i\frac{15}{8}}|1111\rangle\Big)$
$|0010\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0}\cdot e^{-\pi i k_1}\cdot e^{-\pi i\frac{k_2}{2}}\cdot e^{-\pi i\frac{k_3}{4}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{1}{4}}|0001\rangle + e^{-\pi i\frac{2}{4}}|0010\rangle + e^{-\pi i\frac{3}{4}}|0011\rangle + e^{-\pi i\frac{4}{4}}|0100\rangle + e^{-\pi i\frac{5}{4}}|0101\rangle + e^{-\pi i\frac{6}{4}}|0110\rangle + e^{-\pi i\frac{7}{4}}|0111\rangle + e^{-\pi i\frac{8}{4}}|1000\rangle + e^{-\pi i\frac{9}{4}}|1001\rangle + e^{-\pi i\frac{10}{4}}|1010\rangle + e^{-\pi i\frac{11}{4}}|1011\rangle + e^{-\pi i\frac{12}{4}}|1100\rangle + e^{-\pi i\frac{13}{4}}|1101\rangle + e^{-\pi i\frac{14}{4}}|1110\rangle + e^{-\pi i\frac{15}{4}}|1111\rangle\Big)$
$|0011\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0}\cdot e^{-\pi i k_1}\cdot e^{-\pi i\frac{k_2}{2}}\cdot e^{-\pi i\frac{k_3}{4}}\cdot e^{-\pi i k_0}\cdot e^{-\pi i\frac{k_1}{2}}\cdot e^{-\pi i\frac{k_2}{4}}\cdot e^{-\pi i\frac{k_3}{8}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{3}{8}}|0001\rangle + e^{-\pi i\frac{6}{8}}|0010\rangle + e^{-\pi i\frac{9}{8}}|0011\rangle + e^{-\pi i\frac{12}{8}}|0100\rangle + e^{-\pi i\frac{15}{8}}|0101\rangle + e^{-\pi i\frac{18}{8}}|0110\rangle + e^{-\pi i\frac{21}{8}}|0111\rangle + e^{-\pi i\frac{24}{8}}|1000\rangle + e^{-\pi i\frac{27}{8}}|1001\rangle + e^{-\pi i\frac{30}{8}}|1010\rangle + e^{-\pi i\frac{33}{8}}|1011\rangle + e^{-\pi i\frac{36}{8}}|1100\rangle + e^{-\pi i\frac{39}{8}}|1101\rangle + e^{-\pi i\frac{42}{8}}|1110\rangle + e^{-\pi i\frac{45}{8}}|1111\rangle\Big)$
$|0100\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i j_1k_0}\cdot e^{-2\pi i j_1k_1}\cdot e^{-\pi i j_1k_2}\cdot e^{-\pi i\frac{j_1k_3}{2}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{1}{2}}|0001\rangle + e^{-\pi i\frac{2}{2}}|0010\rangle + e^{-\pi i\frac{3}{2}}|0011\rangle + e^{-\pi i\frac{4}{2}}|0100\rangle + e^{-\pi i\frac{5}{2}}|0101\rangle + e^{-\pi i\frac{6}{2}}|0110\rangle + e^{-\pi i\frac{7}{2}}|0111\rangle + e^{-\pi i\frac{8}{2}}|1000\rangle + e^{-\pi i\frac{9}{2}}|1001\rangle + e^{-\pi i\frac{10}{2}}|1010\rangle + e^{-\pi i\frac{11}{2}}|1011\rangle + e^{-\pi i\frac{12}{2}}|1100\rangle + e^{-\pi i\frac{13}{2}}|1101\rangle + e^{-\pi i\frac{14}{2}}|1110\rangle + e^{-\pi i\frac{15}{2}}|1111\rangle\Big)$
$|0101\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0}\cdot e^{-2\pi i k_1}\cdot e^{-\pi i k_2}\cdot e^{-\pi i\frac{k_3}{2}}\cdot e^{-\pi i k_0}\cdot e^{-\pi i\frac{k_1}{2}}\cdot e^{-\pi i\frac{k_2}{4}}\cdot e^{-\pi i\frac{k_3}{8}}\,|k_0k_1k_2k_3\rangle = \frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{5}{8}}|0001\rangle + e^{-\pi i\frac{10}{8}}|0010\rangle + e^{-\pi i\frac{15}{8}}|0011\rangle + e^{-\pi i\frac{20}{8}}|0100\rangle + e^{-\pi i\frac{25}{8}}|0101\rangle + e^{-\pi i\frac{30}{8}}|0110\rangle + e^{-\pi i\frac{35}{8}}|0111\rangle + e^{-\pi i\frac{40}{8}}|1000\rangle + e^{-\pi i\frac{45}{8}}|1001\rangle + e^{-\pi i\frac{50}{8}}|1010\rangle + e^{-\pi i\frac{55}{8}}|1011\rangle + e^{-\pi i\frac{60}{8}}|1100\rangle + e^{-\pi i\frac{65}{8}}|1101\rangle + e^{-\pi i\frac{70}{8}}|1110\rangle + e^{-\pi i\frac{75}{8}}|1111\rangle\Big)$

$$|0110\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i \frac{k_3}{2}} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} \cdot e^{-\pi i \frac{k_3}{4}} \cdot |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{3}{4}}|0001\rangle + e^{-\pi i \frac{6}{4}}|0010\rangle + e^{-\pi i \frac{9}{4}}|0011\rangle + e^{-\pi i \frac{12}{4}}|0100\rangle + e^{-\pi i \frac{15}{4}}|0101\rangle + e^{-\pi i \frac{18}{4}}|0110\rangle$$

$$+ e^{-\pi i \frac{21}{4}}|0111\rangle + e^{-\pi i \frac{24}{4}}|1000\rangle + e^{-\pi i \frac{27}{4}}|1001\rangle + e^{-\pi i \frac{30}{4}}|1010\rangle + e^{-\pi i \frac{33}{4}}|1011\rangle + e^{-\pi i \frac{36}{4}}|1100\rangle$$

$$+ e^{-\pi i \frac{39}{4}}|1101\rangle + e^{-\pi i \frac{42}{4}}|1110\rangle + e^{-\pi i \frac{45}{4}}|1111\rangle\Big)$$

$$|0111\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i \frac{k_3}{2}} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} \cdot e^{-\pi i \frac{k_3}{4}} \cdot e^{-\pi i k_0} \cdot e^{-\pi i \frac{k_1}{2}} \cdot e^{-\pi i \frac{k_2}{4}}$$

$$\cdot e^{-\pi i \frac{k_3}{8}} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{7}{8}}|0001\rangle + e^{-\pi i \frac{14}{8}}|0010\rangle + e^{-\pi i \frac{21}{8}}|0011\rangle + e^{-\pi i \frac{28}{8}}|0100\rangle + e^{-\pi i \frac{35}{8}}|0101\rangle + e^{-\pi i \frac{42}{8}}|0110\rangle$$

$$+ e^{-\pi i \frac{49}{8}}|0111\rangle + e^{-\pi i \frac{56}{8}}|1000\rangle + e^{-\pi i \frac{63}{8}}|1001\rangle + e^{-\pi i \frac{70}{8}}|1010\rangle + e^{-\pi i \frac{77}{8}}|1011\rangle + e^{-\pi i \frac{84}{8}}|1100\rangle$$

$$+ e^{-\pi i \frac{91}{8}}|1101\rangle + e^{-\pi i \frac{98}{8}}|1110\rangle + e^{-\pi i \frac{105}{8}}|1111\rangle\Big)$$

$$|1000\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\big(|0000\rangle + e^{-\pi i}|0001\rangle + e^{-2\pi i}|0010\rangle + e^{-3\pi i}|0011\rangle + e^{-4\pi i}|0100\rangle + e^{-5\pi i}|0101\rangle + e^{-6\pi i}|0110\rangle + e^{-7\pi i}|0111\rangle$$

$$+ e^{-8\pi i}|1000\rangle + e^{-9\pi i}|1001\rangle + e^{-10\pi i}|1010\rangle + e^{-11\pi i}|1011\rangle + e^{-12\pi i}|1100\rangle + e^{-13\pi i}|1101\rangle$$

$$+ e^{-14\pi i}|1110\rangle + e^{-15\pi i}|1111\rangle\big)$$

$$|1001\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-\pi i k_0} \cdot e^{-\pi i \frac{k_1}{2}} \cdot e^{-\pi i \frac{k_2}{4}} \cdot e^{-\pi i \frac{k_3}{8}} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{9}{8}}|0001\rangle + e^{-\pi i \frac{18}{8}}|0010\rangle + e^{-\pi i \frac{27}{8}}|0011\rangle + e^{-\pi i \frac{36}{8}}|0100\rangle + e^{-\pi i \frac{45}{8}}|0101\rangle + e^{-\pi i \frac{54}{8}}|0110\rangle$$

$$+ e^{-\pi i \frac{63}{8}}|0111\rangle + e^{-\pi i \frac{72}{8}}|1000\rangle + e^{-\pi i \frac{81}{8}}|1001\rangle + e^{-\pi i \frac{90}{8}}|1010\rangle + e^{-\pi i \frac{99}{8}}|1011\rangle + e^{-\pi i \frac{108}{8}}|1100\rangle$$

$$+ e^{-\pi i \frac{117}{8}}|1101\rangle + e^{-\pi i \frac{126}{8}}|1110\rangle + e^{-\pi i \frac{135}{8}}|1111\rangle\Big)$$

$$|1010\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} \cdot e^{-\pi i \frac{k_3}{4}} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{5}{4}}|0001\rangle + e^{-\pi i \frac{10}{4}}|0010\rangle + e^{-\pi i \frac{15}{4}}|0011\rangle + e^{-\pi i \frac{20}{4}}|0100\rangle + e^{-\pi i \frac{25}{4}}|0101\rangle + e^{-\pi i \frac{30}{4}}|0110\rangle$$

$$+ e^{-\pi i \frac{35}{4}}|0111\rangle + e^{-\pi i \frac{40}{4}}|1000\rangle + e^{-\pi i \frac{45}{4}}|1001\rangle + e^{-\pi i \frac{50}{4}}|1010\rangle + e^{-\pi i \frac{55}{4}}|1011\rangle + e^{-\pi i \frac{60}{4}}|1100\rangle$$

$$+ e^{-\pi i \frac{65}{4}}|1101\rangle + e^{-\pi i \frac{70}{4}}|1110\rangle + e^{-\pi i \frac{75}{4}}|1111\rangle\Big)$$

$$|1011\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i \frac{k_2}{2}} \cdot e^{-\pi i \frac{k_3}{4}} \cdot e^{-\pi i k_0} \cdot e^{-\pi i \frac{k_1}{2}} \cdot e^{-\pi i \frac{k_2}{4}}$$

$$\cdot e^{-\pi i \frac{k_3}{8}} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{11}{8}}|0001\rangle + e^{-\pi i \frac{22}{8}}|0010\rangle + e^{-\pi i \frac{33}{8}}|0011\rangle + e^{-\pi i \frac{44}{8}}|0100\rangle + e^{-\pi i \frac{55}{8}}|0101\rangle + e^{-\pi i \frac{66}{8}}|0110\rangle$$

$$+ e^{-\pi i \frac{77}{8}}|0111\rangle + e^{-\pi i \frac{88}{8}}|1000\rangle + e^{-\pi i \frac{99}{8}}|1001\rangle + e^{-\pi i \frac{110}{8}}|1010\rangle + e^{-\pi i \frac{121}{8}}|1011\rangle$$

$$+ e^{-\pi i \frac{132}{8}}|1100\rangle + e^{-\pi i \frac{143}{8}}|1101\rangle + e^{-\pi i \frac{154}{8}}|1110\rangle + e^{-\pi i \frac{165}{8}}|1111\rangle\Big)$$

$$|1100\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i \frac{k_3}{2}} |k_0 k_1 k_2 k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i \frac{3}{2}}|0001\rangle + e^{-\pi i \frac{6}{2}}|0010\rangle + e^{-\pi i \frac{9}{2}}|0011\rangle + e^{-\pi i \frac{12}{2}}|0100\rangle + e^{-\pi i \frac{15}{2}}|0101\rangle + e^{-\pi i \frac{18}{2}}|0110\rangle$$

$$+ e^{-\pi i \frac{21}{2}}|0111\rangle + e^{-\pi i \frac{24}{2}}|1000\rangle + e^{-\pi i \frac{27}{2}}|1001\rangle + e^{-\pi i \frac{30}{2}}|1010\rangle + e^{-\pi i \frac{33}{2}}|1011\rangle + e^{-\pi i \frac{36}{2}}|1100\rangle$$

$$+ e^{-\pi i \frac{39}{2}}|1101\rangle + e^{-\pi i \frac{42}{2}}|1110\rangle + e^{-\pi i \frac{45}{2}}|1111\rangle\Big)$$

$$|1101\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i\frac{k_3}{2}} \cdot e^{-\pi i k_0} \cdot e^{-\pi i\frac{k_1}{2}} \cdot e^{-\pi i\frac{k_2}{4}} \cdot e^{-\pi i\frac{k_3}{8}}\,|k_0k_1k_2k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{13}{8}}|0001\rangle + e^{-\pi i\frac{26}{8}}|0010\rangle + e^{-\pi i\frac{39}{8}}|0011\rangle + e^{-\pi i\frac{52}{8}}|0100\rangle + e^{-\pi i\frac{65}{8}}|0101\rangle + e^{-\pi i\frac{78}{8}}|0110\rangle + e^{-\pi i\frac{91}{8}}|0111\rangle + e^{-\pi i\frac{104}{8}}|1000\rangle + e^{-\pi i\frac{117}{8}}|1001\rangle + e^{-\pi i\frac{130}{8}}|1010\rangle + e^{-\pi i\frac{143}{8}}|1011\rangle + e^{-\pi i\frac{158}{8}}|1100\rangle + e^{-\pi i\frac{171}{8}}|1101\rangle + e^{-\pi i\frac{184}{8}}|1110\rangle + e^{-\pi i\frac{197}{8}}|1111\rangle\Big)$$

$$|1110\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i\frac{k_3}{2}} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i\frac{k_2}{2}} \cdot e^{-\pi i\frac{k_3}{4}}\,|k_0k_1k_2k_3\rangle =$$

$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{7}{4}}|0001\rangle + e^{-\pi i\frac{14}{4}}|0010\rangle + e^{-\pi i\frac{21}{4}}|0011\rangle + e^{-\pi i\frac{28}{4}}|0100\rangle + e^{-\pi i\frac{35}{4}}|0101\rangle + e^{-\pi i\frac{42}{4}}|0110\rangle + e^{-\pi i\frac{49}{4}}|0111\rangle + e^{-\pi i\frac{56}{4}}|1000\rangle + e^{-\pi i\frac{63}{4}}|1001\rangle + e^{-\pi i\frac{70}{4}}|1010\rangle + e^{-\pi i\frac{77}{4}}|1011\rangle + e^{-\pi i\frac{84}{4}}|1100\rangle + e^{-\pi i\frac{91}{4}}|1101\rangle + e^{-\pi i\frac{98}{4}}|1110\rangle + e^{-\pi i\frac{105}{4}}|1111\rangle\Big)$$

$$|1111\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-2\pi i k_2} \cdot e^{-\pi i k_3} \cdot e^{-2\pi i k_0} \cdot e^{-2\pi i k_1} \cdot e^{-\pi i k_2} \cdot e^{-\pi i\frac{k_3}{2}} \cdot e^{-2\pi i k_0} \cdot e^{-\pi i k_1} \cdot e^{-\pi i\frac{k_2}{2}} \cdot e^{-\pi i\frac{k_3}{4}} \cdot e^{-\pi i k_0} \cdot e^{-\pi i\frac{k_1}{2}} \cdot e^{-\pi i\frac{k_2}{4}} \cdot e^{-\pi i\frac{k_3}{8}}\,|k_0k_1k_2k_3\rangle =$$

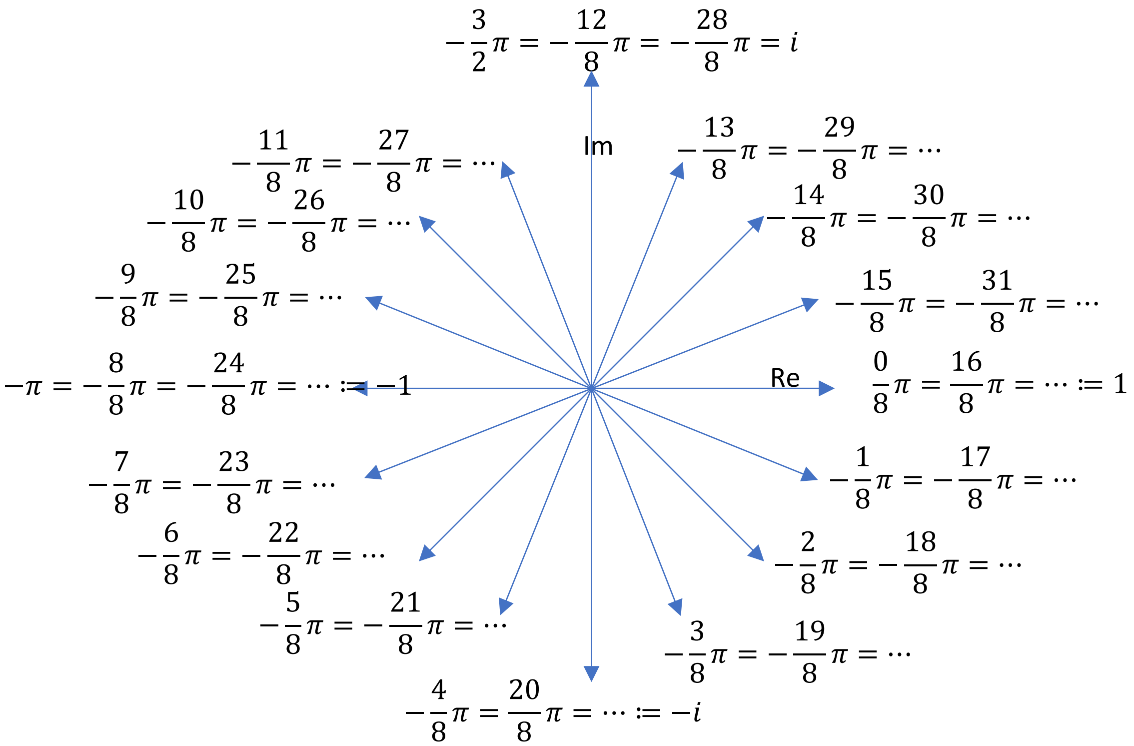
$$\frac{1}{4}\Big(|0000\rangle + e^{-\pi i\frac{15}{8}}|0001\rangle + e^{-\pi i\frac{30}{8}}|0010\rangle + e^{-\pi i\frac{45}{8}}|0011\rangle + e^{-\pi i\frac{60}{8}}|0100\rangle + e^{-\pi i\frac{75}{8}}|0101\rangle + e^{-\pi i\frac{90}{8}}|0110\rangle + e^{-\pi i\frac{105}{8}}|0111\rangle + e^{-\pi i\frac{120}{8}}|1000\rangle + e^{-\pi i\frac{135}{8}}|1001\rangle + e^{-\pi i\frac{150}{8}}|1010\rangle + e^{-\pi i\frac{165}{8}}|1011\rangle + e^{-\pi i\frac{180}{8}}|1100\rangle + e^{-\pi i\frac{195}{8}}|1101\rangle + e^{-\pi i\frac{210}{8}}|1110\rangle + e^{-\pi i\frac{225}{8}}|1111\rangle\Big)$$

We get the matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{4}{8}} & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{8}{8}} & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{12}{8}} & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{15}{8}} \\
1 & e^{-\pi i\frac{1}{4}} & e^{-\pi i\frac{2}{4}} & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{4}{4}} & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{7}{4}} & e^{-\pi i\frac{8}{4}} & e^{-\pi i\frac{9}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{11}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{13}{4}} & e^{-\pi i\frac{14}{4}} & e^{-\pi i\frac{15}{4}} \\
1 & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{12}{8}} & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{18}{8}} & e^{-\pi i\frac{21}{8}} & e^{-\pi i\frac{24}{8}} & e^{-\pi i\frac{27}{8}} & e^{-\pi i\frac{30}{8}} & e^{-\pi i\frac{33}{8}} & e^{-\pi i\frac{36}{8}} & e^{-\pi i\frac{39}{8}} & e^{-\pi i\frac{42}{8}} & e^{-\pi i\frac{45}{8}} \\
1 & e^{-\pi i\frac{1}{2}} & e^{-\pi i\frac{2}{2}} & e^{-\pi i\frac{3}{2}} & e^{-\pi i\frac{4}{2}} & e^{-\pi i\frac{5}{2}} & e^{-\pi i\frac{6}{2}} & e^{-\pi i\frac{7}{2}} & e^{-\pi i\frac{8}{2}} & e^{-\pi i\frac{9}{2}} & e^{-\pi i\frac{10}{2}} & e^{-\pi i\frac{11}{2}} & e^{-\pi i\frac{12}{2}} & e^{-\pi i\frac{13}{2}} & e^{-\pi i\frac{14}{2}} & e^{-\pi i\frac{15}{2}} \\
1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{20}{8}} & e^{-\pi i\frac{25}{8}} & e^{-\pi i\frac{30}{8}} & e^{-\pi i\frac{35}{8}} & e^{-\pi i\frac{40}{8}} & e^{-\pi i\frac{45}{8}} & e^{-\pi i\frac{50}{8}} & e^{-\pi i\frac{55}{8}} & e^{-\pi i\frac{60}{8}} & e^{-\pi i\frac{65}{8}} & e^{-\pi i\frac{70}{8}} & e^{-\pi i\frac{75}{8}} \\
1 & e^{-\pi i\frac{3}{4}} & e^{-\pi i\frac{6}{4}} & e^{-\pi i\frac{9}{4}} & e^{-\pi i\frac{12}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{18}{4}} & e^{-\pi i\frac{21}{4}} & e^{-\pi i\frac{24}{4}} & e^{-\pi i\frac{27}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{33}{4}} & e^{-\pi i\frac{36}{4}} & e^{-\pi i\frac{39}{4}} & e^{-\pi i\frac{42}{4}} & e^{-\pi i\frac{45}{4}} \\
1 & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{21}{8}} & e^{-\pi i\frac{28}{8}} & e^{-\pi i\frac{35}{8}} & e^{-\pi i\frac{42}{8}} & e^{-\pi i\frac{49}{8}} & e^{-\pi i\frac{56}{8}} & e^{-\pi i\frac{63}{8}} & e^{-\pi i\frac{70}{8}} & e^{-\pi i\frac{77}{8}} & e^{-\pi i\frac{84}{8}} & e^{-\pi i\frac{91}{8}} & e^{-\pi i\frac{98}{8}} & e^{-\pi i\frac{105}{8}} \\
1 & e^{-\pi i} & e^{-2\pi i} & e^{-3\pi i} & e^{-4\pi i} & e^{-5\pi i} & e^{-6\pi i} & e^{-7\pi i} & e^{-8\pi i} & e^{-9\pi i} & e^{-10\pi i} & e^{-11\pi i} & e^{-12\pi i} & e^{-13\pi i} & e^{-14\pi i} & e^{-15\pi i} \\
1 & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{18}{8}} & e^{-\pi i\frac{27}{8}} & e^{-\pi i\frac{36}{8}} & e^{-\pi i\frac{45}{8}} & e^{-\pi i\frac{54}{8}} & e^{-\pi i\frac{63}{8}} & e^{-\pi i\frac{72}{8}} & e^{-\pi i\frac{81}{8}} & e^{-\pi i\frac{90}{8}} & e^{-\pi i\frac{99}{8}} & e^{-\pi i\frac{108}{8}} & e^{-\pi i\frac{117}{8}} & e^{-\pi i\frac{126}{8}} & e^{-\pi i\frac{135}{8}} \\
1 & e^{-\pi i\frac{5}{4}} & e^{-\pi i\frac{10}{4}} & e^{-\pi i\frac{15}{4}} & e^{-\pi i\frac{20}{4}} & e^{-\pi i\frac{25}{4}} & e^{-\pi i\frac{30}{4}} & e^{-\pi i\frac{35}{4}} & e^{-\pi i\frac{40}{4}} & e^{-\pi i\frac{45}{4}} & e^{-\pi i\frac{50}{4}} & e^{-\pi i\frac{55}{4}} & e^{-\pi i\frac{60}{4}} & e^{-\pi i\frac{65}{4}} & e^{-\pi i\frac{70}{4}} & e^{-\pi i\frac{75}{4}} \\
1 & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{22}{8}} & e^{-\pi i\frac{33}{8}} & e^{-\pi i\frac{44}{8}} & e^{-\pi i\frac{55}{8}} & e^{-\pi i\frac{66}{8}} & e^{-\pi i\frac{77}{8}} & e^{-\pi i\frac{88}{8}} & e^{-\pi i\frac{99}{8}} & e^{-\pi i\frac{110}{8}} & e^{-\pi i\frac{121}{8}} & e^{-\pi i\frac{132}{8}} & e^{-\pi i\frac{143}{8}} & e^{-\pi i\frac{154}{8}} & e^{-\pi i\frac{165}{8}} \\
1 & e^{-\pi i\frac{3}{2}} & e^{-\pi i\frac{6}{2}} & e^{-\pi i\frac{9}{2}} & e^{-\pi i\frac{12}{2}} & e^{-\pi i\frac{15}{2}} & e^{-\pi i\frac{18}{2}} & e^{-\pi i\frac{21}{2}} & e^{-\pi i\frac{24}{2}} & e^{-\pi i\frac{27}{2}} & e^{-\pi i\frac{30}{2}} & e^{-\pi i\frac{33}{2}} & e^{-\pi i\frac{36}{2}} & e^{-\pi i\frac{39}{2}} & e^{-\pi i\frac{42}{2}} & e^{-\pi i\frac{45}{2}} \\
1 & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{26}{8}} & e^{-\pi i\frac{39}{8}} & e^{-\pi i\frac{52}{8}} & e^{-\pi i\frac{65}{8}} & e^{-\pi i\frac{78}{8}} & e^{-\pi i\frac{91}{8}} & e^{-\pi i\frac{104}{8}} & e^{-\pi i\frac{117}{8}} & e^{-\pi i\frac{130}{8}} & e^{-\pi i\frac{143}{8}} & e^{-\pi i\frac{158}{8}} & e^{-\pi i\frac{171}{8}} & e^{-\pi i\frac{184}{8}} & e^{-\pi i\frac{197}{8}} \\
1 & e^{-\pi i\frac{7}{4}} & e^{-\pi i\frac{14}{4}} & e^{-\pi i\frac{21}{4}} & e^{-\pi i\frac{28}{4}} & e^{-\pi i\frac{35}{4}} & e^{-\pi i\frac{42}{4}} & e^{-\pi i\frac{49}{4}} & e^{-\pi i\frac{56}{4}} & e^{-\pi i\frac{63}{4}} & e^{-\pi i\frac{70}{4}} & e^{-\pi i\frac{77}{4}} & e^{-\pi i\frac{84}{4}} & e^{-\pi i\frac{91}{4}} & e^{-\pi i\frac{98}{4}} & e^{-\pi i\frac{105}{4}} \\
1 & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{30}{8}} & e^{-\pi i\frac{45}{8}} & e^{-\pi i\frac{60}{8}} & e^{-\pi i\frac{75}{8}} & e^{-\pi i\frac{90}{8}} & e^{-\pi i\frac{105}{8}} & e^{-\pi i\frac{120}{8}} & e^{-\pi i\frac{135}{8}} & e^{-\pi i\frac{150}{8}} & e^{-\pi i\frac{165}{8}} & e^{-\pi i\frac{180}{8}} & e^{-\pi i\frac{195}{8}} & e^{-\pi i\frac{210}{8}} & e^{-\pi i\frac{225}{8}}
\end{pmatrix}$$

The matrix corresponds to the general matrix reduced to rank 16.

We use the picture of a rotating unit vector to simplify the entries in this matrix:



We resolve $\pm 1, \pm i$ and calculate modulo 2π:

$$k_0k_1k_2k_3 \\ \downarrow$$

$$\begin{array}{c|cccccccccccccccccc}
j_0j_1j_2j_3 & & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\ \hline
0000 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0001 & & 1 & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{3}{8}} & -i & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{7}{8}} & -1 & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{11}{8}} & i & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{15}{8}} \\
0010 & & 1 & e^{-\pi i\frac{2}{8}} & -i & e^{-\pi i\frac{6}{8}} & -1 & e^{-\pi i\frac{10}{8}} & i & e^{-\pi i\frac{14}{8}} & 1 & e^{-\pi i\frac{2}{8}} & -i & e^{-\pi i\frac{6}{8}} & -1 & e^{-\pi i\frac{10}{8}} & i & e^{-\pi i\frac{14}{8}} \\
0011 & & 1 & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{9}{8}} & i & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{5}{8}} & -1 & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{1}{8}} & -i & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{13}{8}} \\
0100 & & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
0101 & & 1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{15}{8}} & -i & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{11}{8}} & 1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{15}{8}} & -i & e^{-\pi i\frac{11}{8}} \\
0110 & & 1 & e^{-\pi i\frac{6}{8}} & i & e^{-\pi i\frac{2}{8}} & -1 & e^{-\pi i\frac{14}{8}} & -i & e^{-\pi i\frac{10}{8}} & 1 & e^{-\pi i\frac{6}{8}} & i & e^{-\pi i\frac{2}{8}} & -1 & e^{-\pi i\frac{14}{8}} & -i & e^{-\pi i\frac{10}{8}} \\
0111 & \frac{1}{4} & 1 & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{5}{8}} & i & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{1}{8}} & -1 & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{13}{8}} & -i & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{9}{8}} \\
1000 & & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1001 & & 1 & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{11}{8}} & -i & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{15}{8}} & -1 & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{3}{8}} & i & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{7}{8}} \\
1010 & & 1 & e^{-\pi i\frac{10}{8}} & -i & e^{-\pi i\frac{14}{8}} & -1 & e^{-\pi i\frac{2}{8}} & i & -i & e^{-\pi i\frac{14}{8}} & -1 & e^{-\pi i\frac{2}{8}} & i & e^{-\pi i\frac{6}{8}} & 1 & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{6}{8}} \\
1011 & & 1 & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{1}{8}} & i & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{13}{8}} & -1 & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{9}{8}} & -i & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{5}{8}} \\
1100 & & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1101 & & 1 & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{7}{8}} & -i & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{11}{8}} & -1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{15}{8}} & i & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{3}{8}} \\
1110 & & 1 & e^{-\pi i\frac{14}{8}} & i & e^{-\pi i\frac{10}{8}} & -1 & e^{-\pi i\frac{6}{8}} & -i & e^{-\pi i\frac{2}{8}} & 1 & e^{-\pi i\frac{14}{8}} & i & e^{-\pi i\frac{10}{8}} & -1 & e^{-\pi i\frac{6}{8}} & -i & e^{-\pi i\frac{2}{8}} \\
1111 & & 1 & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{13}{8}} & i & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{9}{8}} & -1 & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{5}{8}} & -i & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{1}{8}}
\end{array}$$

Again, we have the task to disassemble this matrix into gates.

We go back to the entry (jk):

$$|j_0j_1j_2j_3\rangle \rightarrow \frac{1}{4}\sum_{k=0}^{7} e^{-\pi i\frac{(8j_0+4j_1+2j_2+j_3)(8k_0+4k_1+2k_2+k_3)}{8}} |k_0k_1k_2k_3\rangle$$

We examine the structure of the entries in the exponent:

$$\frac{(8j_0+4j_1+2j_2+j_3)(8k_0+4k_1+2k_2+k_3)}{8} =$$

$$\frac{64j_0k_0+32j_0k_1+16j_0k_2+8j_0k_3}{8} + \frac{32j_1k_0+16j_1k_1+8j_1k_2+4j_1k_3}{8} +$$

$$\frac{16j_2k_0+8j_2k_1+4j_2k_2+2j_2k_3}{8} + \frac{8j_3k_0+4j_3k_1+2j_3k_2+j_3k_3}{8} =$$

$$8j_0k_0+4j_0k_1+2j_0k_2+j_0k_3+4j_1k_0+2j_1k_1+j_1k_2+\frac{j_1k_3}{2}+$$

$$2j_2k_0+j_2k_1+\frac{j_2k_2}{2}+\frac{j_2k_3}{4}+j_3k_0+\frac{j_3k_1}{2}+\frac{j_3k_2}{4}+\frac{j_3k_3}{8}$$

We get:

$$\underset{(1)}{e^{-2\pi ij_0k_0}} \cdot \underset{(2)}{e^{-2\pi ij_0k_1}} \cdot \underset{(3)}{e^{-2\pi ij_0k_2}} \cdot \underset{(4)}{e^{-\pi ij_0k_3}} \cdot$$

$$\underset{(5)}{e^{-2\pi ij_1k_0}} \cdot \underset{(6)}{e^{-2\pi ij_1k_1}} \cdot \underset{(7)}{e^{-\pi ij_1k_2}} \cdot \underset{(8)}{e^{-\pi i\frac{j_1k_3}{2}}} \cdot$$

$$\underset{(9)}{e^{-2\pi ij_2k_0}} \cdot \underset{(10)}{e^{-\pi ij_2k_1}} \cdot \underset{(11)}{e^{-\pi i\frac{j_2k_2}{2}}} \cdot \underset{(12)}{e^{-\pi i\frac{j_2k_3}{4}}} \cdot$$

$$\underset{(13)}{e^{-\pi ij_3k_0}} \cdot \underset{(14)}{e^{-\pi i\frac{j_3k_1}{2}}} \cdot \underset{(15)}{e^{-\pi i\frac{j_3k_2}{4}}} \cdot \underset{(16)}{e^{-\pi i\frac{j_3k_3}{8}}}$$

We interpret:

(1), (2), (3), (5), (6), (9) are identity transformations that have so far, no effect.

(4), (7), (10), (13) are Hadamards.

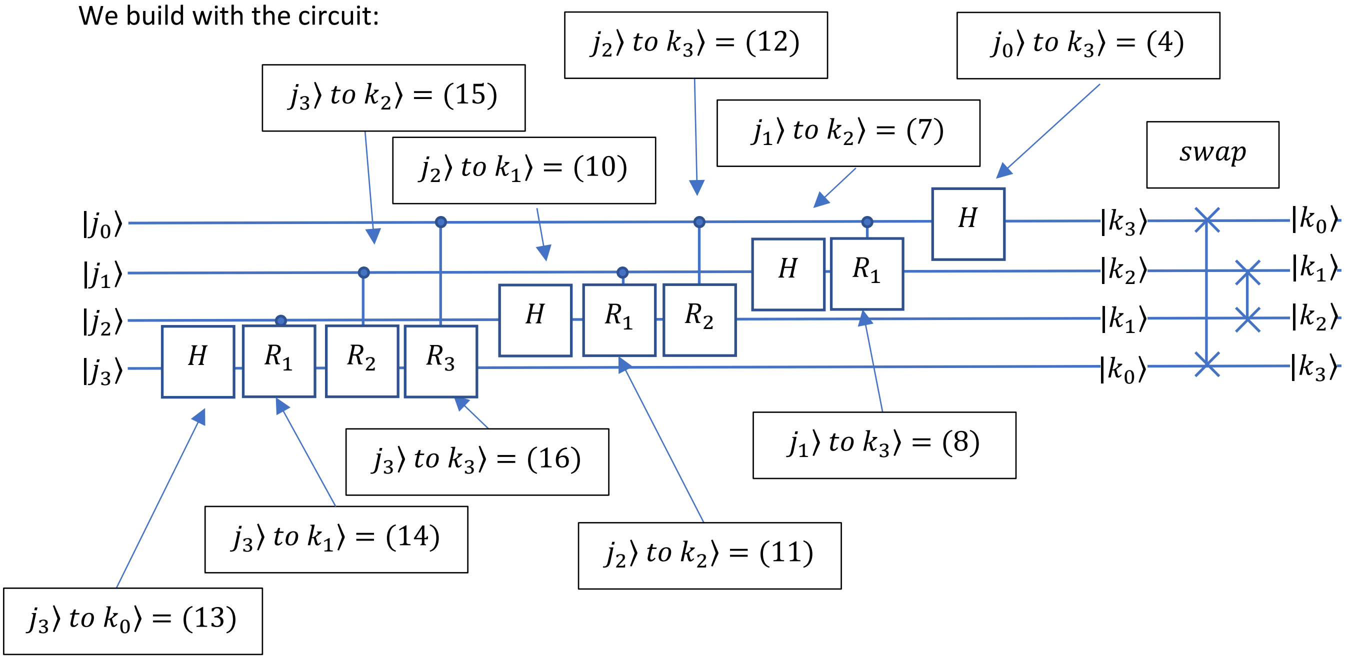
(8), (11), (14) are $S^\dagger$gates or R_1 gates

(12), (15) are $T^\dagger$ gates or R_2 gates.

(16) is the R_3 gate.

(4) is the Hadamard converting j_0 to k_3.	(7) is the Hadamard converting j_1 to k_2.
(10) is the Hadamard converting j_2 to k_1.	(13) is the Hadamard converting j_3 to k_0.
(8) is the controlled R_1 gate from j_1 to k_3	(11) is the controlled R_1 gate from j_2 to k_2
(14) is the controlled R_1 gate from j_3 to k_1	(12) is the controlled R_2 gate from j_2 to k_3
(15) is the controlled R_2 gate from j_3 to k_2	(16) is the controlled R_3 from j_3 to k_3

We build with the circuit:



Note: The gates named R_1 and R_2 are the controlled $S^\dagger$ and $T^\dagger$ gates. They are special forms of the single qubit unitary rotation gate R_k:

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi i}{2^k}} \end{pmatrix}$$

$$R_1 \coloneqq S = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi i}{2}} \end{pmatrix}, R_2 \coloneqq T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi i}{8}} \end{pmatrix}$$

Conceptual level

From the circuit above we can derive the state before the swap gate:

The Hadamard, acting on $\lvert j_0\rangle$:	$\lvert j_0\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle}{\sqrt{2}}$

The Hadamard, acting on $\lvert j_1\rangle$:	$\lvert j_1\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_1}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_1-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_1}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_1} e^{-\pi i \frac{j_0}{2}}\lvert 1\rangle}{\sqrt{2}}$

The Hadamard, acting on $\lvert j_2\rangle$:	$\lvert j_2\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_1-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_2}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_2-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_2} e^{-\pi i \frac{j_1}{2}} e^{-\pi i \frac{j_0}{4}}\lvert 1\rangle}{\sqrt{2}}$

The Hadamard, acting on $\lvert j_3\rangle$:	$\lvert j_2\rangle \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_3}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_1-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_3}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_2-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}\lvert 1\rangle}{\sqrt{2}}$
The controlled R_3-gate:	$\frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}\lvert 1\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 1\rangle}{\sqrt{2}}$

We combine:

$$\left(\frac{\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle}{\sqrt{2}}\right)\left(\frac{\lvert 0\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 1\rangle}{\sqrt{2}}\right)\left(\frac{\lvert 0\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 1\rangle}{\sqrt{2}}\right)\left(\frac{\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 1\rangle}{\sqrt{2}}\right) =$$

$$\frac{1}{4}\left(\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle\right)\left(\lvert 0\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 1\rangle\right)\left(\lvert 0\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 1\rangle\right)\left(\lvert 0\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 1\rangle\right) =$$

$$\begin{aligned}\frac{1}{4}\left(\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle\right)\left(\lvert 0\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 1\rangle\right)\Big(&\lvert 00\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 01\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 10\rangle\\ &+ e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 11\rangle\Big) =\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\left(\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle\right)\left(\lvert 0\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 1\rangle\right)\Big(&\lvert 00\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 01\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 10\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 11\rangle\Big) =\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\left(\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle\right)\Big(&\lvert 000\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 001\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 010\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 011\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 100\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 101\rangle\\ &+ e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 110\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 111\rangle\Big) =\end{aligned}$$

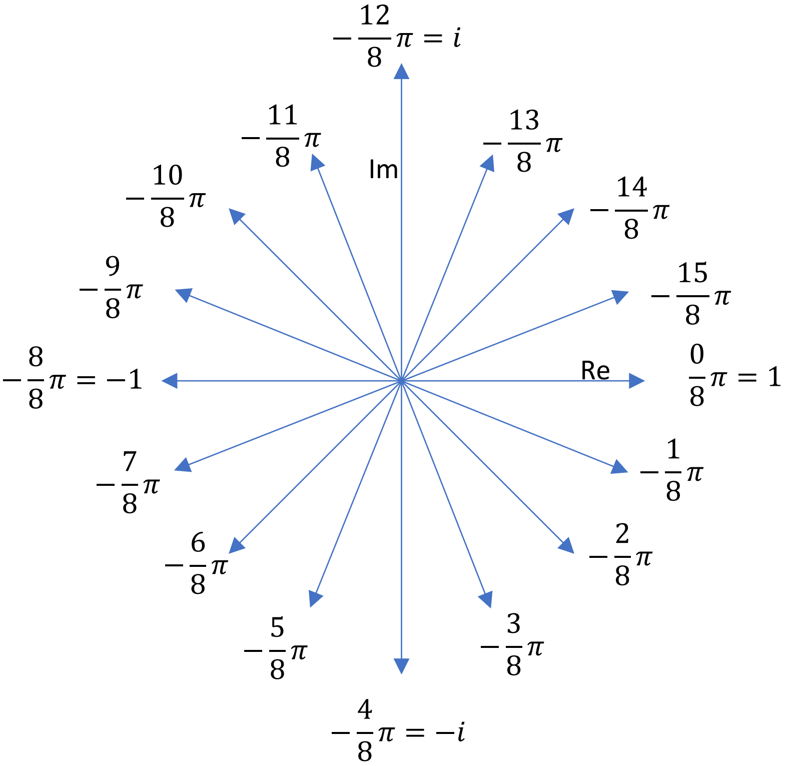
$$\begin{aligned}\frac{1}{4}\left(\lvert 0\rangle + e^{-\pi i j_0}\lvert 1\rangle\right)\Big(&\lvert 000\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 001\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 010\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 011\rangle + e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 100\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{5j_1}{4}}e^{-\pi i\frac{5j_0}{8}}\lvert 101\rangle\\ &+ e^{-\pi i j_2}e^{-\pi i\frac{3j_1}{2}}e^{-\pi i\frac{3j_0}{4}}\lvert 110\rangle + e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{7j_1}{4}}e^{-\pi i\frac{7j_0}{8}}\lvert 111\rangle\Big) =\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\Big(&\lvert 0000\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 0001\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 0010\rangle + e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 0011\rangle\\ &+ e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 0100\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{5j_1}{4}}e^{-\pi i\frac{5j_0}{8}}\lvert 0101\rangle + e^{-\pi i j_2}e^{-\pi i\frac{3j_1}{2}}e^{-\pi i\frac{3j_0}{4}}\lvert 0110\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{7j_1}{4}}e^{-\pi i\frac{7j_0}{8}}\lvert 0111\rangle + e^{-\pi i j_0}\lvert 1000\rangle + e^{-\pi i j_0}e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 1001\rangle\\ &+ e^{-\pi i j_0}e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 1010\rangle + e^{-\pi i j_0}e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 1011\rangle\\ &+ e^{-\pi i j_0}e^{-\pi i j_0}e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 1100\rangle + e^{-\pi i j_0}e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{5j_1}{4}}e^{-\pi i\frac{5j_0}{8}}\lvert 1101\rangle\\ &+ e^{-\pi i j_0}e^{-\pi i j_2}e^{-\pi i\frac{3j_1}{2}}e^{-\pi i\frac{3j_0}{4}}\lvert 1110\rangle + e^{-\pi i j_0}e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{7j_1}{4}}e^{-\pi i\frac{7j_0}{8}}\lvert 1111\rangle\Big) =\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\Big(&\lvert 0000\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{j_0}{8}}\lvert 0001\rangle + e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{j_0}{4}}\lvert 0010\rangle + e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{3j_0}{8}}\lvert 0011\rangle\\ &+ e^{-\pi i j_1}e^{-\pi i\frac{j_0}{2}}\lvert 0100\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{5j_1}{4}}e^{-\pi i\frac{5j_0}{8}}\lvert 0101\rangle + e^{-\pi i j_2}e^{-\pi i\frac{3j_1}{2}}e^{-\pi i\frac{3j_0}{4}}\lvert 0110\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{7j_1}{4}}e^{-\pi i\frac{7j_0}{8}}\lvert 0111\rangle + e^{-\pi i j_0}\lvert 1000\rangle + e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{j_1}{4}}e^{-\pi i\frac{9j_0}{8}}\lvert 1001\rangle\\ &+ e^{-\pi i j_2}e^{-\pi i\frac{j_1}{2}}e^{-\pi i\frac{5j_0}{4}}\lvert 1010\rangle + e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{3j_1}{4}}e^{-\pi i\frac{11j_0}{8}}\lvert 1011\rangle + e^{-\pi i j_1}e^{-\pi i\frac{3j_0}{2}}\lvert 1100\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{j_2}{2}}e^{-\pi i\frac{5j_1}{4}}e^{-\pi i\frac{13j_0}{8}}\lvert 1101\rangle + e^{-\pi i j_2}e^{-\pi i\frac{3j_1}{2}}e^{-\pi i\frac{7j_0}{4}}\lvert 1110\rangle\\ &+ e^{-\pi i j_3}e^{-\pi i\frac{3j_2}{2}}e^{-\pi i\frac{7j_1}{4}}e^{-\pi i\frac{15j_0}{8}}\lvert 1111\rangle\Big)\end{aligned}$$

The picture of the rotating unit vector is helpful to examine the structure of the transitions.

Note: factors of π only.



We calculate the transitions:

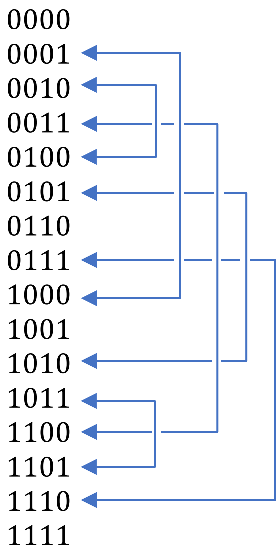
$j_0j_1j_2j_3$	
$\|0000\rangle$	$\frac{1}{4}(\|0000\rangle + \|0001\rangle + \|0010\rangle + \|0011\rangle + \|0100\rangle + \|0101\rangle + \|0110\rangle + \|0111\rangle + \|1000\rangle + \|1001\rangle + \|1010\rangle + \|1011\rangle + \|1100\rangle + \|1101\rangle + \|1110\rangle + \|1111\rangle)$
$\|0001\rangle$	$\frac{1}{4}(\|0000\rangle - \|0001\rangle + \|0010\rangle - \|0011\rangle + \|0100\rangle - \|0101\rangle + \|0110\rangle - \|0111\rangle + \|1000\rangle - \|1001\rangle + \|1010\rangle - \|1011\rangle + \|1100\rangle - \|1101\rangle + \|1110\rangle - \|1111\rangle)$
$\|0010\rangle$	$\frac{1}{4}(\|0000\rangle - i\|0001\rangle - \|0010\rangle + i\|0011\rangle + \|0100\rangle - i\|0101\rangle - \|0110\rangle + i\|0111\rangle + \|1000\rangle - i\|1001\rangle - \|1010\rangle + i\|1011\rangle + \|1100\rangle - i\|1101\rangle - \|1110\rangle + i\|1111\rangle)$
$\|0011\rangle$	$\frac{1}{4}(\|0000\rangle + i\|0001\rangle - \|0010\rangle - i\|0011\rangle + \|0100\rangle + i\|0101\rangle - \|0110\rangle - i\|0111\rangle + \|1000\rangle + i\|1001\rangle - \|1010\rangle - i\|1011\rangle + \|1100\rangle + i\|1101\rangle - \|1110\rangle - i\|1111\rangle)$
$\|0100\rangle$	$\frac{1}{4}\left(\|0000\rangle + e^{-i\frac{\pi}{4}}\|0001\rangle - i\|0010\rangle + e^{-i\frac{3\pi}{4}}\|0011\rangle - \|0100\rangle + e^{-i\frac{5\pi}{4}}\|0101\rangle + i\|0110\rangle + e^{-i\frac{7\pi}{4}}\|0111\rangle + \|1000\rangle + e^{-i\frac{\pi}{4}}\|1001\rangle - i\|1010\rangle + e^{-i\frac{3\pi}{4}}\|1011\rangle - \|1100\rangle + e^{-i\frac{5\pi}{4}}\|1101\rangle + i\|1110\rangle + e^{-i\frac{7\pi}{4}}\|1111\rangle\right)$
$\|0101\rangle$	$\frac{1}{4}\left(\|0000\rangle + e^{-i\frac{5\pi}{4}}\|0001\rangle - i\|0010\rangle + e^{-i\frac{7\pi}{4}}\|0011\rangle - \|0100\rangle + e^{-i\frac{9\pi}{4}}\|0101\rangle + i\|0110\rangle + e^{-i\frac{11\pi}{4}}\|0111\rangle + \|1000\rangle + e^{-i\frac{5\pi}{4}}\|1001\rangle - i\|1010\rangle + e^{-i\frac{7\pi}{4}}\|1011\rangle - \|1100\rangle + e^{-i\frac{9\pi}{4}}\|1101\rangle - \|1110\rangle + e^{-i\frac{11\pi}{4}}\|1111\rangle\right)$
$\|0110\rangle$	$\frac{1}{4}\left(\|0000\rangle + e^{-i\frac{3\pi}{4}}\|0001\rangle + i\|0010\rangle + e^{-i\frac{\pi}{4}}\|0011\rangle - \|0100\rangle + e^{-i\frac{7\pi}{4}}\|0101\rangle - i\|0110\rangle + e^{-i\frac{5\pi}{4}}\|0111\rangle - \|1000\rangle + e^{-i\frac{3\pi}{4}}\|1001\rangle + i\|1010\rangle + e^{-i\frac{\pi}{4}}\|1011\rangle - \|1100\rangle + e^{-i\frac{7\pi}{4}}\|1101\rangle - i\|1110\rangle + e^{-i\frac{5\pi}{4}}\|1111\rangle\right)$
$\|0111\rangle$	$\frac{1}{4}\left(\|0000\rangle + e^{-i\frac{7\pi}{4}}\|0001\rangle + i\|0010\rangle + e^{-i\frac{5\pi}{4}}\|0011\rangle - \|0100\rangle + e^{-i\frac{3\pi}{4}}\|0101\rangle - i\|0110\rangle + e^{-i\frac{\pi}{4}}\|0111\rangle + \|1000\rangle + e^{-i\frac{7\pi}{4}}\|1001\rangle + i\|1010\rangle + e^{-i\frac{5\pi}{4}}\|1011\rangle - \|1100\rangle + e^{-i\frac{3\pi}{4}}\|1101\rangle - i\|1110\rangle + e^{-i\frac{\pi}{4}}\|1111\rangle\right)$

$\vert 1000\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{\pi}{8}}\vert 0001\rangle + e^{-i\frac{\pi}{4}}\vert 0010\rangle + e^{-i\frac{3\pi}{8}}\vert 0011\rangle - i\vert 0100\rangle + e^{-i\frac{5\pi}{8}}\vert 0101\rangle + e^{-i\frac{6\pi}{8}}\vert 0110\rangle + e^{-i\frac{7\pi}{8}}\vert 0111\rangle - \vert 1000\rangle$ $+ e^{-i\frac{9\pi}{8}}\vert 1001\rangle + e^{-i\frac{10\pi}{8}}\vert 1010\rangle + e^{-i\frac{11\pi}{8}}\vert 1011\rangle + i\vert 1100\rangle + e^{-i\frac{13\pi}{8}}\vert 1101\rangle + e^{-i\frac{14\pi}{8}}\vert 1110\rangle$ $+ e^{-i\frac{15\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1001\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{9\pi}{8}}\vert 0001\rangle + e^{-i\frac{\pi}{4}}\vert 0010\rangle + e^{-i\frac{11\pi}{8}}\vert 0011\rangle - i\vert 0100\rangle + e^{-i\frac{13\pi}{8}}\vert 0101\rangle + e^{-i\frac{3\pi}{4}}\vert 0110\rangle + e^{-i\frac{15\pi}{8}}\vert 0111\rangle$ $+ \vert 1000\rangle + e^{-i\frac{1\pi}{8}}\vert 1001\rangle + e^{-i\frac{5\pi}{4}}\vert 1010\rangle + e^{-i\frac{3\pi}{8}}\vert 1011\rangle + i\vert 1100\rangle + e^{-i\frac{5\pi}{8}}\vert 1101\rangle + e^{-i\frac{7\pi}{4}}\vert 1110\rangle$ $+ e^{-i\frac{7\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1010\rangle$	$\frac{1}{4}\Big(\Big\vert 0000\rangle + e^{-i\frac{5\pi}{8}}\Big\vert 0001\rangle + e^{-i\frac{10\pi}{8}}\Big\vert 0010\rangle + e^{-i\frac{15\pi}{8}}\Big\vert 0011\rangle - i\vert 0100\rangle + e^{-i\frac{9\pi}{8}}\vert 0101\rangle + e^{-i\frac{14\pi}{8}}\vert 0110\rangle + e^{-i\frac{3\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{13\pi}{8}}\vert 1001\rangle + e^{-i\frac{2\pi}{8}}\vert 1010\rangle + e^{-i\frac{7\pi}{8}}\vert 1011\rangle + i\vert 1100\rangle + e^{-i\frac{1\pi}{8}}\vert 1101\rangle + e^{-i\frac{6\pi}{8}}\vert 1110\rangle$ $+ e^{-i\frac{11\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1011\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{13\pi}{8}}\vert 0001\rangle + e^{-i\frac{10\pi}{8}}\vert 0010\rangle + e^{-i\frac{7\pi}{8}}\vert 0011\rangle - i\vert 0100\rangle + e^{-i\frac{1\pi}{8}}\vert 0101\rangle + e^{-i\frac{14\pi}{8}}\vert 0110\rangle + e^{-i\frac{11\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{5\pi}{8}}\vert 1001\rangle + e^{-i\frac{2\pi}{8}}\vert 1010\rangle + e^{-i\frac{15\pi}{8}}\vert 1011\rangle + i\vert 1100\rangle + e^{-i\frac{9\pi}{8}}\vert 1101\rangle + e^{-i\frac{6\pi}{8}}\vert 1110\rangle$ $+ e^{-i\frac{3\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1100\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{3\pi}{8}}\vert 0001\rangle + e^{-i\frac{6\pi}{8}}\vert 0010\rangle + e^{-i\frac{9\pi}{8}}\vert 0011\rangle + i\vert 0100\rangle + e^{-i\frac{15\pi}{8}}\vert 0101\rangle + e^{-i\frac{2\pi}{8}}\vert 0110\rangle + e^{-i\frac{5\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{11\pi}{8}}\vert 1001\rangle + e^{-i\frac{14\pi}{8}}\vert 1010\rangle + e^{-i\frac{17\pi}{8}}\vert 1011\rangle - i\vert 1100\rangle + e^{-i\frac{7\pi}{8}}\vert 1101\rangle$ $+ e^{-i\frac{10\pi}{8}}\vert 1110\rangle + e^{-i\frac{13\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1101\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{11\pi}{8}}\vert 0001\rangle + e^{-i\frac{6\pi}{8}}\vert 0010\rangle + e^{-i\frac{\pi}{8}}\vert 0011\rangle + i\vert 0100\rangle + e^{-i\frac{7\pi}{8}}\vert 0101\rangle + e^{-i\frac{2\pi}{8}}\vert 0110\rangle + e^{-i\frac{13\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{3\pi}{8}}\vert 1001\rangle + e^{-i\frac{14\pi}{8}}\vert 1010\rangle + e^{-i\frac{9\pi}{8}}\vert 1011\rangle - i\vert 1100\rangle + e^{-i\frac{15\pi}{8}}\vert 1101\rangle + e^{-i\frac{10\pi}{8}}\vert 1110\rangle$ $+ e^{-i\frac{5\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1110\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{7\pi}{8}}\vert 0001\rangle + e^{-i\frac{14\pi}{8}}\vert 0010\rangle + e^{-i\frac{5\pi}{8}}\vert 0011\rangle + i\vert 0100\rangle + e^{-i\frac{3\pi}{8}}\vert 0101\rangle + e^{-i\frac{10\pi}{8}}\vert 0110\rangle + e^{-i\frac{1\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{15\pi}{8}}\vert 1001\rangle + e^{-i\frac{6\pi}{8}}\vert 1010\rangle + e^{-i\frac{13\pi}{8}}\vert 1011\rangle - i\vert 1100\rangle + e^{-i\frac{11\pi}{8}}\vert 1101\rangle$ $+ e^{-i\frac{2\pi}{8}}\vert 1110\rangle + e^{-i\frac{9\pi}{8}}\vert 1111\rangle\Big)$
$\vert 1111\rangle$	$\frac{1}{4}\Big(\vert 0000\rangle + e^{-i\frac{15\pi}{8}}\vert 0001\rangle + e^{-i\frac{14\pi}{8}}\vert 0010\rangle + e^{-i\frac{13\pi}{8}}\vert 0011\rangle + i\vert 0100\rangle + e^{-i\frac{11\pi}{8}}\vert 0101\rangle + e^{-i\frac{10\pi}{8}}\vert 0110\rangle + e^{-i\frac{9\pi}{8}}\vert 0111\rangle$ $- \vert 1000\rangle + e^{-i\frac{7\pi}{8}}\vert 1001\rangle + e^{-i\frac{6\pi}{8}}\vert 1010\rangle + e^{-i\frac{5\pi}{8}}\vert 1011\rangle - i\vert 1100\rangle + e^{-i\frac{3\pi}{8}}\vert 1101\rangle + e^{-i\frac{2\pi}{8}}\vert 1110\rangle$ $+ e^{-i\frac{\pi}{8}}\vert 1111\rangle\Big)$

From the transitions we get the matrix:

$$\frac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-i\frac{\pi}{4}} & -i & e^{-i\frac{3\pi}{4}} & -1 & e^{-i\frac{5\pi}{4}} & i & e^{-i\frac{7\pi}{4}} & 1 & e^{-i\frac{\pi}{4}} & -i & e^{-i\frac{3\pi}{4}} & -1 & e^{-i\frac{5\pi}{4}} & i & e^{-i\frac{7\pi}{4}} \\
1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & i & e^{-i\frac{3\pi}{4}} & 1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} \\
1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} \\
1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} & 1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} \\
1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{3\pi}{8}} & -i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{7\pi}{8}} & -1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{11\pi}{8}} & i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{15\pi}{8}} \\
1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{1\pi}{4}} & e^{-i\frac{11\pi}{8}} & -i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{3\pi}{4}} & e^{-i\frac{15\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{5\pi}{4}} & e^{-i\frac{3\pi}{8}} & i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{7\pi}{4}} & e^{-i\frac{7\pi}{8}} \\
1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{15\pi}{8}} & -i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{3\pi}{8}} & -1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{7\pi}{8}} & i & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{11\pi}{8}} \\
1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{7\pi}{8}} & -i & e^{-i\frac{\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{11\pi}{8}} & -1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{15\pi}{8}} & i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{3\pi}{8}} \\
1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{9\pi}{8}} & i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{5\pi}{8}} & -1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{\pi}{8}} & -i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{13\pi}{8}} \\
1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{\pi}{8}} & i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{13\pi}{8}} & -1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{9\pi}{8}} & -i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{5\pi}{8}} \\
1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{5\pi}{8}} & i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{13\pi}{8}} & -i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{9\pi}{8}} \\
1 & e^{-i\frac{15\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{13\pi}{8}} & i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{9\pi}{8}} & -1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{5\pi}{8}} & -i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{\pi}{8}}
\end{pmatrix}$$

We need the final swap:



We get:

$$\frac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{3\pi}{8}} & -i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{7\pi}{8}} & -1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{11\pi}{8}} & i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{15\pi}{8}} \\
1 & e^{-i\frac{\pi}{4}} & -i & e^{-i\frac{3\pi}{4}} & -1 & e^{-i\frac{5\pi}{4}} & i & e^{-i\frac{7\pi}{4}} & 1 & e^{-i\frac{\pi}{4}} & -i & e^{-i\frac{3\pi}{4}} & -1 & e^{-i\frac{5\pi}{4}} & i & e^{-i\frac{7\pi}{4}} \\
1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{9\pi}{8}} & i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{5\pi}{8}} & -1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{\pi}{8}} & -i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{13\pi}{8}} \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{15\pi}{8}} & -i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{3\pi}{8}} & -1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{7\pi}{8}} & i & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{11\pi}{8}} \\
1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} \\
1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{5\pi}{8}} & i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{13\pi}{8}} & -i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{9\pi}{8}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{1\pi}{4}} & e^{-i\frac{11\pi}{8}} & -i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{3\pi}{4}} & e^{-i\frac{15\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{5\pi}{4}} & e^{-i\frac{3\pi}{8}} & i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{7\pi}{4}} & e^{-i\frac{7\pi}{8}} \\
1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & i & e^{-i\frac{3\pi}{4}} & 1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} \\
1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{\pi}{8}} & i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{13\pi}{8}} & -1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{15\pi}{8}} & -i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{5\pi}{8}} \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{7\pi}{8}} & -i & e^{-i\frac{\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{13\pi}{8}} & -1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{15\pi}{8}} & i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{3\pi}{8}} \\
1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} & 1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} \\
1 & e^{-i\frac{15\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{13\pi}{8}} & i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{9\pi}{8}} & -1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{5\pi}{8}} & -i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{\pi}{8}}
\end{pmatrix}$$

This is the matrix for the Fourier transformation for four qubits.

Remark

We can see in the matrix for the Fourier transformation the “rotation of the unit vector” clockwise.

In the first row the vector is steady at position 1.

In the second row the vector is rotating slowly:

$$1, e^{-i\frac{\pi}{8}}, e^{-i\frac{2\pi}{8}}, \dots, e^{-i\frac{15\pi}{8}}$$

In the third row the vector rotates twice as fast:

$$1, e^{-i\frac{2\pi}{8}}, e^{-i\frac{4\pi}{8}}, \dots, e^{-i\frac{30\pi}{8}}$$

Note that we must calculate modulo 2π, the last term becomes $e^{-i\frac{14\pi}{8}}$resp. $e^{-i\frac{7\pi}{4}}$.

Due to the discrete nature we get an alias effect. In the last line the rotation seems to be reversed:

$$1, e^{-i\frac{15\pi}{8}}, e^{-i\frac{14\pi}{8}}, \dots, e^{-i\frac{1\pi}{8}}$$

In between this becomes a jumping of the unit vector forward and backward that culminates in line 9 to a jumping between $+1$ and -1.

Basic level

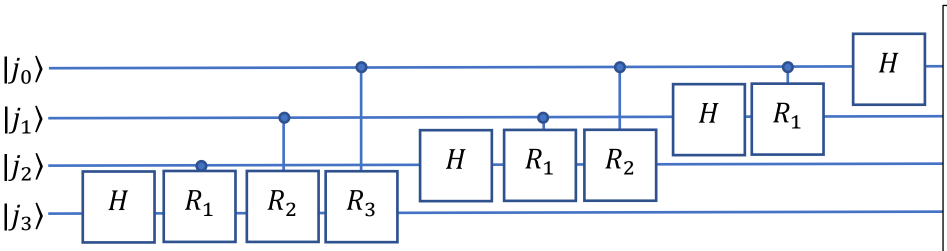
We apply to the 4 qubit Quantum Fourier transform. The matrix becomes 16×16:

We use the unitary matrix:

$$\frac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{3}{8}} & -i & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{7}{8}} & -1 & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{11}{8}} & i & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{15}{8}} \\
1 & e^{-\pi i\frac{2}{8}} & -i & e^{-\pi i\frac{6}{8}} & -1 & e^{-\pi i\frac{10}{8}} & i & e^{-\pi i\frac{14}{8}} & 1 & e^{-\pi i\frac{2}{8}} & -i & e^{-\pi i\frac{6}{8}} & -1 & e^{-\pi i\frac{10}{8}} & i & e^{-\pi i\frac{14}{8}} \\
1 & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{9}{8}} & i & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{5}{8}} & -1 & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{1}{8}} & -i & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{13}{8}} \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{15}{8}} & -i & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{11}{8}} & 1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{15}{8}} & -i & e^{-\pi i\frac{11}{8}} \\
1 & e^{-\pi i\frac{6}{8}} & i & e^{-\pi i\frac{2}{8}} & -1 & e^{-\pi i\frac{14}{8}} & -i & e^{-\pi i\frac{10}{8}} & 1 & e^{-\pi i\frac{6}{8}} & i & e^{-\pi i\frac{2}{8}} & -1 & e^{-\pi i\frac{14}{8}} & -i & e^{-\pi i\frac{10}{8}} \\
1 & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{5}{8}} & i & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{1}{8}} & -1 & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{13}{8}} & -i & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{9}{8}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{11}{8}} & -i & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{15}{8}} & -1 & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{3}{8}} & i & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{7}{8}} \\
1 & e^{-\pi i\frac{10}{8}} & -i & e^{-\pi i\frac{14}{8}} & -1 & e^{-\pi i\frac{2}{8}} & i & -i & e^{-\pi i\frac{14}{8}} & -1 & e^{-\pi i\frac{2}{8}} & i & e^{-\pi i\frac{6}{8}} & 1 & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{6}{8}} \\
1 & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{1}{8}} & i & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{13}{8}} & -1 & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{9}{8}} & -i & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{5}{8}} \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-\pi i\frac{13}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{7}{8}} & -i & e^{-\pi i\frac{1}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{11}{8}} & -1 & e^{-\pi i\frac{5}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{15}{8}} & i & e^{-\pi i\frac{9}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{3}{8}} \\
1 & e^{-\pi i\frac{14}{8}} & i & e^{-\pi i\frac{10}{8}} & -1 & e^{-\pi i\frac{6}{8}} & -i & e^{-\pi i\frac{2}{8}} & 1 & e^{-\pi i\frac{14}{8}} & i & e^{-\pi i\frac{10}{8}} & -1 & e^{-\pi i\frac{6}{8}} & -i & e^{-\pi i\frac{2}{8}} \\
1 & e^{-\pi i\frac{15}{8}} & e^{-\pi i\frac{14}{8}} & e^{-\pi i\frac{13}{8}} & i & e^{-\pi i\frac{11}{8}} & e^{-\pi i\frac{10}{8}} & e^{-\pi i\frac{9}{8}} & -1 & e^{-\pi i\frac{7}{8}} & e^{-\pi i\frac{6}{8}} & e^{-\pi i\frac{5}{8}} & -i & e^{-\pi i\frac{3}{8}} & e^{-\pi i\frac{2}{8}} & e^{-\pi i\frac{1}{8}}
\end{pmatrix}$$

This is a symmetric matrix.

We need to build this matrix out of a series of gates (matrices) in order to let it run on a quantum computer. We will calculate all gates "from left to the right" and strictly follow the wiring diagram.



Note: no implicit swaps, we use the wiring diagram "as is".

We also avoid to distinguish between j_i and k_i and simply name all wires j_i.

The first Hadamard is the Kronecker product:

$$Id \otimes Id \otimes Id \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

The second, R_1-gate. We multiply $j_2 = 1 = j_3$ by $-i$.

$$\begin{array}{c|cccccccccccccccc}
 & & & & & & & & j_0j_1j_2j_3 \downarrow & & & & & & & & \\
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{array}$$

The third, R_2-gate. We multiply $j_1 = 1 = j_3$ by $e^{-\frac{\pi i}{4}}$.

$$\begin{array}{c|cccccccccccccccc}
 & & & & & & & & j_0j_1j_2j_3 \downarrow & & & & & & & & \\
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}}
\end{array}$$

The fourth, R_3-gate.

We multiply $j_0 = 1 = j_3$ by $e^{-\frac{\pi i}{8}}$. $\overset{j_0j_1j_2j_3}{\downarrow}$

$$\begin{array}{c|cccccccccccccccc}
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{8}} & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{8}} & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{8}} & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{8}}
\end{array}$$

The fifth, Hadamard is the Kronecker product. We calculate:

$$Id \otimes Id \otimes H \otimes Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix}$$

The sixth, R_1-gate. We multiply $j_1 = 1 = j_2$ by $-i$.

$$
\begin{array}{c|cccccccccccccccc}
 & & & & & & & & j_0j_1j_2j_3 \downarrow & & & & & & & & \\
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{array}
$$

The seventh, R_2-gate. We multiply $j_0 = 1 = j_2$ by $e^{-\frac{\pi i}{4}}$.

$$
\begin{array}{c|cccccccccccccccc}
 & & & & & & & & j_0j_1j_2j_3 \downarrow & & & & & & & & \\
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}} & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{\pi i}{4}}
\end{array}
$$

The eighth, Hadamard is the Kronecker product. We remember that they perform an implicit swap and calculate:

$$Id \otimes H \otimes Id \otimes Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{array}\right)$$

The ninth, R_1-gate. We multiply $j_0 = 1 = j_1$ by $-i$.

$j_0j_1j_2j_3$

↓

$$\begin{array}{c|cccccccccccccccc}
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0100 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{array}$$

The tenth, Hadamard is the Kronecker product:

$$H \otimes Id \otimes Id \otimes Id = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1
\end{pmatrix}$$

The final swap:

$$\overset{j_0j_1j_2j_3}{\downarrow}$$

$$\begin{array}{c|cccccccccccccccc}
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111\\
\hline
0000 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0001 & 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0010 & 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0011 & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0100 & 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0101 & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0110 & 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0111 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1000 & 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1001 & 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1010 & 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1011 & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
1100 & 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1101 & 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1110 & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
1111 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}$$

In contrast to the two and three qubit case we strictly followed the wiring diagram:

$$Temp \coloneqq H_4 \cdot R1_3 \cdot H_3 \cdot R2_2 \cdot R1_2 \cdot H_2 \cdot R3_1 \cdot R2_1 \cdot R1_1 \cdot H_1$$

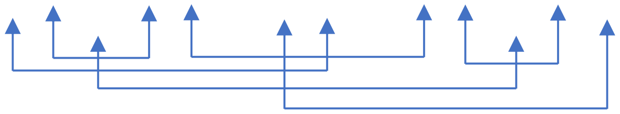
Note: " · " denotes the matrix product.

Note: indices for better distinguishing between gates of the same name.

A consequence of our access is that we have to swap columns, not rows and in the end multiply the swap matrix from the right:

$$FT = Temp \cdot SWAP$$

$$
\begin{array}{c|cccccccccccccccc}
 & & & & & & & & j_0j_1j_2j_3 \downarrow & & & & & & & & \\
j_0j_1j_2j_3 & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0010 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0100 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0110 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1000 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1010 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1011 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1100 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$



We get the result: $FT =$

$$
\frac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{3\pi}{8}} & -i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{7\pi}{8}} & -1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{11\pi}{8}} & i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{15\pi}{8}} \\
1 & e^{-i\frac{2\pi}{8}} & -i & e^{-i\frac{6\pi}{8}} & -1 & e^{-i\frac{10\pi}{8}} & i & e^{-i\frac{14\pi}{8}} & 1 & e^{-i\frac{2\pi}{8}} & -i & e^{-i\frac{6\pi}{8}} & -1 & e^{-i\frac{10\pi}{8}} & i & e^{-i\frac{14\pi}{8}} \\
1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{9\pi}{8}} & i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{5\pi}{8}} & -1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{\pi}{8}} & -i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{13\pi}{8}} \\
1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{15\pi}{8}} & -i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{3\pi}{8}} & -1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{7\pi}{8}} & i & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{11\pi}{8}} \\
1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & i & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{7\pi}{4}} & -i & e^{-i\frac{5\pi}{4}} \\
1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{5\pi}{8}} & i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{13\pi}{8}} & -i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{9\pi}{8}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{-i\frac{9\pi}{8}} & e^{-i\frac{1\pi}{4}} & e^{-i\frac{11\pi}{8}} & -i & e^{-i\frac{13\pi}{8}} & e^{-i\frac{3\pi}{4}} & e^{-i\frac{15\pi}{8}} & -1 & e^{-i\frac{\pi}{8}} & e^{-i\frac{5\pi}{4}} & e^{-i\frac{3\pi}{8}} & i & e^{-i\frac{5\pi}{8}} & e^{-i\frac{7\pi}{4}} & e^{-i\frac{7\pi}{8}} \\
1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & i & e^{-i\frac{3\pi}{4}} & 1 & e^{-i\frac{5\pi}{4}} & -i & e^{-i\frac{7\pi}{4}} & -1 & e^{-i\frac{\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} \\
1 & e^{-i\frac{11\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{\pi}{8}} & i & e^{-i\frac{7\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{13\pi}{8}} & -1 & e^{-i\frac{3\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{15\pi}{8}} & -i & e^{-i\frac{15\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{5\pi}{8}} \\
1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & e^{-i\frac{13\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{7\pi}{8}} & -i & e^{-i\frac{\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{13\pi}{8}} & -1 & e^{-i\frac{5\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{15\pi}{8}} & i & e^{-i\frac{9\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{3\pi}{8}} \\
1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} & 1 & e^{-i\frac{7\pi}{4}} & i & e^{-i\frac{5\pi}{4}} & -1 & e^{-i\frac{3\pi}{4}} & -i & e^{-i\frac{\pi}{4}} \\
1 & e^{-i\frac{15\pi}{8}} & e^{-i\frac{14\pi}{8}} & e^{-i\frac{13\pi}{8}} & i & e^{-i\frac{11\pi}{8}} & e^{-i\frac{10\pi}{8}} & e^{-i\frac{9\pi}{8}} & -1 & e^{-i\frac{7\pi}{8}} & e^{-i\frac{6\pi}{8}} & e^{-i\frac{5\pi}{8}} & -i & e^{-i\frac{3\pi}{8}} & e^{-i\frac{2\pi}{8}} & e^{-i\frac{\pi}{8}}
\end{pmatrix}
$$

This is the quantum Fourier transform matrix for four qubits.

For your convenience there is a wxmaxima-file available on this net site. You may find all matrices there and can check my results. As the syntax for the matrices is used by other CAS too, you may copy the matrix definitions in a CAS of your choice and verify the results. This may need some “work by hand” to get the definitions out of the surrounding context.

Appendix

Discrete Fourier Transform

We use a binary sequence:

$$x_0, x_1, \dots, x_{n-1} \quad n = 2^l$$

Note: we use n as a power of 2, $n = 1,2,4,8, \dots$ in order to get quadratic matrices.

We perform a discrete Fourier transform and calculate a new sequence:

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j e^{-2\pi i \frac{jk}{n}}$$

$$y_0 = \frac{1}{\sqrt{n}} \left(x_0 e^{-2\pi i \frac{0 \cdot 0}{n}} + x_1 e^{-2\pi i \frac{1 \cdot 0}{n}} + \dots + x_{n-1} e^{-2\pi i \frac{(n-1) \cdot 0}{n}} = x_0 + x_1 + \dots + x_{n-1} \right)$$

$$y_1 = \frac{1}{\sqrt{n}} \left(x_0 + x_1 e^{-2\pi i \frac{1 \cdot 1}{n}} + \dots + x_{n-1} e^{-2\pi i \frac{(n-1) \cdot 1}{n}} \right)$$

$$\dots$$

$$y_{n-1} = \frac{1}{\sqrt{n}} \left(x_0 + x_1 e^{-2\pi i \frac{1 \cdot (n-1)}{n}} + \dots + x_{n-1} e^{-2\pi i \frac{(n-1) \cdot (n-1)}{n}} \right)$$

The bit k of the resulting bit-vector is the sum over all bits of the input vector multiplied by a factor.

We express this with matrix multiplication:

$$\begin{pmatrix} u_{00} & u_{01} & \dots & u_{0(n-2)} & u_{0(n-1)} \\ u_{10} & u_{11} & \dots & u_{1(n-2)} & u_{1(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ u_{(n-2)0} & u_{(n-2)1} & \dots & u_{(n-2)(n-2)} & u_{(n-2)(n-1)} \\ .\,u_{(n-1)0} & u_{(n-1)1} & \dots & u_{(n-1)(n-2)} & u_{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \dots \\ y_{n-2} \\ y_{n-1} \end{pmatrix}$$

The corresponding matrix U:

$$\begin{array}{cc|c} & & \begin{array}{c} k \\ \downarrow \end{array} \\ & & \begin{array}{ccccc} 0 & 1 & \dots & n-2 & n-1 \end{array} \\ \hline j \rightarrow & \begin{array}{c} 0 \\ 1 \\ \dots \\ n-2 \\ n-1 \end{array} & \frac{1}{\sqrt{n}} \begin{pmatrix} e^{-2\pi i \frac{0 \cdot 0}{n}} & e^{-2\pi i \frac{1 \cdot 0}{n}} & \dots & e^{-2\pi i \frac{(n-2) \cdot 0}{n}} & e^{-2\pi i \frac{(n-1) \cdot 0}{n}} \\ e^{-2\pi i \frac{0 \cdot 1}{n}} & e^{-2\pi i \frac{1 \cdot 1}{n}} & \dots & e^{-2\pi i \frac{(n-2) \cdot 1}{n}} & e^{-2\pi i \frac{(n-1) \cdot 1}{n}} \\ \dots & \dots & \dots & \dots & \dots \\ e^{-2\pi i \frac{0 \cdot (n-2)}{n}} & e^{-2\pi i \frac{1 \cdot (n-2)}{n}} & \dots & e^{-2\pi i \frac{(n-2) \cdot (n-2)}{n}} & e^{-2\pi i \frac{(n-1) \cdot (n-2)}{n}} \\ e^{-2\pi i \frac{0 \cdot (n-1)}{n}} & e^{-2\pi i \frac{1 \cdot (n-1)}{n}} & \dots & e^{-2\pi i \frac{(n-2) \cdot (n-1)}{n}} & e^{-2\pi i \frac{(n-1) \cdot (n-1)}{n}} \end{pmatrix} \end{array}$$

For a unitary quantum Fourier transform we work with kets (states) in the $|0/1\rangle$-basis:

$$|j\rangle \rightarrow \sum_{k=0}^{n-1} e^{-2\pi i \frac{jk}{n}} |k\rangle$$

$|j\rangle = |00 \dots 0\rangle$ is a n-bit state.

We get:

$$|00\dots0\rangle \rightarrow \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi i\frac{jk}{n}}\,|k\rangle = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi i0}\,|k\rangle =$$

$$\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1}|k\rangle\right) = \frac{1}{\sqrt{n}}(|00\dots00\rangle + |00\dots01\rangle + \dots + |11\dots10\rangle + |11\dots11\rangle)$$

This is a superposition over all basis states, a kind of Hadamard.

Note: If n is a power of two the exponentials contain binary fractions.

In order to make this kind of calculation available for quantum computing this must be an operation represented by a unitary matrix.

We take the matrix U:

$$\begin{array}{cc|c}
 & & \begin{array}{cccccc} & & & k & & \\ & & & \downarrow & & \\ 0 & 1 & 2 & \dots & n-2 & n-1 \end{array} \\ \hline
j \rightarrow & \begin{array}{c} 0 \\ 1 \\ 2 \\ \dots \\ n-2 \\ n-1 \end{array} & \frac{1}{\sqrt{n}}\begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
1 & e^{-2\pi i\frac{1}{n}} & e^{-2\pi i\frac{2}{n}} & \dots & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-1)}{n}} \\
1 & e^{-2\pi i\frac{2}{n}} & e^{-2\pi i\frac{4}{n}} & \dots & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
1 & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-2)(n-2)}{n}} & e^{-2\pi i\frac{(n-2)(n-1)}{n}} \\
1 & e^{-2\pi i\frac{(n-1)}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-1)(n-2)}{n}} & e^{-2\pi i\frac{(n-1)(n-1)}{n}}
\end{pmatrix}
\end{array}$$

A matrix U is unitary if $U^{\dagger}U = UU^{\dagger} = Id$.

Note: Id is the identity matrix.

Note: U is symmetric

Note: $U^{\dagger}$ is the transposed and complex conjugated version of U.

Note: U contains binary fractions.

We build $U^{\dagger}$:

$$\frac{1}{\sqrt{n}}\begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 1 \\
1 & e^{2\pi i\frac{1}{n}} & e^{2\pi i\frac{2}{n}} & \dots & e^{2\pi i\frac{(n-2)}{n}} & e^{2\pi i\frac{(n-1)}{n}} \\
1 & e^{2\pi i\frac{2}{n}} & e^{2\pi i\frac{4}{n}} & \dots & e^{2\pi i\frac{(n-2)\cdot 2}{n}} & e^{2\pi i\frac{(n-1)\cdot 2}{n}} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
1 & e^{2\pi i\frac{(n-2)}{n}} & e^{2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{2\pi i\frac{(n-2)(n-2)}{n}} & e^{2\pi i\frac{(n-2)(n-1)}{n}} \\
1 & e^{2\pi i\frac{(n-1)}{n}} & e^{2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{2\pi i\frac{(n-1)(n-2)}{n}} & e^{2\pi i\frac{(n-1)(n-1)}{n}}
\end{pmatrix}$$

We calculate $U^\dagger U$:

$$\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & e^{-2\pi i\frac{1}{n}} & e^{-2\pi i\frac{2}{n}} & \dots & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-1)}{n}} \\ 1 & e^{-2\pi i\frac{2}{n}} & e^{-2\pi i\frac{4}{n}} & \dots & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{-2\pi i\frac{(n-2)}{n}} & e^{-2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-2)(n-2)}{n}} & e^{-2\pi i\frac{(n-2)(n-1)}{n}} \\ 1 & e^{-2\pi i\frac{(n-1)}{n}} & e^{-2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{-2\pi i\frac{(n-1)(n-2)}{n}} & e^{-2\pi i\frac{(n-1)(n-1)}{n}} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & e^{2\pi i\frac{1}{n}} & e^{2\pi i\frac{2}{n}} & \dots & e^{2\pi i\frac{(n-2)}{n}} & e^{2\pi i\frac{(n-1)}{n}} \\ 1 & e^{2\pi i\frac{2}{n}} & e^{2\pi i\frac{4}{n}} & \dots & e^{2\pi i\frac{(n-2)\cdot 2}{n}} & e^{2\pi i\frac{(n-1)\cdot 2}{n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{2\pi i\frac{(n-2)}{n}} & e^{2\pi i\frac{(n-2)\cdot 2}{n}} & \dots & e^{2\pi i\frac{(n-2)(n-2)}{n}} & e^{2\pi i\frac{(n-2)(n-1)}{n}} \\ 1 & e^{2\pi i\frac{(n-1)}{n}} & e^{2\pi i\frac{(n-1)\cdot 2}{n}} & \dots & e^{2\pi i\frac{(n-1)(n-2)}{n}} & e^{2\pi i\frac{(n-1)(n-1)}{n}} \end{pmatrix}$$

The first row times the first column obviously gives 1.

The first row times the second column:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{2\pi i\frac{l}{n}}\right)$$

We see that the product is a geometric series:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{2\pi i\frac{l}{n}}\underset{\bigstar}{}\right) = \frac{1}{n}\left(\sum_{l=0}^{n-1}\left(e^{2\pi i\frac{1}{n}}\right)^l\right) = \left(\sum_{l=0}^{n-1}\frac{1}{n}\left(e^{2\pi i\frac{1}{n}}\right)^l\right)$$

We multiply by $\left(1 - e^{2\pi i\frac{1}{n}}\right) \neq 0$:

$$\left(1 - e^{2\pi i\frac{1}{n}}\right)\sum_{l=0}^{n-1}\frac{1}{n}\left(e^{2\pi i\frac{1}{n}}\right)^l =$$

$$\frac{1}{n}\left(1 - e^{2\pi i\frac{1}{n}}\right)\left(\left(e^{2\pi i\frac{1}{n}}\right)^0 + \left(e^{2\pi i\frac{1}{n}}\right)^1 + \left(e^{2\pi i\frac{1}{n}}\right)^2 + \dots + \left(e^{2\pi i\frac{1}{n}}\right)^{n-2} + \left(e^{2\pi i\frac{1}{n}}\right)^{n-1}\right) =$$

$$\frac{1}{n}\left(\left(e^{2\pi i\frac{1}{n}}\right)^0 \underbrace{- \left(e^{2\pi i\frac{1}{n}}\right)^1 + \left(e^{2\pi i\frac{1}{n}}\right)^1} \underbrace{- \left(e^{2\pi i\frac{1}{n}}\right)^2 + \dots} \underbrace{+ \left(e^{2\pi i\frac{1}{n}}\right)^{n-2} - \dots} - \underbrace{\left(e^{2\pi i\frac{1}{n}}\right)^{n-1} + \left(e^{2\pi i\frac{1}{n}}\right)^{n-1}} - \left(e^{2\pi i\frac{1}{n}}\right)^n\right) =$$

$$\frac{1}{n}\left(\left(e^{2\pi i\frac{1}{n}}\right)^0 - \left(e^{2\pi i\frac{1}{n}}\right)^n\right) = \frac{1}{n}\left(1 - \left(e^{2\pi i\frac{1}{n}}\right)^n\right) =$$

$$\frac{1}{n}\left(1 - e^{2\pi i}\right) = \frac{1}{n}(1-1) = 0$$

The result is 0.

We can apply this procedure to any combination of row b of the first matrix and column c of the second matrix:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{-2\pi i\frac{bl}{n}} e^{2\pi i\frac{cl}{n}}\right) = \frac{1}{n}\left(\sum_{l=0}^{n-1} e^{2\pi i\frac{(c-b)l}{n}}\right)$$

We name $c - b \coloneqq d$ and rewrite:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{2\pi i d\frac{l}{n}}\right)$$

This is the same expression as ★ so we can conclude:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{2\pi i d\frac{l}{n}}\right) = \begin{cases} 0 \; if \; d \neq 0 \; resp. b \neq c \\ 1 \; if \; d = 0 \; resp. b = c \end{cases}$$

Note: $\left(1 - e^{2\pi i\frac{d}{n}}\right) \neq 0$ for all $c \neq b$ because $d = (c - b) < n$.

For the case $c = d$ we can calculate the sum directly and get:

$$\frac{1}{n}\left(\sum_{l=0}^{n-1} e^{0}\right) = 1$$

Result:

$$U^{\dagger}U = UU^{\dagger} = Id$$

The matrices for the three-qubit case

The first Hadamard:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The first R_1-gate and the Hadamard:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & i \end{pmatrix}$$

The first Hadamard, R_1-gate and R_2-gate

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & e^{-\pi i\frac{1}{4}} & 0 & 0 & 0 & e^{-\pi i\frac{3}{4}} & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & e^{-\pi i\frac{3}{4}} & 0 & 0 & 0 & e^{-\pi i\frac{7}{4}} \end{pmatrix}$$

The first Hadamard, R_1-gate and R_2-gate and the second Hadamard:

$$\frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -i & 0 & -1 & 0 & i & 0 \\ 0 & e^{-\pi i\frac{1}{4}} & 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{7}{4}} \\ 1 & 0 & i & 0 & -1 & 0 & -i & 0 \\ 0 & e^{-\pi i\frac{1}{4}} & 0 & e^{-\pi i\frac{7}{4}} & 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{3}{4}} \end{pmatrix}$$

The first Hadamard, R_1-gate and R_2-gate, the second Hadamard and the second R_1-gate:

$$\frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i & 0 & -i & 0 & i \\ 1 & 0 & -i & 0 & -1 & 0 & i & 0 \\ 0 & e^{-\pi i\frac{1}{4}} & 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{7}{4}} \\ 1 & 0 & i & 0 & -1 & 0 & -i & 0 \\ 0 & e^{-\pi i\frac{3}{4}} & 0 & e^{-\pi i\frac{1}{4}} & 0 & e^{-\pi i\frac{5}{4}} & 0 & e^{-\pi i\frac{3}{4}} \end{pmatrix}$$

The first Hadamard, R_1-gate and R_2-gate, the second Hadamard, the second R_1-gate and the last Hadamard:

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{7}{4}} \\ 1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\ 1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{3}{4}} \\ 1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{1}{4}} \end{pmatrix}$$

The final swap delivers:

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{7}{4}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & e^{-\pi i\frac{3}{4}} & i & e^{-\pi i\frac{1}{4}} & -1 & e^{-\pi i\frac{5}{4}} & -i & e^{-\pi i\frac{3}{4}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{-\pi i\frac{3}{4}} & -i & e^{-\pi i\frac{7}{4}} & -1 & e^{-\pi i\frac{1}{4}} & i & e^{-\pi i\frac{3}{4}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & e^{-\pi i\frac{7}{4}} & i & e^{-\pi i\frac{3}{4}} & -1 & e^{-\pi i\frac{1}{4}} & -i & e^{-\pi i\frac{1}{4}} \end{pmatrix}$$