We use a function $f(x): \{0,1\} \rightarrow \{0,1\}$.

We use a two-bit quantum computer starting in the state $|x, y\rangle$.

All states $|x\rangle$, $|y\rangle$ are the basis states either $|0\rangle$ or $|1\rangle$.

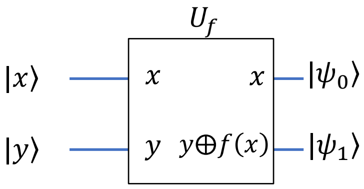
We use logic gates to transform this state into $|x, y \oplus f(x)\rangle$.

$\oplus$ denotes addition modulo 2.

$\otimes$ denotes the Kronecker product or tensor product.

We call the unitary transformation matrix $U_f$.

$U_f$ maps $|x, y\rangle \rightarrow |x, y \oplus f(x)\rangle$.




Input: $|xy\rangle = |x\rangle \otimes |y\rangle$

Output: $|\psi_0 \psi_1\rangle = |\psi_0\rangle \otimes |\psi_1\rangle$

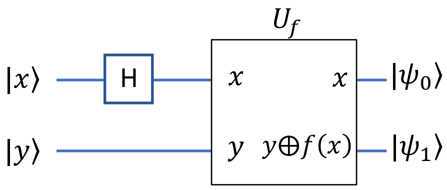
| | input | | | |
|---|---|---|---|---|
| | $\lvert 00\rangle$ | $\lvert 01\rangle$ | $\lvert 10\rangle$ | $\lvert 11\rangle$ |
| Let $f$ be constant, $f(x) = 0 \; \forall \, x \in \{0,1\}$ | $\lvert 00\rangle \rightarrow \lvert 00\rangle$ | $\lvert 01\rangle \rightarrow \lvert 00\rangle$ | $\lvert 10\rangle \rightarrow \lvert 10\rangle$ | $\lvert 11\rangle \rightarrow \lvert 10\rangle$ |
| Let $f$ be constant, $f(x) = 1 \; \forall \, x \in \{0,1\}$ | $\lvert 00\rangle \rightarrow \lvert 01\rangle$ | $\lvert 01\rangle \rightarrow \lvert 01\rangle$ | $\lvert 10\rangle \rightarrow \lvert 11\rangle$ | $\lvert 11\rangle \rightarrow \lvert 11\rangle$ |
| Let $f$ be balanced, $f(0) = 0, f(1) = 1$ | $\lvert 00\rangle \rightarrow \lvert 00\rangle$ | $\lvert 01\rangle \rightarrow \lvert 01\rangle$ | $\lvert 10\rangle \rightarrow \lvert 10\rangle$ | $\lvert 11\rangle \rightarrow \lvert 11\rangle$ |
| Let $f$ be balanced, $f(0) = 1, f(1) = 0$ | $\lvert 00\rangle \rightarrow \lvert 01\rangle$ | $\lvert 01\rangle \rightarrow \lvert 00\rangle$ | $\lvert 10\rangle \rightarrow \lvert 11\rangle$ | $\lvert 11\rangle \rightarrow \lvert 10\rangle$ |

For each input we get two times the same output connected. Thus, the output is not sufficient to decide what function $f(x)$ we used.

We bring the input into superposition by use of the Hadamard operator:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \; H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

We use:




Input: $|xy\rangle = H|x\rangle \otimes |y\rangle$.

Output: $|\psi_0 \psi_1\rangle = |\psi_0\rangle \otimes |\psi_1\rangle$

Note: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ etc.

| input | Let $f$ be constant, $f(x) = 0 \ \forall \ x \in \{0,1\}$<br>$y = y \oplus f(x)$<br>$y = 0 \rightarrow 0, y = 1 \rightarrow 1$ | Let $f$ be constant, $f(x) = 1 \ \forall \ x \in \{0,1\}$<br>$y = y \oplus f(x)$<br>$y = 0 \rightarrow 1, y = 1 \rightarrow 0$ |
|---|---|---|
| $\|00\rangle$<br>$\|01\rangle$<br>$\|10\rangle$<br>$\|11\rangle$ | $\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle)$ | $\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle)$ |
| input | Let $f$ be balanced,<br>$f(0) = 0, f(1) = 1$<br>$y = y \oplus f(x)$<br>$x = 0, y = 0 \rightarrow 0$<br>$x = 0, y = 1 \rightarrow 1$<br>$x = 1, y = 0 \rightarrow 1$<br>$x = 1, y = 1 \rightarrow 0$ | Let $f$ be balanced,<br>$f(0) = 1, f(1) = 0$<br>$y = y \oplus f(x)$<br>$x = 0, y = 0 \rightarrow 1$<br>$x = 0, y = 1 \rightarrow 0$<br>$x = 1, y = 0 \rightarrow 0$<br>$x = 1, y = 1 \rightarrow 1$ |
| $\|00\rangle$<br>$\|01\rangle$<br>$\|10\rangle$<br>$\|11\rangle$ | $\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$ | $\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle + \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$<br>$\frac{1}{\sqrt{2}}(\|00\rangle - \|10\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$<br>$\frac{1}{\sqrt{2}}(\|01\rangle - \|11\rangle) \rightarrow \frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$ |

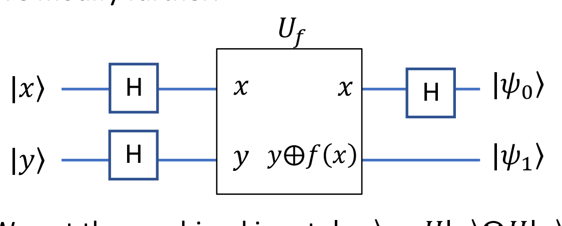
Now something has changed. As before we know the input, e. g. $|00\rangle$ resp. $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ after the Hadamard, but the output is unique coupled with the kind of function we use.

There is still one problem that we can not use the output as it is. Take e. g.

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is not separable meaning it cannot be written as a Kronecker product $|x\rangle \otimes |y\rangle$. Measuring the first qubit in this case will give random results zero or one. The same holds for all other possibilities.

We modify further:




We get the combined input: $|xy\rangle = H|x\rangle \otimes H|y\rangle$.

We combine the two qubits $|x\rangle$ and $|y\rangle$ via the Kronecker product, expressed in the $|0\rangle$ $|1\rangle$ basis:

$$|x\rangle = a \cdot |0\rangle + b \cdot |1\rangle, \qquad |y\rangle = c \cdot |0\rangle + d \cdot |1\rangle$$

Note: $a, b, c, d \in \{0,1\}$

The Kronecker product:

$$H|x\rangle \otimes H|y\rangle = H(a|0\rangle + b|1\rangle) \otimes H(c|0\rangle + d|1\rangle) =$$

$$\left(a\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + b\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \otimes \left(c\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + d\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) =$$

$$\left(a\frac{1}{\sqrt{2}}|0\rangle + a\frac{1}{\sqrt{2}}|1\rangle + b\frac{1}{\sqrt{2}}|0\rangle - b\frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(c\frac{1}{\sqrt{2}}|0\rangle + c\frac{1}{\sqrt{2}}|1\rangle + d\frac{1}{\sqrt{2}}|0\rangle - d\frac{1}{\sqrt{2}}|1\rangle\right) =$$

$$\left(\frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{c+d}{\sqrt{2}}|0\rangle + \frac{c-d}{\sqrt{2}}|1\rangle\right) =$$

$$\frac{(a+b)(c+d)}{2}|00\rangle + \frac{(a+b)(c-d)}{2}|01\rangle + \frac{(a-b)(c+d)}{2}|10\rangle + \frac{(a-b)(c-d)}{2}|11\rangle$$

We have four possible combinations for $a$ and $b$ and get after the Hadamard:

| | | |
|---|---|---|
| ① | $a = 1, b = 0, c = 0, d = 1 \rightarrow 1\ 0\ 0\ 1$ | $\frac{1}{2}(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle)$ |
| ② | $a = 1, b = 0, c = 1, d = 0 \rightarrow 1\ 0\ 1\ 0$ | $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle)$ |
| ③ | $a = 0, b = 1, c = 0, d = 1 \rightarrow 0\ 1\ 0\ 1$ | $\frac{1}{2}(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle)$ |
| ④ | $a = 0, b = 1, c = 1, d = 0 \rightarrow 0\ 1\ 1\ 0$ | $\frac{1}{2}(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle)$ |

We apply $U_f$ to the possible combinations.

| | | |
|---|---|---|
| Let $f$ be constant, $f(x) = 0\ \forall x \in \{0,1\}$ $y = y \oplus f(x)$ $y = 0 \rightarrow 0,$ $y = 1 \rightarrow 1$ | ① | $\frac{1}{2}(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle) = \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ |
| | ② | $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle) = \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| | ③ | $\frac{1}{2}(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle) = \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ |
| | ④ | $\frac{1}{2}(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle) = \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| Let $f$ be constant, $f(x) = 1\ \forall x \in \{0,1\}$ $y = y \oplus f(x)$ $y = 0 \rightarrow 1,$ $y = 1 \rightarrow 0$ | ① | $\frac{1}{2}(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle) \rightarrow \frac{1}{2}(\|01\rangle - \|00\rangle + \|11\rangle - \|10\rangle) =$ $-\frac{1}{2}(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle) = -\frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ |
| | ② | $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle) \rightarrow \frac{1}{2}(\|01\rangle + \|00\rangle + \|11\rangle + \|10\rangle) =$ $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle) = \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| | ③ | $\frac{1}{2}(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle) \rightarrow \frac{1}{2}(\|01\rangle - \|00\rangle - \|11\rangle + \|10\rangle) =$ $-\frac{1}{2}(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle) = -\frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ |
| | ④ | $\frac{1}{2}(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle) \rightarrow \frac{1}{2}(\|01\rangle + \|00\rangle - \|11\rangle - \|10\rangle) =$ $\frac{1}{2}(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle) = \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}} \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |

| | | |
|---|---|---|
| Let $f$ be balanced,<br>$f(0)=0, f(1)=1$<br>$y=y\oplus f(x)$<br>$x=0, y=0\rightarrow 0$<br>$x=0, y=1\rightarrow 1$<br>$x=1, y=0\rightarrow 1$<br>$x=1, y=1\rightarrow 0$ | ① | $\frac{1}{2}(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)\rightarrow\frac{1}{2}(\|00\rangle-\|01\rangle+\|11\rangle-\|10\rangle)=$<br>$\frac{1}{2}(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)=\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ |
| | ② | $\frac{1}{2}(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)\rightarrow\frac{1}{2}(\|00\rangle+\|01\rangle+\|11\rangle+\|10\rangle)=$<br>$\frac{1}{2}(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)=\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| | ③ | $\frac{1}{2}(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)\rightarrow\frac{1}{2}(\|00\rangle-\|01\rangle-\|11\rangle+\|10\rangle)=$<br>$\frac{1}{2}(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)=\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ |
| | ④ | $\frac{1}{2}(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)\rightarrow\frac{1}{2}(\|00\rangle+\|01\rangle-\|11\rangle-\|10\rangle)=$<br>$\frac{1}{2}(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)=\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |

| | | |
|---|---|---|
| Let $f$ be balanced,<br>$f(0)=1, f(1)=0$<br>$y=y\oplus f(x)$<br>$x=0, y=0\rightarrow 1$<br>$x=0, y=1\rightarrow 0$<br>$x=1, y=0\rightarrow 0$<br>$x=1, y=1\rightarrow 1$ | ① | $\frac{1}{2}(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)\rightarrow\frac{1}{2}(\|01\rangle-\|00\rangle+\|10\rangle-\|11\rangle)=$<br>$-\frac{1}{2}(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)=-\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ |
| | ② | $\frac{1}{2}(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)\rightarrow\frac{1}{2}(\|01\rangle+\|00\rangle+\|10\rangle+\|11\rangle)=$<br>$\frac{1}{2}(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)=\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| | ③ | $\frac{1}{2}(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)\rightarrow\frac{1}{2}(\|01\rangle-\|00\rangle-\|10\rangle+\|11\rangle)=$<br>$-\frac{1}{2}(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)=-\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| | ④ | $\frac{1}{2}(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)\rightarrow\frac{1}{2}(\|01\rangle+\|00\rangle-\|10\rangle-\|11\rangle)=$<br>$\frac{1}{2}(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)=\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |

We summarize the results:

| | $a\ b\ c\ d=;$ | | | |
|---|---|---|---|---|
| | 1 0 0 1 | 1 0 1 0 | 0 1 0 1 | 0 1 1 0 |
| $f(x)=0$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| $f(x)=1$ | $-\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ | $-\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| $f(0)=0$<br>$f(1)=1$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |
| $f(0)=1$<br>$f(1)=0$ | $-\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ | $-\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}$ | $\frac{(\|0\rangle-\|1\rangle)}{\sqrt{2}}\cdot\frac{(\|0\rangle+\|1\rangle)}{\sqrt{2}}$ |

We apply the final Hadamard to the first qubit.

We notice:

$$H(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) = \frac{2}{\sqrt{2}}|0\rangle = \sqrt{2}|0\rangle$$

$$H(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle - |0\rangle + |1\rangle) = \frac{2}{\sqrt{2}}|1\rangle = \sqrt{2}|1\rangle$$

We get as final state:

| | $a\ b\ c\ d =;$ | | | |
|---|---|---|---|---|
| | $1\ 0\ 0\ 1$ | $1\ 0\ 1\ 0$ | $0\ 1\ 0\ 1$ | $0\ 1\ 1\ 0$ |
| $f(x) = 0$ | $\|0\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|0\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ | $\|1\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|1\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| $f(x) = 1$ | $-\|0\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|0\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ | $-\|1\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|1\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| $f(0) = 0$ $f(1) = 1$ | $\|1\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|0\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ | $\|0\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|1\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |
| $f(0) = 1$ $f(1) = 0$ | $-\|1\rangle \cdot \frac{(\|0\rangle - \|1\rangle)}{\sqrt{2}}$ | $\|0\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ | $-\|1\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ | $\|1\rangle \cdot \frac{(\|0\rangle + \|1\rangle)}{\sqrt{2}}$ |

The combinations 1001 and 0101 have a special property. By measuring the first qubit we get:

| | |
|---|---|
| in the case 1 0 0 1: | in the case 0 1 0 1: |
| $\pm\|0\rangle$ if the function is constant,<br>$\pm\|1\rangle$ if the function is balanced. | $\pm\|1\rangle$ if the function is constant,<br>$\pm\|0\rangle$ if the function is balanced. |

With one measurement we can decide whether the function in question is balanced or constant.

Interpretation (with respect to case 1001):

Noting that $f(0) \oplus f(1) = 0$ if $f(x)$ is constant, $f(0) \oplus f(1) = 1$ if $f(x)$ is balanced we can rewrite:

Output: $|\psi_0\psi_1\rangle = \pm|f(0) \oplus f(1)\rangle \left(\frac{(|0\rangle - |1\rangle)}{\sqrt{2}}\right)$

Citing Nielsen/Chuang[1]: "… the quantum circuit has given us the ability to determine a global property of $f(x)$, namely $f(0) \oplus f(1)$, using only one evaluation of $f(x)$. … in a quantum computer it is possible for the two alternatives to interfere with one another to yield some global property of the function f, …"

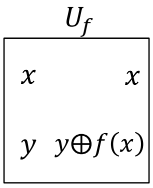
Please note that we never modified qubit one directly. The superposition transports effects of the modification of qubit two to qubit one. We are not dealing with two isolated qubits of dimension two each but with a four-dimensional entity after applying the Hadamards.

We must destroy this four-dimensional entity by the process of measurement to get a two-dimensional entity back.

[1] *Quantum Computation and Quantum Information, Nielsen/Chang, Cambridge University Press, ISBN 978-1-107-00217-3, page 33.*

## Basic level

We will go to back to the function and its implementation in the quantum circuit, the unitary operator $U_f$.




Depending on what function we use the operator looks different.

We have four possibilities:

| | |
|---|---|
| ① | $f(x)\ constant, f(x) = 0$ |
| ② | $f(x)\ constant, f(x) = 1$ |
| ③ | $f(x)\ balanced, f(0) = 0, f(1) = 1$ |
| ④ | $f(x)\ balanced, f(0) = 1, f(1) = 0$ |

① If $f(x) = 0$ then $y \oplus f(x) = y$. In this case $U_f$ is the identity matrix:

$$U_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{input} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \text{ output} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

② If $f(x) = 1$ then $y \oplus f(x) = \overline{y}$. In this case $U_f$ is the matrix:

$$U_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{input} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \text{ output} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli $X$-matrix.

This is a NOT of qubit two. $U_f$ is unitary:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

③ If $f(0) = 0 \wedge f(1) = 1$ then

$$x = 0, y = 0 \rightarrow y = 0$$

$$x = 0, y = 1 \rightarrow y = 1$$

$$x = 1, y = 0 \rightarrow y = 1$$

$$x = 1, y = 1 \rightarrow y = 0$$

In this case $U_f$ is the matrix:

$$U_f = \text{input} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \text{ output} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note: $U_f$ cannot be decomposed into a Kronecker product of two $2 \times 2$ matrices. $U_f$ is a CNOT.

The CNOT is unitary:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

④ If $f(0) = 1 \wedge f(1) = 0$ then

$$x = 0, y = 0 \rightarrow y = 1$$

$$x = 0, y = 1 \rightarrow y = 0$$

$$x = 1, y = 0 \rightarrow y = 0$$

$$x = 1, y = 1 \rightarrow y = 1$$

In this case $U_f$ is the matrix:

$$U_f = \text{input} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \text{ output} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note: $U_f$ cannot be decomposed into a Kronecker product of two $2 \times 2$ matrices. $U_f$ is a kind of inverted CNOT, $\overline{C}NOT$. It is unitary:

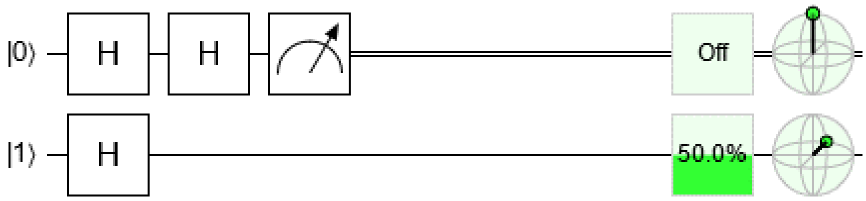
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We summarize:

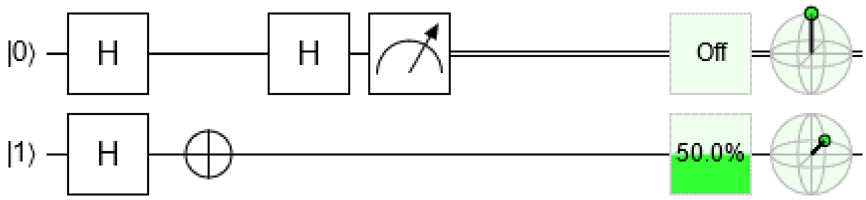
| | | |
|---|---|---|
| ① | $f(x)\ constant, f(x) = 0$ | $U_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| ② | $f(x)\ constant, f(x) = 1$ | $U_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ |
| ③ | $f(x)\ balanced, f(0) = 0, f(1) = 1$ | $U_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ |
| ④ | $f(x)\ balanced, f(0) = 1, f(1) = 0$ | $U_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |

We check by help of quirk: https://algassert.com/quirk

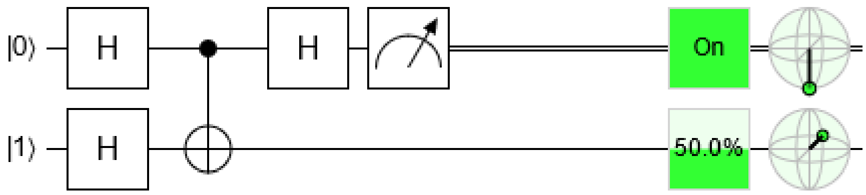
①



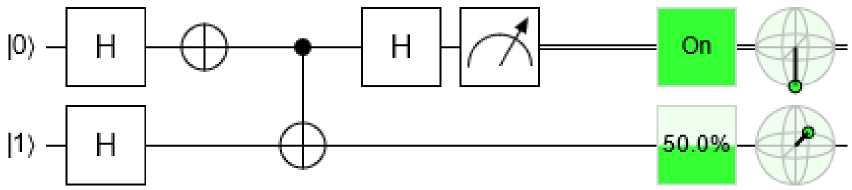

②




③




④



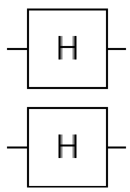

Note: We compose the $\overline{CNOT}$ by first inverting qubit one and then applying a $CNOT$.

We express in terms of vectors and matrices.

$$|01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

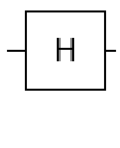
The double Hadamard:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{pmatrix}$$



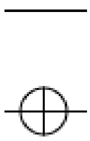

The Hadamard on line one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{pmatrix}$$



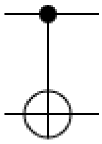
The NOT on line two:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$



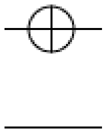
The CNOT from line one to line two:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

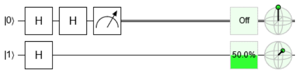


The NOT on line one:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$



We applicate:




① single Hadamard double Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=$$

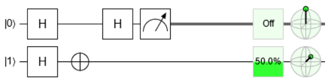
$$\frac{1}{2\sqrt{2}}\begin{pmatrix}2\\-2\\0\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

We decompose into a Kronecker product:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

This corresponds to the solution on the conceptual level:

$$|0\rangle\left(\frac{(|0\rangle-|1\rangle)}{\sqrt{2}}\right)$$




② single Hadamard NOT double Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}-2\\2\\0\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$$

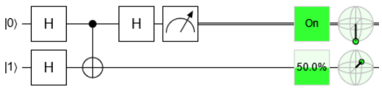
We decompose into a Kronecker product:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}=-\binom{1}{0}\otimes\frac{1}{\sqrt{2}}\binom{1}{-1}$$

This corresponds to the solution on the conceptual level:

$$-|0\rangle\left(\frac{(|0\rangle-|1\rangle)}{\sqrt{2}}\right)$$

③




single Hadamard CNOT double Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}0\\0\\2\\-2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

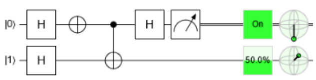
We decompose into a Kronecker product:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}=\binom{0}{1}\otimes\frac{1}{\sqrt{2}}\binom{1}{-1}$$

This corresponds to the solution on the conceptual level:

$$|1\rangle\left(\frac{(|0\rangle-|1\rangle)}{\sqrt{2}}\right)$$

④




single Hadamard CNOT NOT double Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\ 1\\ -1\\ 1\end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\begin{pmatrix}-1\\ 1\\ 1\\ -1\end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}}\begin{pmatrix}0\\ 0\\ -2\\ 2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ -1\\ 1\end{pmatrix}$$

We decompose into a Kronecker product:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ -1\\ 1\end{pmatrix} = -\begin{pmatrix}0\\ 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

This corresponds to the solution on the conceptual level:

$$-|1\rangle\left(\frac{(|0\rangle - |1\rangle)}{\sqrt{2}}\right)$$