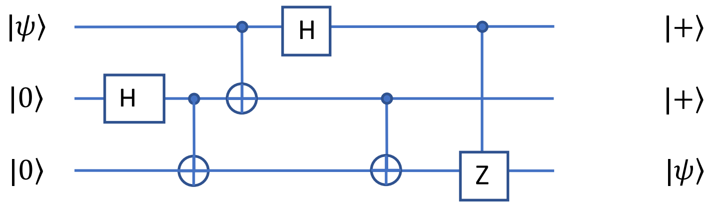
Quantum teleportation denotes the process of exchanging the state of one qubit with that of another in a system of several qubits without performing the exchange explicitly. This is possible because the qubits form a multidimensional state space via the Kronecker product. Operators do not act on a single qubit, but on the entire state space, modifying implicit all other qubits. The measurement in the end is necessary to break the multidimensional object into three independent qubits again.

The following circuit performs a quantum teleportation, changing the input from $|\psi 00\rangle$ to $|++\psi\rangle$



$|\psi\rangle$ is a random input state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The input state $|\psi\rangle$ is being "teleported" to the bottom line, the two qubits $|0\rangle$ on line 2 and 3 change to $|+\rangle$ after the procedure.

The output:

$$|+\rangle \otimes |+\rangle \otimes |\psi\rangle \; resp. \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

The gates

The Hadamard gate changes:

$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle)$	$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle - |1\rangle)$

The controlled Z-gate from line one to line three reverses the sign of line three if line one is $|1\rangle$ and line three is $|1\rangle$:

$Z|1x0\rangle = |1x0\rangle$	$Z|1x1\rangle = -|1x1\rangle$

Conceptual level

We use the input state:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle$$

We apply the first Hadamard on line two:

$$H(|\psi\rangle \otimes |0\rangle \otimes |0\rangle) = (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle =$$

$$\frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)(|0\rangle + |1\rangle)|0\rangle =$$

$$\frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)(|00\rangle+|10\rangle)=$$

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|010\rangle+\beta|100\rangle+\beta|110\rangle)$$

We apply the first CNOT from line two to line three:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle)$$

We apply the second CNOT from line one to line two:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle) \qquad ①$$

We apply the second Hadamard on line one:

$$H\left(\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle)\right)=$$

$$H\left(\frac{1}{\sqrt{2}}(|0\rangle(\alpha|00\rangle+\alpha|11\rangle)+|1\rangle(\beta|10\rangle+\beta|01\rangle))\right)=$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cdot(|0\rangle+|1\rangle)(\alpha|00\rangle+\alpha|11\rangle)+\frac{1}{\sqrt{2}}\cdot(|0\rangle-|1\rangle)(\beta|10\rangle+\beta|01\rangle)\right)=$$

$$\frac{1}{2}(\alpha|000\rangle+\alpha|100\rangle+\alpha|011\rangle+\alpha|111\rangle+\beta|010\rangle+\beta|001\rangle-\beta|110\rangle-\beta|101\rangle) \qquad ②$$

We apply the CNOT from line two to line three:

$$\frac{1}{2}(\alpha|000\rangle+\alpha|100\rangle+\alpha|010\rangle+\alpha|110\rangle+\beta|011\rangle+\beta|001\rangle-\beta|111\rangle-\beta|101\rangle)$$

We apply the controlled Z-gate from line one to line three:

$$\frac{1}{2}(\alpha|000\rangle+\alpha|100\rangle+\alpha|010\rangle+\alpha|110\rangle+\beta|011\rangle+\beta|001\rangle+\beta|111\rangle+\beta|101\rangle)$$

We disassemble the qubits:

$$\frac{1}{\sqrt{2}}|0\rangle\frac{1}{\sqrt{2}}|0\rangle\alpha|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\frac{1}{\sqrt{2}}|0\rangle\alpha|0\rangle+\frac{1}{\sqrt{2}}|0\rangle\frac{1}{\sqrt{2}}|1\rangle\alpha|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\frac{1}{\sqrt{2}}|1\rangle\alpha|0\rangle+$$

$$\frac{1}{\sqrt{2}}|0\rangle\frac{1}{\sqrt{2}}|1\rangle\beta|1\rangle+\frac{1}{\sqrt{2}}|0\rangle\frac{1}{\sqrt{2}}|0\rangle\beta|1\rangle+\frac{1}{\sqrt{2}}|1\rangle\frac{1}{\sqrt{2}}|1\rangle\beta|1\rangle+\frac{1}{\sqrt{2}}|1\rangle\frac{1}{\sqrt{2}}|0\rangle\beta|1\rangle=$$

$$|+\rangle\frac{1}{\sqrt{2}}|0\rangle\alpha|0\rangle+|+\rangle\frac{1}{\sqrt{2}}|1\rangle\alpha|0\rangle+|+\rangle\frac{1}{\sqrt{2}}|1\rangle\beta|1\rangle+|+\rangle\frac{1}{\sqrt{2}}|0\rangle\beta|1\rangle=$$

$$|+\rangle|+\rangle\alpha|0\rangle+|+\rangle|+\rangle\beta|1\rangle=$$

$$|+\rangle|+\rangle(\alpha|0\rangle+\beta|1\rangle)=$$

$$|+\rangle|+\rangle|\psi\rangle$$

Basic level.

We can do the same calculation on the basic level.

We have three qubits:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \otimes \begin{pmatrix}1\\ 0\end{pmatrix} \otimes \begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \otimes \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\\ 0\\ 0\\ \beta\\ 0\\ 0\\ 0\end{pmatrix}$$

The Hadamard acting on the first line:

$$H \otimes I \otimes I =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\ \hline 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\end{array}\right)$$

The Hadamard acting on the second line:

$$I \otimes H \otimes I =$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\left(\begin{array}{cccc|cccc}1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\end{array}\right)$$

The Z-gate on line three:

$$\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right)$$

Note: The Z-gate is constructed by the same method as the following CNOT.

The CNOT acting from line one to line two. We adopt the method from

https://quantumcomputing.stackexchange.com/users/14597/rajiv-krishnakumar

https://quantumcomputing.stackexchange.com/questions/28315/how-to-calculate-cnot-gate-for-three-qubit-1-control-2-target-states-as-a-mat

This method uses the conceptual level to build the matrix on the basis level

$|000\rangle\ |001\rangle\ |010\rangle\ |011\rangle\ |100\rangle\ |101\rangle\ |110\rangle\ |111\rangle$ ⟵ output

$$\text{input} \rightarrow \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix} \quad \begin{pmatrix}
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x
\end{pmatrix}$$

The CNOT acting from line one to line two:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

The CNOT acting from line two to line three:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

We assemble the gates one by one. The input state:

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first Hadamard (acting on the second line):

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ \alpha \\ 0 \\ \beta \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

The CNOT from line two to line three:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ \alpha \\ 0 \\ \beta \\ 0 \\ \beta \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

The CNOT from line one to line two:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{pmatrix}$$

The second Hadamard (acting on the first line):

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix}$$

The CNOT from line two to line three:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}$$

The Z-gate on line three:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix}$$

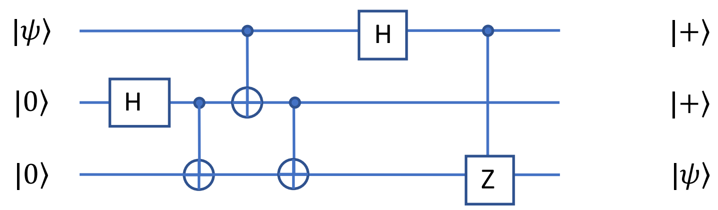
This is a separable state. You may find it in the paper “Kronecker_product” on this website on page 6, searching for $|{+}{+}.\rangle$ and building the sum of $|{+}+0\rangle$ and $|{+}+1\rangle$. Replace the "1" by α and β. Alternatively, we can verify this by calculating the Kronecker product of the output wanted:

$$|+\rangle \otimes |+\rangle \otimes |\psi\rangle =$$

$$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} =$$

$$\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix}$$

Let us work with the example. What happens if we pull the third CNOT forward end execute it before we apply the Hadamard on the first line?



We have the input state:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle$$

We apply the first Hadamard on line two:

$$H(|\psi\rangle \otimes |0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|010\rangle + \beta|100\rangle + \beta|110\rangle)$$

We apply the first CNOT from line two to line three:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

We apply the second CNOT from line one to line two:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

We apply the third CNOT from line two to line three:

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|010\rangle + \beta|111\rangle + \beta|101\rangle)$$

We apply the second Hadamard on line one:

$$H\left(\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|010\rangle + \beta|111\rangle + \beta|101\rangle)\right) =$$

$$H\left(\frac{1}{\sqrt{2}}(|0\rangle(\alpha|00\rangle + \alpha|10\rangle) + |1\rangle(\beta|11\rangle + \beta|01\rangle))\right) =$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle)(\alpha|00\rangle + \alpha|10\rangle) + \frac{1}{\sqrt{2}} \cdot (|0\rangle - |1\rangle)(\beta|11\rangle + \beta|01\rangle)\right) =$$

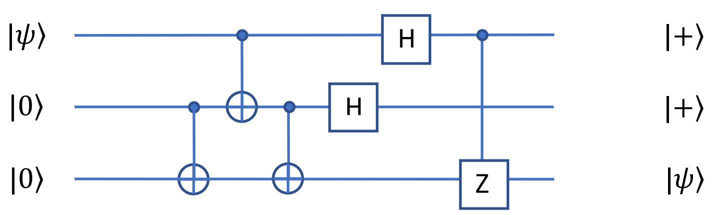
$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|010\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|001\rangle - \beta|111\rangle - \beta|101\rangle)$$

We compare this with the state of the original circuit (the CNOT after the Hadamard):

$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|010\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|001\rangle - \beta|111\rangle - \beta|101\rangle)$$

This is the same state. Pulling the CNOT forward doesn't change the result.

We're getting braver. What happens if we execute the first Hadamard after the series of three CNOT?



We have the input state:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle = \alpha|000\rangle + \beta|100\rangle$$

We apply the first CNOT from line two to line three:

$$\alpha|000\rangle + \beta|100\rangle$$

We apply the second CNOT from line one to line two:

$$\alpha|000\rangle + \beta|110\rangle$$

We apply the third CNOT from line two to line three:

$$\alpha|000\rangle + \beta|111\rangle$$

We apply the first Hadamard on line two:

$$H(\alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle) =$$

$$\frac{1}{\sqrt{2}}(\alpha|0\rangle(|0\rangle + |1\rangle)|0\rangle + \beta|1\rangle(|0\rangle - |1\rangle)|1\rangle) =$$

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|010\rangle + \beta|101\rangle - \beta|111\rangle)$$

We compare with the original state ① on page 2:

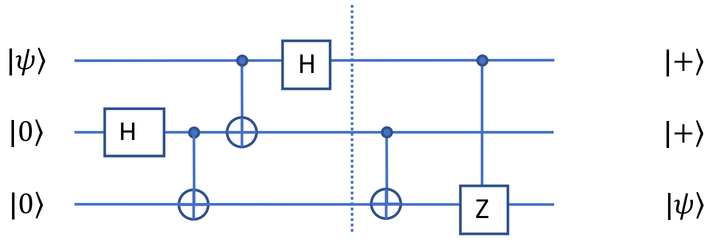
$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|010\rangle + \beta|111\rangle + \beta|101\rangle)$$

There is a sign switch.

We conclude that we can move operators as long as we don't skip hot spots on the lines.

A new scenario

We take a closer look at the circuit and stop at the dotted line:



The state of the qubits we get from ②, page 2:

$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle)$$

We regroup this state:

$$\frac{1}{2}\big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\big) \quad ③$$

Obviously, the CNOT and the controlled Z are designed to react to this state.

In case qubits one and two are $|00\rangle$ they do not act, the state on line three already is the input state $|\psi\rangle$.

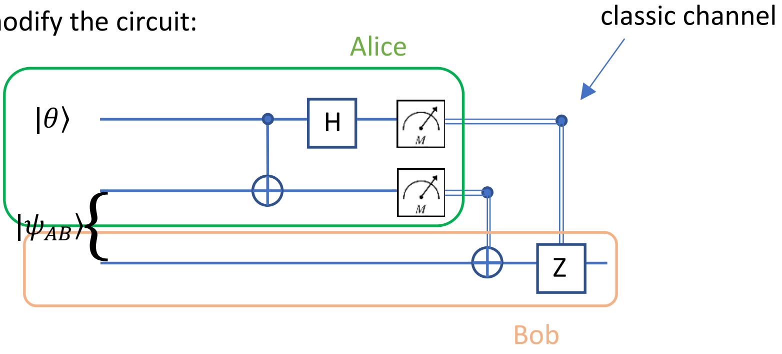
In case qubits one and two are $|01\rangle$ we need to switch $|0\rangle$ and $|1\rangle$ to get the state wanted, $|\psi\rangle$. The CNOT does this.

In case qubits one and two are $|10\rangle$ we only need to change the sign of $\beta|1\rangle$ to get back $|\psi\rangle$. The controlled Z does this.

In case qubits one and two are $|11\rangle$ we must apply both changes to get back $|\psi\rangle$, the CNOT and the controlled Z.

Spatial teleportation

We modify the circuit:



Alice has qubit $|\theta\rangle$ and Alice and Bob share two entangled qubits $|\psi_{AB}\rangle$:

$$|\theta\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Note: $|\psi_{AB}\rangle$ is an entangled state, a Bell-state meaning it is not separable and cannot be composed by a single application of the Kronecker product to two basis states.

We calculate the initial state:

$$|\theta\rangle|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\big(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\big)$$

We apply the CNOT from line one to line two:

$$\frac{1}{\sqrt{2}}\big(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\big)$$

We apply the Hadamard on line one:

$$\frac{1}{2}\big(\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\big) =$$

$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle) =$$

$$\frac{1}{2}\big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\big)$$

This is the same state we got in ③ on page 8.

Now, Alice can tell Bob the state of her qubit. She measures qubit one and her half of the entangled pair and sends the result via a classical channel to Bob.

If Bob receives the two classical bits 00, he knows that he already has the qubit $|\theta\rangle$ of Alice.

If Bob receives the two classical bits 01, he needs to apply a NOT to his half of the entangled pair to get the qubit $|\theta\rangle$ of Alice.

If Bob receives the two classical bits 10, he needs to apply a Z to his half of the entangled pair to get the qubit $|\theta\rangle$ of Alice.

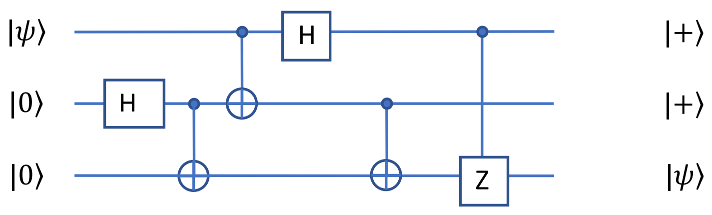
If Bob receives the two classical bits 11, he needs to perform the two operations NOT and Z to his half of the entangled pair to get the qubit $|\theta\rangle$ of Alice.

With this teleportation, Alice can send the information what qubit she had in the beginning to Bob, using two classical bits.

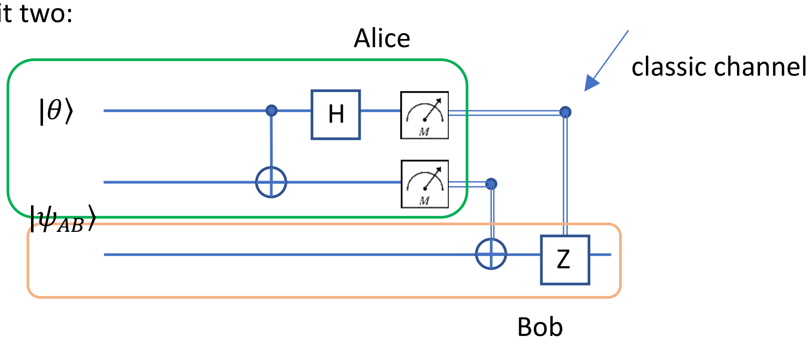
Comparing the circuits

We used two circuits:

Circuit one:



Circuit two:



We check what happens to the input of circuit one after the first Hadamard and the first CNOT:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle \to \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

We compare with the Bell-state of circuit two:

$$\frac{1}{\sqrt{2}}\big(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\big)$$

This is the same state. The Hadamard and the first CNOT in circuit one are needed to transform the qubits two and three $|00\rangle$ into the Bell-state.