Alice has a qubit $|\psi\rangle$ she wants to transmit to Bob. Alice don't know the state of this qubit.

Alice is unable to physically send her qubit $|\psi\rangle$ to Bob (say, by Fed-Ex …).
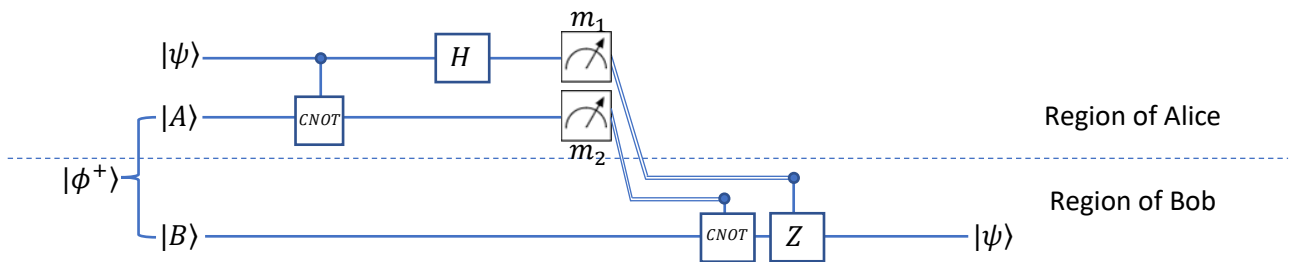
Alice is able to send classical information.

Alice and Bob share an entangled qubit, $|\phi^+\rangle$, one of the Bell states:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice possesses the second qubit of $|\phi^+\rangle$, Bob the first one.

Now the following procedure takes place:

Note:

Single lines denote qubits

Double lines denote classical bits

| H | Denote the Hadamard gate |

| X | Denote a bit flip (the Pauli x-gate) |

| Z | Denote a phase flip (the Pauli z-gate) |

CNOT   Denote a controlled Not gate

Note: The choice of the gates on the line $|B\rangle$ will become clear at the end of the process.

The procedure goes as follows.

First Alice:

- Alice performs a $CNOT$ from $|\psi\rangle$ to $|A\rangle$
- Alice performs a Hadamard on $|\psi\rangle$
- Alice measures $|A\rangle$ and $|\psi\rangle$ obtaining the classical bits m_1 and m_2
- Alice sends classical bits m_1 and m_2 to Bob

Then Bob:

- Bob first performs a $CNOT$ controlled by m_2 on $|B\rangle$
- Bob second performs a phase flip controlled by m_1 on $|B\rangle$

The operation of Bob:

$$
\begin{aligned}
&Id && if\ ab = 00 \\
&Z && if\ ab = 01 \\
&X && if\ ab = 10 \\
&ZX && if\ ab = 11
\end{aligned}
$$

The result is the teleportation:

$$|B\rangle\ becomes\ |\psi\rangle$$

Let us take a closer look at the scenario. The initial state of the three qubits $|\psi\rangle$ and $|\phi^+\rangle$ we name $|\pi_0\rangle$.

In detail:

$|\pi_0\rangle = |B\rangle|A\rangle|\psi\rangle$ or $|BA\psi\rangle$

The qubit $|\psi\rangle$ of Alice is in the state $\alpha|0\rangle + \beta|1\rangle$:

We get:

$$|\pi_0\rangle = |\phi^+\rangle(\alpha|0\rangle + \beta|1\rangle)$$

We expand $|\phi^+\rangle$:

$$|\pi_0\rangle = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)(\alpha|0\rangle + \beta|1\rangle) =$$

$$\frac{\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle}{\sqrt{2}}$$

We apply the $CNOT$ on $|\pi_0\rangle$ and get $|\pi_1\rangle$.

Note that the $CNOT$ is controlled by the last qubit and acts on the second qubit.

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}}$$

We rewrite:

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}} = \frac{\alpha|00\rangle|0\rangle + \alpha|11\rangle|0\rangle + \beta|01\rangle|1\rangle + \beta|10\rangle|1\rangle}{\sqrt{2}} =$$

$$\frac{(\alpha|00\rangle + \alpha|11\rangle)|0\rangle + (\beta|01\rangle + \beta|10\rangle)|1\rangle}{\sqrt{2}}$$

We apply the Hadamard on $|\psi\rangle$:

$$H|\pi_1\rangle = \frac{(\alpha|00\rangle + \alpha|11\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + (\beta|01\rangle + \beta|10\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}{\sqrt{2}} =$$

$$\frac{(\alpha|00\rangle + \alpha|11\rangle)(|0\rangle + |1\rangle) + (\beta|01\rangle + \beta|10\rangle)(|0\rangle - |1\rangle)}{2} =$$

$$\frac{\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle + \beta|100\rangle - \beta|101\rangle}{2} = \pi_2$$

This is the state after Alice applied the $CNOT$ and the Hadamard onto her two qubits, $|\psi\rangle$ and her half of $|\phi^+\rangle$.

We remember that constants can float freely through tensor products and regroup π_2:

$$(\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle + \beta|100\rangle - \beta|101\rangle) \rightarrow$$

$$\frac{1}{2}(\alpha|000\rangle + \beta|100\rangle + \alpha|001\rangle - \beta|101\rangle + \alpha|110\rangle + \beta|010\rangle + \alpha|111\rangle - \beta|011\rangle) =$$

$$\frac{1}{2}((\alpha|0\rangle + \beta|1\rangle)|00\rangle + (\alpha|0\rangle - \beta|1\rangle)|01\rangle + (\alpha|1\rangle + \beta|0\rangle)|10\rangle + (\alpha|1\rangle - \beta|0\rangle)|11\rangle)$$

We remember the ordering: $|\pi_2\rangle = |B\rangle|A\rangle|\psi\rangle$

The effect of the reordering: It seems as if $|B\rangle$ has changed and depends upon α and β.

Now Alice measures. We take a look at the possible results of m_1 and m_2:

$$prob(m_1 m_2 = 00) = \frac{1}{4}\|\alpha|0\rangle + \beta|1\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1 m_2 = 01) = \frac{1}{4}\|\alpha|0\rangle - \beta|1\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1 m_2 = 10) = \frac{1}{4}\|\alpha|1\rangle + \beta|0\rangle\|^2 = \frac{1}{4}$$
$$prob(m_1 m_2 = 11) = \frac{1}{4}\|\alpha|1\rangle + \beta|0\rangle\|^2 = \frac{1}{4}$$

All outcomes on Alice' side has equal probability. We have no dependency on α and β meaning that the choice of α and β have no impact on the result of the measurement.

Note: After the measurement the qubit of Alice is destroyed so we will get no conflict with the no-clone theorem.

We build a list of all measurement outcomes Alice takes and the conditional state of $|B\rangle|A\rangle|\psi\rangle$:

$m_1 m_2$	probability	Conditional state $|B\rangle|A\rangle|\psi\rangle$
00	$\frac{1}{4}$	$(\alpha|0\rangle + \beta|1\rangle)|00\rangle$
01	$\frac{1}{4}$	$(\alpha|0\rangle - \beta|1\rangle)|01\rangle$
10	$\frac{1}{4}$	$(\alpha|1\rangle + \beta|0\rangle)|10\rangle$
11	$\frac{1}{4}$	$(\alpha|1\rangle - \beta|0\rangle)|11\rangle$

Note: No information about α and β could be gained by measurement on Bob's side.

Now Bob performs his operations on qubit $|B\rangle$ according to the measurement results he got from Alice. We add this to our table:

$m_1 m_2$	probability	Conditional state $|B\rangle|A\rangle|\psi\rangle$	Operation on $|B\rangle$	Final state of $|B\rangle$
00	$\frac{1}{4}$	$(\alpha|0\rangle + \beta|1\rangle)|00\rangle$	Id	$\alpha|0\rangle + \beta|1\rangle$
01	$\frac{1}{4}$	$(\alpha|0\rangle - \beta|1\rangle)|01\rangle$	Z	$\alpha|0\rangle + \beta|1\rangle$
10	$\frac{1}{4}$	$(\alpha|1\rangle + \beta|0\rangle)|10\rangle$	X	$\alpha|0\rangle + \beta|1\rangle$
11	$\frac{1}{4}$	$(\alpha|1\rangle - \beta|0\rangle)|11\rangle$	Z X	$\alpha|0\rangle + \beta|1\rangle$

We see that Bob's qubit $|B\rangle$ now is in the state Alice's qubit $|\psi\rangle$ was. The state of the qubit has been "teleported", $|\psi\rangle$ and $|\phi^+\rangle$ are destroyed.

Note: You may find more information at:

https://learning.quantum.ibm.com/course/basics-of-quantum-information/entanglement-in-action

So far, we have worked on the conceptual level with bras and kets.

How does this look like if we work with state vectors? Let us try this too.

We have in total three qubits so our state vector will have 8 dimensions.

Note: You may refer to "Teleportation_1" if you want to derive the explicit forms of the gates $CNOT$, X and Z.

Using the same names as at the conceptual level we have:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note: $|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$|\pi_0\rangle = |\psi\rangle|\phi^+\rangle = |\psi\rangle \otimes |\phi^+\rangle =$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

The CNOT from line one to line two:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We apply the CNOT from line one to line two:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{pmatrix}$$

Hadamard on line one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We apply the Hadamard gate onto line one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$$

We compare this with the conceptual level, there we had:

$$\frac{1}{2}\big((\alpha|0\rangle + \beta|1\rangle)|00\rangle + (\alpha|0\rangle - \beta|1\rangle)|01\rangle + (\alpha|1\rangle + \beta|0\rangle)|10\rangle + (\alpha|1\rangle - \beta|0\rangle)|11\rangle\big)$$

We see the correspondence:

$\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ $\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix}$	collapse to $\lvert 00\rangle \to \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ remains	$\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ $\begin{pmatrix} 0 \\ \alpha \\ 0 \\ 0 \\ 0 \\ -\beta \\ 0 \\ 0 \end{pmatrix}$	collapse to $\lvert 01\rangle \to \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ remains
$\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix}$	collapse to $\lvert 10\rangle \to \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ remains	$\frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\beta \\ 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$	collapse to $\lvert 11\rangle \to \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$ remains

Measuring collapses the state vector leaving different residues. They are determined so Bob can choose the gates he must use to restore the original vector $|\psi\rangle$ of Alice.